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On the geometry of Riemannian hypersurfaces: uniqueness, nonexistence, stability and bifurcation

por

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Campina Grande - PB
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sob orientação do

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Tese apresentada ao Corpo Docente do Programa Associado de Pós-Graduação em Matemática - UFPB/UFCG, como requisito parcial para obtenção do título de Doutor em Matemática.

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Resumo

Neste trabalho estudamos alguns problemas relacionados à geometria de hipersuperfícies Riemannianas imersas em variedades semi-Riemannianas (com índice zero ou um) equipadas com uma função densidade e que podem ser modeladas por uma certa classe de produtos warped. Inicialmente, assumindo condições razoáveis na curvatura média ponderada de tais hipersuperfícies e considerando certas restrições no espaço ambiente, estabelecermos alguns resultados de unicidade e não-existência. Também estabelecermos resultados de estabilidade, bifurcação e rigidez local associados à problemas variacionais que envolvem o funcional 1-área e o funcional área ponderada de tais hipersuperfície.

Palavras-chave: variedades ponderadas; produtos *warped*; hipersuperfícies Riemannianas; tensor de Bakry-Émery-Ricci; curvatura média ponderada; f-Lapaciano; f-parabolicidade; estabilidade, bifurcação; rigidez local.

Abstract

In this work we study some problems related to the geometry of Riemannian hypersurfaces immersed in semi-Riemannian manifolds (with index zero or one) equipped with a density function and that can be modeled by a certain class of warped products. Initially, assuming reasonable conditions in the weighted mean curvature of such hypersurfaces and considering certain restrictions in the ambient space, we establish some results of uniqueness and non-existence. We also establish results of stability, bifurcation and local rigidity associated with variational problems involving the functional 1-area and the functional weighted area of such a hypersurface.

Keywords: wighted manifolds; warped products; Riemannian hypersurfaces; Bakry-Émery-Ricci tensor, weighted mean curvature; f-Lapacian; f-parabolicity; estability, bifurcation; local rigidity.

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" ... E você estava esperando voar, mas como chegar até as nuvens com os pés no chão?"

Renato Russo

Dedicatória

A Juarez Cavalcante Brito Júnior
in memoriam

Contents

Introduction

A *weighted manifold* $\overline{M}_f^{n+1}$ is a semi-Riemannian manifold $(\overline{M}^{n+1}, g)$ endowed with a weighted volume form $d\mu = e^{-f}dv$, where the *weight function* f is a real-valued smooth function on $\overline{M}^{n+1}$ and dv is the volume element induced by the metric g (for details see, for instance, [13, 60]). In this setting, as a crucial ingredient to understand the geometry of a weighted manifold $\overline{M}_f^{n+1}$, the so-called *Bakry-Émery-Ricci tensor* $\overline{\mathrm{Ric}}_f$ is introduced as being the following extension of the standard Ricci tensor $\overline{\mathrm{Ric}}$ of $\overline{M}^{n+1}$:

$$\overline{\mathrm{Ric}}_f \;=\; \overline{\mathrm{Ric}} + \overline{\mathrm{Hess}}\, f,$$

where $\overline{\mathrm{Hess}}$ is the Hessian tensor in $\overline{M}^{n+1}$. Other weighted objects, such as the *weighted mean curvature* and the *f-divergence* can also be considered. A natural line of investigation that appears into this thematic is the problem of extend results stated in terms of the Ricci curvature, the mean curvature or the divergence, for example, to analogous results for the Bakry-Émery-Ricci tensor, the weghted mean curvature or the f-divergence.

It is also interesting to remark that weighted manifolds are closely related to some classical mathematical concepts, as they can be used as a powerful mathematical tool in order to obtain new results related to them. For some results of geometric investigations concerning these weighted manifolds, we also refer the reader to the articles of Morgan [61] and Wei-Wylie [68].

A theme that has been widely approached in isometric immersion theory in recent years is the study of the geometry of semi-Riemannian manifolds that can be regarded as warped products of the tipe $\big(M^n \times_\alpha I, \pi_{M^n}^*(\langle\cdot\,,\cdot\rangle_{M^n}) + (\alpha\circ\pi_{M^n})^2\pi_{\mathbb{R}}^*(\epsilon dt^2)\big)$

or $\left(I \times_\alpha\ M^n, \pi_{\mathbb{R}}^*(\epsilon dt^2) + (\alpha \circ \pi_{\mathbb{R}})^2 \pi_{M^n}^*(\langle\cdot,\cdot\rangle_{M^n})\right)$, where M^n is a Riemannian manifold, $I \subset \mathbb{R}$ is an open interval, π_{M^n} and $\pi_{\mathbb{R}}$ denote the canonical projections from $M^n \times I$ or $I \times M^n$ onto each factor, $\langle\cdot,\cdot\rangle_{M^n}$ is the Riemannian metric of M^n, α is a positive function defined at the *base* (i.e. at the first factor) of the product and $\epsilon \in \{-1, 1\}$ is a constant that defines the causal character of the product (the warped product is a Lorentzian manifold when $\epsilon = -1$ and a Riemannian manifold when $\epsilon = 1$). These ambient spaces are naturaly foliated by a family of totally umbilical (spacelike, in the Lorentzian case) hypersurfaces $\Sigma_t^n := \Sigma^n \times \{t\}$ or $\Sigma_t^n := \{t\} \times \Sigma^n, t \in I$, that will be called *slices*. In this setting, an interesting question is to investigate the uniqueness of such slices among (spacelike) hypersurfaces of the warped product, under reasonable assumptions on their geometric data.

This branch of study, currently known as Bernstein (Calabi-Bernstein, in the Lorentzian case) tipe results or also rigidity results, had its beginnings when Bernstein [15] proved that the only entire minimal graphs in the 3-dimensional Euclidean space $\mathbb{R}^3$ are the planes. In the Lorentzian setting, there is an analogue result to Bernstein's theorem, which states that the only entire maximal graphs in the 3-dimensional Lorentz-Minkowski space $\mathbb{L}^3$ are the spacelike planes. This result was firstly proved by Calabi [18], and extended to the general n-dimensional case by Cheng and Yau [26]. A natural extension to the Benstein and Calabi-Bernstein problems is to determine a reasonable set of sufficient conditions which guarantee the uniqueness (or nonexistence) of complete (spacelike) hypersurfaces immersed into a certain ambient space.

Another theme that has ben aroused the interest of some geometers it is the study of variational questions associated to the area functional in Riemannian manifolds. A example of this branch is the study of stability of hypersurfaces with constant mean curvature H (shortly, H*-hypersurfaces*) in Riemannian manifolds $\overline{M}^{n+1}$ $(n \geq 2)$, wich began with Barbosa and do Carmo in [9], and Barbosa, do Carmo and Eschenburg in [8]. In these papers, they introduced the notion of stability and proved that any closed H-hypersurface immersed into $\overline{M}^{n+1}$ is a critical point of the variational problem of minimizing the area functional for volume-preserving variations.

In this thesis we will present unicity and nonexistence results related to spacelike hypersurfaces immesed in semi-Riemannian manifolds that can be regarded as one of the weighted warped products decribed above. We will also do a study of stability, local rigidity and bifurcation for variational problems associated with the functional 1-area and the functional weighted area in Riemannian spaces. The results that integrate the present work correspond to the contents of papers [27], [31] [32], [33], [34] and [35].

In the Chapter 1, we describe the ambient spaces that will appear throughout this work, recall some facts about hypersurfaces immersed in such spaces and we have also establish most of the notations that will be used.

In the Chapter 2, based in the paper [35], done in collaboation with H. F. de Lima, A. M. Oliveira, M. S. Santos and M. A. L. Velásquez, we investigate the geometry of conformal Killing graphs in a weighted Riemannian manifold $\overline{M}^{n+1}_f$ endowed with a complete conformal Killing vector field V, which are defined via the global flow associated to V over an integral leaf of the distribution $V^\perp$ (for more details see Section 1.4.1). Taking into account the Cheeger-Gromoll type splitting theorems due to Wei and Wylie [68], we assume that the weight function f does not depend on the parameter of the flow associated to unit vector field $\nu = -V/|V|$ (see Remark 1.5). In these circumstances, we calculate a formula for the f-Laplacian of the support function $g(N,V)$ (cf. Lemma 2.4), where N is the Gauss map of the conformal Killing graph $\Sigma(z)$. Afterwards, in Section 2.2, under a suitable restriction on the norm of the gradient of the function z, which determines such a graph $\Sigma(z)$, we establish sufficient conditions to ensure that $\Sigma(z)$ is totally umbilical and, in particular, an integral leaf of $V^\perp$ (cf. Theorems 2.6, 2.7, 2.10 and 2.11 and Corollaries 2.8, 2.9, 2.12 and 2.13). Our approach is based on the use of the f-Laplacian of the supported function $g(N,V)$, the f-divergence of the tangent part of V on $\Sigma(z)$, jointly with a weighted version of Stoke's Theorem to the context of complete weighted Riemannian manifolds (see Lemma 2.1).

In Section 2.3 we study the stability of f-minimal conformal Killing graphs of $\overline{M}^{n+1}_f$ according to the behavior of the derivative of the conformal factor ψ_V of V, obtaining sufficient conditions to guarantee that an f-minimal conformal Killing graphs be L_f-stable, where L_f stands for the weighted Jacobi operator (cf. Theorem 2.16 and Corollary 2.17). Finally, in Section 2.4 our goal is to investigate the strong f-stability of closed conformal Killing graphs in $\overline{M}^{n+1}_f$ with constant f-mean curvature. More specifically, we get sufficient conditions to a strong f-stable closed conformal Killing graphs be either f-minimal or isometric to a leaf of $V^\perp$ (cf. Theorem 2.19 and

Corollary 2.20).

As it is well known, an $(n+1)$-dimensional Riemannian space $(\overline{M}^{n+1}, \langle \cdot , \cdot \rangle)$ endowed with a suitable Killing vector field Y can be regard as a *Killing warped product* $(M^n \times_\alpha \mathbb{R}, \langle \cdot , \cdot \rangle)$, for an appropriate n-dimensional Riemannian manifold M^n and a certain warping function α (for more details, see Section 1.4.2). In the Chapter 3, based in the paper [32], caried out in collaboation with H. F. de Lima and M. A. L. Velásquez, we obtain uniqueness results related to the mean curvature equation for entire Killing graphs $\Sigma^n(z)$ constructed over the base M^n of a weighted Killing warped product $M^n_f \times_\alpha \mathbb{R}$ with warping function α and whose weight function f does not depend on the parameter $t \in \mathbb{R}$, that is, $\langle \overline{\nabla} f, \partial/\partial t \rangle = 0$ (see Theorem 3.9, Theorem 3.10, Theorem 3.11 and Corollary 3.12 in Section 3.3). For this, in Section 3.1 we establish a suitable f-parabolicity criterion (see Proposition 3.3 and Corollary 3.4) and, under appropriate constraints on the Bakry-Émery-Ricci tensor and on the f-mean curvature, in Section 3.2 we prove some rigidity results concerning complete two-sided hypersurfaces immersed into $M^n_f \times_\alpha \mathbb{R}$ (see Theorem 3.5, Corollary 3.6, Theorem 3.7 and Theorem 3.8).

In the Chapter 4, based in the paper [33], which was done in collaboration with E. L. de Lima, H. F. de Lima and M. A. L. Velásquez, our objective is to carry out a study on the uniqueness, nonexistence and stability of spacelike hypersurfaces immersed into a weighted standard static spacetime $M^n_f \times_\alpha \mathbb{R}_1$, the Lorentzian dual of the weighted Killing warped product space dealt with in the Chapter 3, endowed with a weighted function f does not depend on the parameter $t \in \mathbb{R}$.

We start by obtaining explicit formulas for the Laplacian of the height function h (see Proposition 4.1) and the drift Laplacian of the angle function Θ (see Proposition 4.12), both functions naturally related to a spacelike hypersurface Σ^n immersed into $M^n_f \times_\alpha \mathbb{R}_1$. Then, applying some analytical results to subharmonic smooth functions on complete Riemannian manifolds (for example: some parabolocity criteria, a weak form of the Omori-Yau maximum principle and an extension of the Hopf's Theorem due to Yau) and considering suitable constraints on the f-mean curvature of Σ^n, on the height function h, sometimes on angle function Θ and on the Bakry-Émery-Ricci tensor of M^n, we establish some uniqueness results (see Theorems 4.3, 4.7, 4.10, 4.13 and 4.17, and Corollary 4.4) and some nonexistence results (see Corollaries 4.8, 4.14 and 4.18). In Remark 4.15 we exhibit a large family of standard static spacetimes that verify the hypotheses adopted in Theorem 4.13 and also in their corollary. Next, in Corollaries 4.11, 4.16 and 4.19 we make a particular study on the Calabi-Bernstein type properties of entire Killing graphs $\Sigma^n(z)$ constructed from a smooth function z defined on the base M^n of $M_f^n \times_\alpha \mathbb{R}_1$.

Proceeding, in Section 4.3, we show that closed spacelike hypersurfaces immersed with constant f-mean curvature in a weighted standard static spacetime $M_f^n \times_\alpha \mathbb{R}_1$ are solutions of the variational problem of maximizing the weighted area functional for all variations that keeps the balance of weighted volume equal to zero (see Proposition 4.22). As a consequence, we establish the notion of f-stability for such hypersurfaces (Definition 4.23) and provide an appropriate stability criterion (Proposition 4.24). Finally, in Theorem 4.26 we obtain a characterization of f-stable closed spacelike hypersurfaces of $M_f^n \times_\alpha \mathbb{R}_1$ through the first nonzero eigenvalue of the drift Laplacian.

In the Chapter 5, based in the paper [34], carried out in collaboration with H. de Lima and M. A. L. Velásquez, we obtain uniqueness results related to the mean curvature equation for entire Killing graphs constructed over the Riemannian base M^n of a weighted standard static spacetime $M_f^n \times_\alpha \mathbb{R}_1$. As in the Riemannian case, dealt with in the Chapter 3, in Section 5.1 we establish a suitable f-parabolicity criterion and, in the Section 5.2, assuming certain control over the Bakry-Émery Ricci tensor and over the f-mean curvature, we study the rigidity of spacelike hypersurfaces immersed in $M_f^n \times_\alpha \mathbb{R}_1$. Finally, we point out that, in Section 5.3, applications of our main results to weighted standard static spacetimes of the type $\mathbb{G}^n \times_\alpha \mathbb{R}_1$, where $\mathbb{G}^n$ stands for the so-called *Gaussian space* which is nothing but that the Euclidian space $\mathbb{R}^n$ endowed with the Gaussian probability density $e^{-f(y)} = (2\pi)^{-\frac{n+1}{2}} e^{-\frac{|y|^2}{2}}$, $y \in \mathbb{R}^n$, are also given.

As observed in [3, 16, 17], the set of trial maps for the variational problem of minimizing the area functional for volume-preserving variations should be a collection of embeddings of H-hypersufcaces Σ^n into $\overline{M}^{n+1}$; in order to detect solutions that are not isometrically congruent, one should take into consideration the action of the diffeomorphism group of Σ^n, acting by right composition in the space of embeddings, and the action of the isometry group of $\overline{M}^{n+1}$, acting by left composition on the space of embeddings. The action of the diffeomorphism group of Σ^n on any set of embeddings of H-hypersufcaces Σ^n into $\overline{M}^{n+1}$ is free, which suggests that one should consider a quotient of the space of embeddings by this action. This means that two embeddings of H-hypersufcaces $x_1 : \Sigma_1^n \looparrowright \overline{M}^{n+1}$ and $x_2 : \Sigma_2^n \looparrowright \overline{M}^{n+1}$ will be considered *equivalent* if there exists a diffeomorphism $\phi : \Sigma_1^n \to \Sigma_2^n$ such that $x_2 = x_1 \circ \phi$. As to the left action of the isometry group of $\overline{M}^{n+1}$, this is not free; nevertheless, the group is compact and one can study a *bifurcation* problem for its critical orbits. Thus, the variational problem described above provides us with a framework where we can study the *equivariant bifurcation* (cf. [3, 16, 17, 66]) in a set of equivalence classes of embeddings of H-hypersufcaces Σ^n into $\overline{M}^{n+1}$.

In this context, our purpose in the Chapter 6 is to study the notions of local rigidity, bifurcation instants and stability associated with the 1-area and f-area functional for a family of open sets in certain warped products, using equivariant bifurcation theory in order to establish sufficient conditions that allow us to guarantee the existence of bifurcation instants or the local rigidity of such families.

In Section 6.2, based in the paper [27], which is a collaboration with J. Q. Oliveira, J. F. da Silva and M. A. L. Velásquez, considering a warped product $I \times_\alpha M^n$ with compact (without boundary) Riemannian fiber M^n, our purpose is to investigate the existence of bifurcation instants or the local rigidity of a certain family $\{\Omega_\tau\}_{\tau\in(\tau_1,\tau_2]}$ of open sets whose boundaries are H_2-hypersurfaces. For this, initially, in a Riemannian manifold $\overline{M}^{n+1}$ we consider the space of open subsets $\Omega \subset \overline{M}^{n+1}$ with compact closure and whose smooth compact boundary $\partial\Omega$ is an orientable hypersurface and we study the variational problem of

$$\text{(VP-1)}: \textit{ minimizing the } 1\textit{-area functional } \mathcal{A}_1(t) \textit{ for all variations of } \\ \partial\Omega \textit{ that preserve the volume of } \Omega.$$

We assemble the Jacobi functional $\mathcal{F}^\lambda(t) = \mathcal{A}_1(t) + \lambda\mathcal{V}(t)$ (see (6.6)) associated with the variational problem, where $\mathcal{V}(t)$ (see (6.4)) is the balance of volume and λ is a real constant, we calculate its first variation $\frac{d}{dt}\mathcal{F}^\lambda(0)$ (see (6.12)), and as a consequence we get that the open subsets Ω of $\overline{M}^{n+1}$ whose boundary $\partial\Omega$ is a compact H_2-hypersurface are characterized as critical points of (VP-1) since the Ricci curvature $\mathrm{Ric}_{\overline{M}}(\,,\,)$ of $\overline{M}^{n+1}$ in the normal directions N_t of the volume-preserving variations is constant (full details can be found in Proposition 6.5 and its subsequent comments). It is immediate to note that any Einstein manifold verifies the adopted condition on the Ricci curvature, and thus we are obtaining a type of extension for the variational characterization of compact H_2-hypersurfaces obtained in [1] and [44]. When we change our variational problem for the of

$$\text{(VP-2)}: \textit{ minimizing the } 1\textit{-area functional } \mathcal{A}_1(t) \textit{ for all variations of } \partial\Omega, \textit{ not necessarily volume-preserving variations of } \Omega,$$

in Proposition 6.5 we observed that the respective critical points of (VP-2) coincide with the same critical points of the initial variational problem (VP-1). For each of these critical points, in Proposition 6.7 we calculate the second variation $\frac{d^2}{dt^2}\mathcal{F}^\lambda(0)$ in terms of the Jacobi differential operator $\mathcal{J}$ (see (6.18)). Furthermore, for a family of critical points $\{\Omega_\tau\} \subset \overline{M}^{n+1}$ associated with our variational problem (VP-2), in Subsection 6.2.2 we use the equivariant bifurcation theory to establish our notions of bifurcation instants and local rigidity, as well as to relate these two concepts to the *Morse index* $\mathrm{Ind}\left(\mathcal{F}^{\lambda(\tau)}, \Omega_\tau\right)$ of each Ω_τ, which in turn can be understood as the number of negative eigenvalues (counted with multiplicity) of the Jacobi operator $\mathcal{J}_\tau$ on Ω_τ.

We begin the Section 6.2.3 by listing in Table 6.1 all Riemmanian warped products of the type $I \times_\alpha M^n$ that satisfy the condition on the Ricci curvature that we are assuming. Then, in these products, we consider the family $\{\Omega_\tau\}_{\tau\in(\tau_1,\tau_2]}$ of open subsets of $(I \times_\alpha M^n, d\tau^2 + \alpha(\tau)^2\langle\cdot\,,\cdot\rangle_M)$ given by

$$\Omega_\tau = (\tau_1, \tau) \times M^n\,, \quad \text{with} \quad \tau \in (\tau_1, \tau_2],$$

where τ_1 and τ_2 are fixed numbers in $I \subset \mathbb{R}$. Thus, assuming M^n to be compact (without boundary), we have that the boundary $\partial\Omega_\tau$ of each Ω_τ is the disjoint union $\partial\Omega_\tau = \Sigma^n_{\tau_1} \cup \Sigma^n_\tau$ of two compact hypersurfaces $\Sigma^n_{\tau_1} = \{\tau_1\} \times M^n$ (fixed) and $\Sigma^n_\tau = \{\tau\} \times M^n$. Since the variations of $\partial\Omega_\tau$ only affects Σ^n_τ and taking into account that Σ^n_τ is a compact H_2^τ-hypersurface with constant second mean curvature $H_2^\tau = (f'(\tau)/f(\tau))^2$, we have that each element of $\{\Omega_\tau\}_{\tau\in(\tau_1,\tau_2]}$ is a critical point for the variational problem (VP-2).

Next, in Proposition 6.8 we collect all the elements that are sufficient to get an explicit expression for the eigenvalues of the Jacobi operator $\mathcal{J}_\tau$ of each element of $\{\Omega_\tau\}_{\tau\in(\tau_1,\tau_2]}$, an expression that we will allow to calculate the Morse index $\mathrm{Ind}\left(\mathcal{F}^{\lambda(\tau)}, \Omega_\tau\right)$. Then, in Theorem 6.9, considering appropriate conditions of the spectrum of the Laplacian on M^n and the warped function α, we establish the local rigidity of the family $\{\Omega_\tau\}_{\tau\in(\tau_1,\tau_2]} \subset I \times_\alpha M^n$ (see also Corollary 6.10). Furthermore, in Theorem 6.11 and in Theorem 6.13 we establish some sufficient conditions in terms of α and the behavior of eigenvalues of the Laplacian on M^n to obtain bifurcation instants of $\{\Omega_\tau\}_{\tau\in(\tau_1,\tau_2]} \subset I \times_\alpha M^n$ (see also Corollary 6.12 and Corollary 6.14). Finally, in Tables 6.2, 6.3 and 6.4 we list examples that verify all the conditions we are assuming.

In Section 6.3, we based in the paper [31], also carried out in collaboration with H. F. de Lima and M. A. L. Velásquez. There, our purpose is to study the notions of local rigidity, bifurcation instants and stability for a family of open sets $\{\Omega_\gamma\}_\gamma$ of a weighted Killing warped product $M^n_f \times_\alpha \mathbb{R}$ whose boundaries $\partial\Omega_\gamma$ are closed hypersurfaces with constant weighted mean curvature $H_f(\gamma)$ (in abbreviation, we say that $\partial\Omega_\gamma$ is a closed $H_f(\gamma)$-*hypersufcace*), where γ varies on a prescribed interval $I \subset \mathbb{R}$.

For this we consider the variational problems:

(VP-3) : *Minimizing the weighted area functional $\mathcal{A}_f$ (see (6.36)) for all variations of $\partial\Omega_\gamma$ that preserve the weighted volume of Ω_γ,*

(VP-4) : *Minimizing the weighted area functional $\mathcal{A}_f$ (see (6.36)) for all variations of $\partial\Omega_\gamma$, not necessarily weighted volume-preserving variations of Ω_γ.*

By an analysis of the first variation of the associated *weighted Jacobi functional*

$$\mathcal{F}_f^{\lambda(\gamma)} = \mathcal{A}_f + \lambda(\gamma)\mathcal{V}_f, \quad \text{with } \lambda(\gamma) \in \mathbb{R}$$

(see (6.37)), where $\mathcal{V}_f$ is the weighted volume functional (see (6.35)), we obtain in Proposition 6.18 that the critical points of (VP-3) and (VP-4) are the open sets Ω_γ whose boundary $\partial\Omega_\gamma$ is a closed $H_f(\gamma)$-hypersurface with constant weighted mean curvature $H_f(\gamma) = \lambda(\gamma)/n$. For these critical points, in Proposition 6.21 we obtain the formula of the second variation of $\mathcal{F}_f^{\lambda(\gamma)}$.

Concerning the variational problem (VP-4), in Subsection 6.3.2 we use the equivariant bifurcation theory (cf. [3, 17, 16, 66]) to establish our notions of bifurcation instants and local rigidity in terms of the *Morse index* of the *weighted Jacobi operator* $\mathcal{J}_{f;\gamma}$ (see (6.52)). Then, in Section 6.3.3 we get some results of local rigidity and bifurcation instants in $M^n_f \times_\alpha \mathbb{R}$ via the analysis the number of negative eigenvalues of $\mathcal{J}_{f;\gamma}$.

Chapter 1

Preliminaries

In this chapter, our aim is to establish the major part of the notations that will be used and describe the ambient spaces that will be appear throughout this work.

1.1 Riemannian setting

Let $\overline{M}^{n+1}$ be a $(n+1)$-dimensional orientable Riemannian manifold ($n \geq 2$) with metric tensor $\langle \cdot\,,\cdot \rangle$, Levi-Civita connection $\overline{\nabla}$ and curvature tensor $\overline{R}$. We denote by $\mathfrak{X}(\overline{M})$ the set of vector fields of class C^∞ on $\overline{M}^{n+1}$, by $C^\infty(\overline{M})$ the ring of real functions of class C^∞ on $\overline{M}^{n+1}$ and by $C_0^\infty(\overline{M})$ the set of all smooth functions defined in $\overline{M}^{n+1}$ with compact support. In this context, we consider *hypersurfaces* $x : \Sigma^n \looparrowright \overline{M}^{n+1}$, namely, isometric immersions from a connected, n-dimensional orientable Riemannian manifold Σ^n into $\overline{M}^{n+1}$. Since Σ^n is orientable, one can choose a globally defined unit normal vector field N on Σ^n, which will be called the *Gauss map* of $x : \Sigma^n \hookrightarrow \overline{M}_f^{n+1}$. The *shape operator* of $x : \Sigma^n \looparrowright \overline{M}^{n+1}$ with respect to N is given by

$$\begin{array}{rcl} A : \mathfrak{X}(\Sigma^n) & \to & \mathfrak{X}(\Sigma^n) \\ Y & \mapsto & A(Y) = -\overline{\nabla}_Y N. \end{array}$$

Since, for each fixed $p \in \Sigma^n$, $A_p : T_p\Sigma \to T_p\Sigma$ is a self-adjoint linear map, the spectral theorem allows us to choose on $T_p\Sigma$ an orthonormal basis $\{e_1, \ldots, e_n\}$ of eigenvectors of A_p, with corresponding eigenvalues $\kappa_1(p), \ldots, \kappa_n(p)$, respectively. The functions $\kappa_1, \ldots, \kappa_n$ on Σ^n thus defined are called *principal curvatures* of $x : \Sigma^n \looparrowright \overline{M}^{n+1}$. Moreover, it is well known that the curvature tensor R of Σ^n is described in terms of A and $\overline{R}$ by the so called *Gauss equation*, which can be written as

$$R(U,V)W = (\overline{R}(U,V)W)^\top + \langle A(U), W\rangle A(V) - \langle A(V), W\rangle A(U) \tag{1.1}$$

for all $U, V, W \in \mathfrak{X}(\Sigma)$, where $(\,\cdot\,)^\top$ stands for tangential components on Σ^n.

We will deal with the *first three mean curvatures* of the hypersurface $x : \Sigma^n \looparrowright \overline{M}^{n+1}$, namely

$$\begin{cases} H_1 = \dfrac{1}{n}\displaystyle\sum_{i=1}^{n} \kappa_i, \\ H_2 = \dfrac{2}{n(n-1)}\displaystyle\sum_{i<j} \kappa_i\kappa_j, \\ H_3 = \dfrac{6}{n(n-1)(n-2)}\displaystyle\sum_{i<j<k} \kappa_i\kappa_j\kappa_k. \end{cases} \tag{1.2}$$

We have that H_1 is the *mean curvature* of $x : \Sigma^n \looparrowright \overline{M}^{n+1}$, which is the main extrinsic curvature of Σ^n and when there is no danger of confusion it will be denote simply by H. On the other hand, the *second mean curvature* H_2 defines a geometric quantity which is related to the *scalar curvature* S of $x : \Sigma^n \looparrowright \overline{M}^{n+1}$. Indeed, it follows from the Gauss equation (1.1) that the (non-normalized) Ricci curvature Ric_Σ of $x : \Sigma^n \looparrowright \overline{M}^{n+1}$ is given by

$$\mathrm{Ric}_\Sigma(U,V) = \mathrm{Ric}_{\overline{M}}(U,V) - \langle \overline{R}(U,N)V, N\rangle + nH_1\langle A(U),V\rangle - \langle A(U),A(V)\rangle,$$

for $U,V \in \mathfrak{X}(\Sigma^n)$, where $\mathrm{Ric}_{\overline{M}}$ stands for the Ricci curvature of $\overline{M}^{n+1}$. Therefore, S obeys the relation

$$\mathrm{S} = \overline{\mathrm{S}} - 2\,\mathrm{Ric}_{\overline{M}}(N,N) + n(n-1)H_2, \tag{1.3}$$

where $\overline{\mathrm{S}}$ stands for the scalar curvature of $\overline{M}^{n+1}$. For instance, if there is one $\overline{\varrho} \in \mathbb{R}$ such that the Ricci curvature of $\overline{M}^{n+1}$ verifies the condition

$$\mathrm{Ric}_{\overline{M}}(N,N) = \overline{\varrho} = \text{const.} \quad \text{on} \quad \Sigma^n, \tag{1.4}$$

we get from (1.3) and (1.4) that S and H_2 are related by

$$\mathrm{S} = (n-1)\left(\overline{\varrho} + nH_2\right). \tag{1.5}$$

When required, if a hypersurface $x : \Sigma^n \looparrowright \overline{M}^{n+1}$ has constant second mean curvature H_2, for short we will simply say that $x : \Sigma^n \looparrowright \overline{M}^{n+1}$ is an *H_2-hypersurface*.

One also let the *Newton transformation* $T : \mathfrak{X}(\Sigma^n) \to \mathfrak{X}(\Sigma^n)$ associated with $x : \Sigma^n \looparrowright \overline{M}^{n+1}$ be given by setting

$$T = nH_1\,\mathrm{Id} - A, \tag{1.6}$$

where $\mathrm{Id} : \mathfrak{X}(\Sigma^n) \to \mathfrak{X}(\Sigma^n)$ denotes the identity map.

Associated to the Newton transformation T one has the well known *Cheng-Yau's square operator* [25]

$$\begin{array}{rccl} \Box : & C^\infty(\Sigma^n) & \to & C^\infty(\Sigma^n) \\ & u & \mapsto & \Box(u) = \mathrm{tr}\left(T \circ \mathrm{Hess}_\Sigma\, u\right), \end{array} \tag{1.7}$$

that is a second order differential operator, where Hess_Σ stands for the Hessian operator on Σ^n.

1.2 Lorentzian setting

Let $(\overline{M}^{n+1}, \langle\cdot\,,\cdot\rangle)$ be a $(n+1)$-dimentional Lorentzian manifold. We mean by $C^\infty(\overline{M})$ the ring of real functions of class C^∞ on $\overline{M}^{n+1}$ and by $\mathfrak{X}(\overline{M})$ the $C^\infty(\overline{M})$-module of vector fields of class C^∞ on $\overline{M}^{n+1}$. We recall (cf. [62, Chapter 3]) that a vector field $X \in \mathfrak{X}(\overline{M})$ is said to be *timelike* if $\langle X, X\rangle < 0$ on $\overline{M}^{n+1}$; *spacelike* if $\langle X, X\rangle > 0$ on $\overline{M}^{n+1}$ and a *unit* vector field if $\langle X, X\rangle = \pm 1$ on $\overline{M}^{n+1}$. Furthermore, a Lorentzian manifold $\overline{M}^{n+1}$ is said to be *time-orientable* if there exist a timelike vector field globaly defined on $\overline{M}^{n+1}$. Consider a *spacelike hypersurface* $x : \Sigma^n \hookrightarrow \overline{M}_f^{n+1}$. It means that the induced metric on Σ^n via the immersion x is a Riemannian metric. When $\overline{M}^{n+1}$ is time-orientable by a timelike vector field (cf. [62, Lemma 5.32]), say a certain $K \in \mathfrak{X}(\overline{M})$, and Σ^n is a spacelike hypersurface, then Σ^n is orientable and one can choose a globally defined unit normal vector field N on Σ^n having the same time-orientation of $\overline{M}^{n+1}$, that is, $\langle K, N\rangle < 0$. Such N is said the *future-pointing Gauss map* of Σ^n. If we let

$$\begin{array}{rclcl} A & : & \mathfrak{X}(\Sigma^n) & \to & \mathfrak{X}(\Sigma^n) \\ & & Y & \mapsto & A(Y) \;=\; -\overline{\nabla}_Y N \end{array} \tag{1.8}$$

denote the *shape operator* of Σ^n with respect to N, then the *mean curvature* H of Σ^n is defined by

$$\begin{array}{rclcl} H & : & \Sigma^n & \to & \mathbb{R} \\ & & p & \mapsto & H(p) \;=\; -\dfrac{1}{n}\operatorname{tr}(A_p)\,. \end{array} \tag{1.9}$$

The choice of the sign in our definition of H is motivated by the fact that in that case the mean curvature vector is given by $\mathbf{H} = HN$ and, therefore, $H(p) > 0$ at a point $p \in \Sigma^n$ if, and only if, $\mathbf{H}(p)$ is in the same time-orientation as $N(p)$, and hence as $K(p)$.

1.3 Weighted manifolds

On a complete Riemannian manifold $\overline{M}^{n+1}$, let us remember that the classical Laplace operator Δ on $\overline{M}^{n+1}$ can be defined as the differential operator associated to the standard Dirichlet form

$$\mathcal{Q}(\varphi) = \int_{\overline{M}} |\nabla\varphi|^2 dv, \quad \varphi \in C_0^\infty(\overline{M}) \subset \mathcal{L}^2(dv),$$

where $|\cdot|$ is the norm induced by the Riemannian metric of $\overline{M}^{n+1}$, dv is the volume element on $\overline{M}^{n+1}$ and $\mathcal{L}^2(dv)$ denotes the set of measurable functions u on $\overline{M}^{n+1}$ such that the Lebesgue integral (with respect to dv) of $|\varphi|^2$ exists and is finite.

Now let $f \in C^\infty(\overline{M})$ be a real valued smooth function, that will be referred as a *weight function* (or *density function*). If we replace the measure dv with the *weighted* *measure*

$$d\mu = e^{-f} dv \tag{1.10}$$

in the definition of $\mathcal{Q}$, we obtain a new quadratic form $\mathcal{Q}_f$, and we will denote by Δ_f the elliptic operator on $C_0^\infty(\overline{M}) \subset \mathcal{L}^2(d\mu)$ induced by $\mathcal{Q}_f$. In this sense, Δ_f arises as a natural generalization of the Laplacian. It is clearly symmetric and positive and extends to a positive operator on $\mathcal{L}^2(d\mu)$. By Stokes theorem,

$$\Delta_f(\varphi) = \Delta\varphi - \langle\nabla\varphi, \nabla f\rangle, \quad \varphi \in C_0^\infty(\overline{M}).$$

The triple $(\overline{M}^{n+1}, \langle\cdot\,,\cdot\rangle, d\mu)$ and the differential operator Δ_f defined above and acting in $C^\infty(\overline{M})$ will be called, respectively, the *weighted manifold* associated with $\overline{M}^{n+1}$ and f, which we simply denote by $\overline{M}_f^{n+1}$, and the *f-Laplacian* (or *drift Laplacian*).

Let us remember that a Riemannian manifold Σ^n is *parabolic* if every bounded solution of $\Delta u \geq 0$ must be identically constant. We recall that a smooth function u on a weighted manifold $\overline{M}_f^{n+1}$ is said to be *f-superharmonic* if $\Delta_f(u) \leq 0$. Taking this into account, the weighted manifold $\overline{M}_f^{n+1}$ is called *f-parabolic* if the only nonnegative and f-superharmonic functions on $\overline{M}_f^{n+1}$ are the constant ones.

Let us remember that a notion of curvature for weighted manifolds goes back to Lichnerowicz [58, 57] and it was later developed by Bakry and Émery in their seminal work [6], where they introduced the following modified Ricci curvature

$$\overline{\mathrm{Ric}}_f = \overline{\mathrm{Ric}} + \overline{\mathrm{Hess}}\, f, \tag{1.11}$$

where $\overline{\mathrm{Ric}}$ and $\overline{\mathrm{Hess}}$ are the standard Ricci tensor and the Hessian on $\overline{M}_f^{n+1}$, respectively. As it is common in the current literature, we will refer to this tensor as being the *Bakry-Émery-Ricci tensor* of $\overline{M}_f^{n+1}$. We note that the interplay between the geometry of $\overline{M}^{n+1}$ and the behavior of the weighted function f is mostly taken into account by means of its Bakry-Émery-Ricci tensor Ric_f (cf. [68]).

1.3.1 Hypersurfaces in a class of weighted warped products

Let $\overline{M}_f^{n+1}$ be a weighted warped product of the tipe

$$\Big(M^n \times_\alpha I\,,\, \langle\cdot\,,\cdot\rangle := \pi_{M^n}^*(\langle\cdot\,,\cdot\rangle_{M^n}) + (\alpha\circ\pi_{M^n})^2\pi_{\mathbb{R}}^*(\epsilon dt^2)\,,\, d\mu\Big)$$

or

$$\Big(I \times_\alpha M^n\,,\, \langle\cdot\,,\cdot\rangle := \pi_{\mathbb{R}}^*(\epsilon dt^2) + (\alpha\circ\pi_{\mathbb{R}^n})^2\pi_{M^n}^*(\langle\cdot\,,\cdot\rangle_{M^n})\,,\, d\mu\Big).$$

We will often refer to the first factor of the product as being the *base* and the second factor as being the *fiber* of the warped product. Here M^n is a Riemannian manifold, $I \in \mathbb{R}$ is an open interval, π_{M^n} and $\pi_{\mathbb{R}}$ denote the canonical projections from $\overline{M}_f^{n+1}$

onto each factor, α is a positive function defined at the base of the product, $\epsilon \in \{-1, 1\}$ and $d\mu = e^{-f} dv$ is the weighted volume form associated with the real-valued smooth function f, where dv is the volume element induced by the metric $\langle \cdot\,, \cdot \rangle$.

In the case that $\overline{M}^{n+1}$ is a Riemannian manifold (i.e. when $\epsilon = 1$), we will consider *two-sided hypersurfaces* $x : \Sigma^n \hookrightarrow \overline{M}_f^{n+1}$. This condition means that there is a globally defined unit normal vector field N. On the other hand, in the Lorentzian case (i.e., when $\epsilon = -1$), Σ^n will be considered a spacelike hypersurface and, in this case, there exist a normal timelike vector field N globaly defined on Σ.

Let us denote by $\overline{\nabla}$, ∇ and $\widetilde{\nabla}$ the Levi-Civita connections of $\overline{M}^{n+1}$, Σ^n and M^n, respectively. The *f-mean curvature* of Σ^n is the function H_f given by

$$nH_f = nH + \epsilon\langle \overline{\nabla} f, N\rangle, \tag{1.12}$$

where $H = \epsilon\frac{1}{n}\,\mathrm{tr}(A)$ denotes the classical mean curvature of Σ^n with respect to N.

The *f-divergence* on Σ^n, for any $X \in \mathfrak{X}(\Sigma)$, is defined by

$$\mathrm{div}_f X = \mathrm{div} X - \langle \nabla f, X\rangle, \tag{1.13}$$

where $\mathrm{div}(X) = \mathrm{trace}\{Y \mapsto \nabla_Y X\}$ denotes the divergence relative to Σ^n. A direct calculation assures us that

$$\mathrm{div}_f(\varphi X) = \varphi\,\mathrm{div}_f X + \langle \nabla\varphi, X\rangle \tag{1.14}$$

for all $X \in \mathfrak{X}(\Sigma)$ and any $\varphi \in C^\infty(\Sigma)$. We define the *$f$-Laplacian* (or *drift Laplacian*) relative to Σ^n by

$$\Delta_f(\varphi) = \mathrm{div}_f(\nabla\varphi) = \Delta\varphi - \langle \nabla f, \nabla\varphi\rangle, \tag{1.15}$$

for all $\varphi \in C^\infty(\Sigma)$, where Δ is the standard Laplacian relative to Σ^n. From (1.14) and (1.15) we can obtain the expression

$$\Delta_f(\varrho\,\varphi) = \varrho\Delta_f(\varphi) + \varphi\Delta_f(\varrho) + 2\langle \nabla\varrho, \nabla\varphi\rangle, \tag{1.16}$$

which is valid for any pair of functions $\varrho, \varphi \in C^\infty(\Sigma)$.

We recall that a *slice* of $\overline{M}^n$ is a hipersurface $M^n_{t_0}$ obtained by fixing some $t_0 \in I$, that is, $M^n_{t_0} = M^n \times \{t_0\}$ or $M^n_{t_0} = \{t_0\} \times M^n$; and a *slab* of $\overline{M}^n$ is the region lying between two slices, that is, a region of the type

$$M^n \times_\alpha [t_1, t_2] \;=\; \{\, (q, t) \in M^n \times_\alpha I \;\;:\;\; t_1 \le t \le t_2 \,\}$$

or

$$[t_1, t_2] \times_\alpha M^n \;=\; \{\, (t, q) \in I \times_\alpha M^n \;\;:\;\; t_1 \le t \le t_2 \,\}\,.$$

1.4 Ambient spaces and immersed hypersurfaces

In what follows we will introduce the ambient spaces that will appear throughout the forthcoming chapters. Namely, we will describe certain weighted semi-Riemannian manifolds with index zero or one that can be regarded as weighted warped products for which one of the factors is a n-dimensional Riemannian manifold M^n and the other is an open interval $I \subset \mathbb{R}$ whose metric defines the causal character of the product.

1.4.1 Weighted Riemannian spaces furnished wiht a conformal Killing vector field

Let us consider an $(n+1)$-dimensional weighted Riemannian manifold $\overline{M}_f^{n+1}$ endowed with a *conformal Killing vector field* V whose orthogonal distribution $\mathcal{D}$ is integrable. Thus, there exists a smooth function $\psi_V \in C^\infty(\overline{M})$ such that

$$\mathcal{L}_V \langle \cdot\, , \cdot \rangle = 2\psi_V \, \langle \cdot\, , \cdot \rangle, \tag{1.17}$$

where $\mathcal{L}_V$ stands for the Lie derivative in the direction of V. The function ψ_V is called the *conformal factor* of V.

In this setting, we denote by $\Phi : I \times M^n \to \overline{M}_f^{n+1}$ the flow generated by V, where $I = (-\infty, a)$ is an interval with $a > 0$ and M^n is an arbitrarily fixed integral leaf of $\mathcal{D}$ labeled as $t = 0$, which we will suppose to be connected and complete. It may happen that $a = +\infty$, i.e., the vector field V is *complete*. Since $\Phi_t = \Phi(t, .)$ is a conformal map for any fixed $t \in \mathbb{R}$, there exists a positive function $\lambda \in C^\infty(I \times M^n)$ such that $\lambda(0, u) = 1$ and $\Phi_t^* \langle \cdot\, , \cdot \rangle(u) = \lambda^2(t, u)\langle \cdot\, , \cdot \rangle(u)$, for any $u \in M^n$.

We restrict ourselves to the case where the function λ depends only on the variable t, that is, $\lambda \in C^\infty(I)$. Geometrically, as it was already observed in [28], this hypothesis allows us to relate the induced metrics in distinct leaves of the foliation orthogonal to V, which we will denote by $V^\perp$.

From (1.17) we deduce the conformal Killing equation

$$\langle \overline{\nabla}_X V, Y \rangle + \langle X, \overline{\nabla}_Y V \rangle = 2\psi_V \langle X, Y \rangle,$$

for any $X, Y \in \mathfrak{X}(\overline{M})$.

An interesting particular case of a conformal Killing vector field V is that in which

$$\overline{\nabla}_X V = \psi_V X \tag{1.18}$$

for all $X \in \mathfrak{X}(\overline{M})$; in this case we say that V is *closed*, an allusion to the fact that its dual 1-form is closed. Yet more particularly, a closed and conformal Killing vector field V is said to be *parallel* if its conformal factor ψ_V vanishes identically, and *homothetic* if ψ_V is constant.

Let $M_t^n = \Phi_t(M^n)$ be a leaf of $V^\perp$ furnished with the induced metric. From (1.18) we get

$$\overline{\nabla}\langle V, V\rangle = 2\psi_V V. \tag{1.19}$$

Consequently, $|V|^2$ is constant on the leaves of $V^\perp$. Moreover, computing covariant derivatives in (1.19), we have that

$$(\,\overline{\text{Hess}}\,\langle V, V\rangle\,)(X, Y) = 2X(\psi_V)\langle V, Y\rangle + 2\psi_V^2\langle X, Y\rangle$$

and, since both $\overline{\text{Hess}}$ and the metric $\langle\cdot\,,\cdot\rangle$ are symmetric tensors, we get

$$X(\psi_V)\langle V, Y\rangle = Y(\psi_V)\langle V, X\rangle,$$

for all $X, Y \in \mathfrak{X}(\overline{M})$. Now, taking $Y = V$ we arrive at

$$\overline{\nabla}\psi_V = \frac{V(\psi_V)}{|V|^2}V = \nu(\psi_V)\nu, \tag{1.20}$$

where $\nu = -\dfrac{V}{|V|}$ and, hence, ψ_V is also constant on the leaves of $V^\perp$.

Furthermore, with a straightforward computation, we verify that the shape operator A_t of a leaf $M_t^n \in V^\perp$ with respect to ν is given by

$$A_t(X) = \overline{\nabla}_X\nu = \frac{\psi_V}{|V|}X,$$

for any $X \in \mathfrak{X}(M_t^n)$ and, hence, the leaves M_t^n are totally umbilical hypersurfaces with constant mean curvature $\mathcal{H} = \mathcal{H}(t)$ with respect to ν given by

$$\mathcal{H} = \frac{\psi_V}{|V|}. \tag{1.21}$$

Under the additional condition that the weight function f of $\overline{M}_f^{n+1}$ does not depend on the parameter of the flow associated to the unit vector field ν, which means that $\langle\overline{\nabla}f, \nu\rangle = 0$ on $\overline{M}_f^{n+1}$, we obtain from (1.12) and (1.21) that the f-mean curvature of a leaf $M_t^n \in V^\top$ is given by

$$\mathcal{H}_f = \frac{\psi_V}{|V|}. \tag{1.22}$$

Remark 1.1 *We observe that the following result is a consequence of a Cheeger-Gromoll type splitting theorem due to G. Wei and W. Wylie (cf. Theorem* 6.1 *of of [68], , see also Theorem* 1.1 *of [47]):*

> *"Let $\overline{M}_f^{n+1}$ be a weighted Riemannian manifold that contains a line. If the Bakry-Émery-Ricci tensor of $\overline{M}_f^{n+1}$ is nonnegative and the weight function f is bounded then f must be constant along the line."*

Taking into account the Remark 1.1, in any weighted Riemannian manifold $\overline{M}_f^{n+1}$ endowed with complete closed conformal Killing vector field V, having nonnegative Bakry-Émery-Ricci tensor and with bounded weight function f, we have that f does not depend on the parameter of the flow associated with the unit vector field ν.

A particular class of Riemannian manifolds provided with a closed conformal Killing vector field is the so-called warped product of the type $I \times_\alpha M^n$, that is, the product manifold $M^n \times \mathbb{R}$ endowed with the warping metric

$$\langle \cdot\,,\cdot\rangle := \pi_{\mathbb{R}}^*(dt^2) + (\alpha \circ \pi_{\mathbb{R}^n})^2 \pi_{M^n}^*(\langle\,\cdot,\cdot\rangle_{M^n}).$$

A warped product $I \times_\alpha M^n$ endowed with a weight function f will be called a *weighted warped product* and it will be denoted by

$$(\,I \times_\alpha M^n\,)_f\,.$$

For such a space, if π_I is the canonical projection onto I, then the vector field $V = (\alpha \circ \pi_I)\,\partial_t$ is conformal Killing and closed, with conformal factor $\psi_V = \alpha' \circ \pi_I$, where the line denotes differentiation with respect to $t \in I$. Moreover (see [59]), for $t \in I$, the slice $M_t^n = \{t\} \times M^n$ is totally umbilical, with constant mean curvature with respect to $-\partial_t$ given by

$$\mathcal{H}(t) = \frac{\alpha'(t)}{\alpha(t)}.$$

Conversely, let $\overline{M}_f^{n+1}$ be a weighted Riemannian manifold endowed with closed conformal Killing vector field V. If $p \in \overline{M}_f^{n+1}$ and M_p^n is the leaf of $V^\perp$ passing through p, then we can find a neighborhood $\mathcal{U}_p$ of p in M_p^n and an open interval $I \subset \mathbb{R}$ containing 0 such that the flow Φ of V is defined on $\mathcal{U}_p$ for every $t \in I$. Besides, if V is complete, following the ideas in Section 3 of [59], one can prove that

$$\begin{array}{ccc} \left(\mathbb{R} \times M_p^n\right)_f & \longrightarrow & \overline{M}_f^{n+1} \\ (t,u) & \mapsto & \Phi(t,u) \end{array} \tag{1.23}$$

is a global parametrization on $\overline{M}_f^{n+1}$, so that $\overline{M}_f^{n+1}$ is isometric to the weighted warped product

$$\left(\mathbb{R} \times_\alpha M_p^n\right)_f, \tag{1.24}$$

where $\alpha(t) = |V(\Phi(t,u))|$, $t \in \mathbb{R}$ and $u \in M_p^n$ is an arbitrary point.

When the weight function f considered in a warped product of the type $I \times_\alpha M^n$ does not depend on the parameter $t \in \mathbb{R}$, we will explicit this condition simply writing

$$I \times_\alpha M_f^n \tag{1.25}$$

and, in what follows, this notation will be used without further comments. In this case, from (1.22) we get that the f-mean curvature of the slice $\{t\} \times M^n$ with respect to the orientation given by $-\partial_t$ is given by

$$\mathcal{H}_f(t) = \frac{\alpha'(t)}{\alpha(t)} \tag{1.26}$$

At the end of this section, our purpose will be to give a description of one of our objects of study: conformal Killing graphs immersed in a weighted Riemannian manifoldf $\overline{M}_f^{n+1}$ endowed with closed conformal Killing vector field V. In this sense, following the ideas established in [28], given a domain Ω in $M^n = M_0^n$, we define the *conformal Killing graph* $\Sigma(z)$ of a smooth function z on $\overline{\Omega}$ as the hypersurface of $\overline{M}_f^{n+1}$ given by

$$\Sigma(z) = \{\, \Phi(z(u), u) \, : \, u \in \overline{\Omega} \,\},$$

where Φ is the flow generated by V. When $\Omega = M^n$, $\Sigma(z)$ is said to be *entire*.

If we assign coordinates $x_0 = t, x_1, \ldots, x_n$ to points in $\overline{M}_f^{n+1}$ of the form $\overline{u} = \Phi(t, u)$, where $x_1, \ldots, x_n$ are local coordinates in M^n, then the corresponding coordinate vector fields are

$$\partial_0|_{\overline{u}} = V(t) \quad \text{and} \quad \partial_i|_{\overline{u}} = \Phi_{t*}\partial_i|_u, \quad \text{for all } i \in \{1, \ldots, n\}.$$

Thus, the conformal Killing graph $\Sigma(z)$ is parameterized in terms of local coordinates by $z(x_1, \ldots, x_n), x_1, \ldots, x_n$ and the tangent space to $\Sigma(z)$ is spanned by the vectors

$$\frac{\partial z}{\partial x_i}\, \partial_0|_{\Phi(z(u),u)} + \partial_i|_{\Phi(z(u),u)}, \quad \text{for all } i \in \{1, \ldots, n\}. \tag{1.27}$$

Hence, from (1.27) we see that the metric induced on $\Sigma(z)$ is given by

$$\lambda^2(z(u)) \left(\frac{1}{\gamma}\, dz^2 + d\sigma^2 \right),$$

where $\gamma = \dfrac{1}{|V(0)|^2}$ and $d\sigma^2$ stands for the metric of the leaf M^n.

Moreover, denoting by Dz the gradient of the function z with respect the metric $d\sigma^2$, with a straightforward computation we verify that

$$N = \frac{1}{\lambda(z(u))\sqrt{\gamma + |Dz(u)|^2}} \left(\Phi_{z(u)*} Dz(u) - \gamma\, \partial_0|_{\Phi(z(u),u)} \right) \tag{1.28}$$

gives an orientation on $\Sigma(z)$ such that $\langle N, V \rangle < 0$.

In this scenario, we will consider the support function η_V on a conformal Killing graph $\Sigma(z)$ immersed in $\overline{M}_f^{n+1}$, which is defined by

$$\begin{aligned} \eta_V : \Sigma(z) \quad &\rightarrow \quad \mathbb{R} \\ p \quad &\mapsto \quad \eta_V(p) = \langle V(p), N(p) \rangle, \end{aligned} \tag{1.29}$$

where N is the Gauss map of $\Sigma(z)$ given in (1.28). We have that η_V is negative and

$$\nabla \eta_V = -A(V^\top), \tag{1.30}$$

where A is the shape operator of $\Sigma(z)$ with respect to N and $V^\top$ is the projection of vector field V on the tangent bundle of $\Sigma(z)$.

1.4.2 Generalized Robertson-Walker Spacetime

According to the terminology introduced in [4], a particular class of time-oriented Lorentzian manifolds is that of *generalized Robertson-Walker* (GRW) spacetimes denoted by $I_1 \times_\alpha M^n$ $(n \geq 2)$, namely, product manifolds $I \times M^n$ endowed with warped metric tensor

$$\langle \cdot , \cdot \rangle = -\pi_{\mathbb{R}}^*(dt^2) + (\alpha \circ \pi_{M^n})^2 \pi_{M^n}^*(\langle \cdot , \cdot \rangle_{M^n}).$$

In other words, $I_1 \times_\alpha M^n$ is nothing but a warped product with Lorentzian base $(I, -dt^2)$, Riemannian fiber $(M^n, \langle \cdot , \cdot \rangle_M)$ and warping function α.

1.4.3 Weighted Killing warped products

Let $(\overline{M}^{n+1}, g)$ be a $(n+1)$-dimensional Riemannian manifold endowed with a Killing vector field Y which never vanishes. We recall that Y is a Killing vector field Killing if $\mathcal{L}_Y g = 0$, where $\mathcal{L}_Y$ stands for the Lie derivative in the direction of Y. Let us suppose in addition that Y has complete flow lines and that the associated orthogonal distribution $\mathcal{D}$ is integrable. In this setting, we denote by $\Phi : M^n \times \mathbb{R} \to \overline{M}^{n+1}$ the flow generated by Y, where M^n is an arbitrarily fixed integral leaf of $\mathcal{D}$, labeled as $t = 0$, which we will suppose to be connected.

In this setting, $\overline{M}^{n+1}$ can be regarded as the *Killing warped product* $M^n \times_\alpha \mathbb{R}$, that is, the product manifold $M^n \times \mathbb{R}$ endowed with the warping metric

$$\langle \cdot , \cdot \rangle \;=\; \pi_M^*(\langle \cdot , \cdot \rangle_M) + (\alpha \circ \pi_M)^2 \pi_{\mathbb{R}}^*(dt^2), \tag{1.31}$$

where the warping function is given by $\alpha = |Y| > 0$. In particular, when $\alpha = 1$ in (1.31) we have that the space $(M^n \times \mathbb{R}, \langle \cdot , \cdot \rangle)$ is just a standard product space.

Now, let $(M^n \times_\alpha \mathbb{R})_f$ be a *weighted Killing warped product* associated with the density function f. We say that $x : \Sigma^n \looparrowright (M^n \times_\alpha \mathbb{R})_f$ is *f-minimal* when its f-mean curvature vanishes identically. It is a well-known fact that minimal hypersurfaces of $M^n \times_\alpha \mathbb{R}$ arise as critical points of the *area functional* (under compactly supported variations)

$$\mathrm{Vol}(\Sigma^n) \;=\; \int_{\Sigma^n} dv,$$

where dv is the volume element of the hypersurface Σ^n induced via immersion x. Since the weighted structure on $M^n \times_\alpha \mathbb{R}$ also induces a weighted structure on Σ^n, we can consider the similar variational problem for the *weighted area functional*

$$\mathrm{Vol}_f(\Sigma^n) \;=\; \int_{\Sigma^n} e^{-f} dv.$$

From variational formulas (see for instance [13]) one can see that $x : \Sigma^n \looparrowright (M^n \times_\alpha \mathbb{R})_f$ is f-minimal, namely a critical point of the weighted area functional, if and only if H_f vanishes identically.

Remark 1.2 *We observe that the Killing vector field Y determines in $M^n \times_\alpha \mathbb{R}$ a codimension one foliation by totally geodesic slices $M^n \times \{t_0\}$, $t_0 \in \mathbb{R}$, with respect to orientation determined by Y. Moreover, assuming that the weighted function $f \in C^\infty(M^n \times_\alpha \mathbb{R})$ is invariant along the flow determinate by Y, that is, $\langle \overline{\nabla} f, Y\rangle = 0$, from (1.12) we get that each slice $M^n \times \{t\}$ is f-minimal.*

As a consequence of Remark 1.1, in any weighted Killing warped product $(M^n \times_\alpha \mathbb{R})_f$ having nonnegative Bakry-Émery-Ricci tensor and with bounded weighted function f, we have that f does not depend on the parameter of the flow associated to the Killing vector field Y. For sake of simplicity, in what follows, Killing warped products $M^n \times_\alpha \mathbb{R}$ endowed with a weighted function f which does not depend on the parameter $t \in \mathbb{R}$ will denoted by

$$M^n_f \times_\alpha \mathbb{R}$$

and this notation will be used without further comments.

Associated to a two-sided hypersurface $x : \Sigma^n \looparrowright (M^n \times_\alpha \mathbb{R})_f$, we will consider two particular smooth functions, namely, the (vertical) *height function*

$$h := (\pi_{\mathbb{R}})\Big|_{\Sigma^n} : \Sigma^n \to \mathbb{R} \tag{1.32}$$

and the *angle function*

$$\begin{array}{rcl} \Theta : \Sigma^n & \to & \mathbb{R} \\ p & \mapsto & \Theta(p) := \langle N(p), Y(p)\rangle, \end{array} \tag{1.33}$$

where N is the Gauss map of Σ^n and Y is the Killing vector field on $M^n \times_\alpha \mathbb{R}$.

We have that

$$\nabla h = \frac{1}{\alpha^2} Y^\top, \tag{1.34}$$

where $(\,\cdot\,)^\top$ denotes the projection of a smooth vector field in $\mathfrak{X}(M^n \times_\alpha \mathbb{R})$ onto $\mathfrak{X}(\Sigma^n)$. Moreover, we have

$$N^* = N - \frac{1}{\alpha^2}\Theta Y, \tag{1.35}$$

where $(\,\cdot\,)^*$ denotes the projection of a smooth vector field in $\mathfrak{X}(M^n \times_\alpha \mathbb{R})$ onto $\mathfrak{X}(M^n)$. From (1.34) and (1.35) we get the following relation:

$$|\nabla h|^2 = \frac{1}{\alpha^2}|N^*|^2_{M^n}. \tag{1.36}$$

Indeed, we have that

$$\begin{aligned} \langle \nabla h, \nabla h\rangle &= \frac{1}{\alpha^4}\langle Y^\top, Y^\top\rangle = \frac{1}{\alpha^4}\langle Y - \Theta N, Y - \Theta N\rangle \\ &= \frac{1}{\alpha^2}\left(1 - \frac{\Theta^2}{\alpha^2}\right) = \frac{1}{\alpha^2}\langle N - \frac{\Theta}{\alpha^2}Y, N - \frac{\Theta}{\alpha^2}Y\rangle \\ &= \frac{1}{\alpha^2}\langle N^*, N^*\rangle = \frac{1}{\alpha^2}\langle N^*, N^*\rangle_{M^n}. \end{aligned}$$

In what follows, we define the *entire Killing graph* $\Sigma^n(z)$ associated to a smooth function $z \in C^\infty(M^n)$, according to [30], as being the hypersurface of $M^n_f \times_\alpha \mathbb{R}$ given by

$$\Sigma^n(z) \;=\; \{\, \Phi(y, z(y)) \;:\; y \in M^n \,\} \;\subset\; M^n \times_\alpha \mathbb{R}.$$

The induced metric on M^n from the Riemannian metric (1.39) via $\Sigma^n(z)$ is given by

$$\langle\cdot\,,\cdot\rangle_z \;=\; \langle\cdot\,,\cdot\rangle_M + \alpha^2 dz^2.$$

On the other hand, the function

$$\begin{array}{rccl} G: & M^n \times \mathbb{R} & \to & \mathbb{R} \\ & (y,t) & \mapsto & G(y,t) := t - z(y), \end{array}$$

is such that

$$\Sigma^n(z) \;=\; x(G^{-1}(0)).$$

Then, for all $X \in \mathfrak{X}(M^n \times_\alpha \mathbb{R})$ we have

$$X(G) \;=\; X^*(G) + \frac{1}{\alpha^2}\langle X, \nu\rangle\, \nu(G) \;=\; \Big\langle \frac{1}{\alpha^2}\nu - Dz, X \Big\rangle,$$

where ν is the unit vector field given by $\nu = \frac{Y}{|Y|}$, Dz denotes the gradient of a function z with respect to the metric $\langle\cdot\,,\cdot\rangle_M$ of M^n and X^* is the orthogonal projection of X on $\mathfrak{X}(M^n)$. Thus,

$$\overline{\nabla} G \;=\; \frac{1}{\alpha^2}\nu - Dz$$

is a normal vector field on $G^{-1}(0)$ and, consequently,

$$N_0 \;=\; x_*(\overline{\nabla} G) \;=\; \frac{1}{\alpha^2} Y - x_*(Dz)$$

is a normal vector field on $\Sigma^n(z)$. Since,

$$|N_0| \;=\; \frac{(1 + \alpha^2 |Dz|^2_M)^{1/2}}{\alpha},$$

it follows that

$$N \;=\; \frac{N_0}{|N_0|} \;=\; \frac{1}{\alpha\,(1 + \alpha^2 |Dz|^2_M)^{1/2}} \big(Y - \alpha^2 x_*(Dz)\big) \tag{1.37}$$

gives an unit normal vector field on $\Sigma^n(z)$, which we will consider as being its Gauss map, for which the angle function Θ defined in (1.33) is given by

$$\Theta \;=\; \langle N, Y\rangle \;=\; \frac{\alpha}{(1 + \alpha^2 |Dz|^2_M)^{1/2}} \;>\; 0. \tag{1.38}$$

1.4.4 Weighted standard static spacetimes

Consider an $(n+1)$-dimensional Lorentzian manifold $\overline{M}^{n+1}$ with Lorentzian metric $g = g(\cdot,\cdot)$ and endowed with a timelike Killing vector field Y. Here timelike referred to a vector field means that $Y_p \in T_p\overline{M}$ is a timelike vector (and so nonzero) for each $p \in \overline{M}^{n+1}$.

We observe that the distribution $\mathcal{D}$ of all smooth vector fields of $\overline{M}^{n+1}$ that are orthogonal to Y, defined at each point by

$$\overline{M}^{n+1} \ni p \ \longmapsto \ \mathcal{D}(p) = \left\{ v \in T_p\overline{M} \ : \ g(v, Y_p) = 0 \right\},$$

is of constant rank and integrable. Given a Riemannian integral leaf M^n of that distribution $\mathcal{D}$, let $\Psi : I \times M^n \to \overline{M}^{n+1}$ be the flow generated by Y with initial values in M^n, where I is a maximal interval of definition. Without loss of generality, in what follows we will consider $I = \mathbb{R}$. In this setting, our space $\overline{M}^{n+1}$ can be regarded as the *standard static spacetime* $M^n \times_\alpha \mathbb{R}_1$ (cf. Proposition 12.38 of [62]), that is, the product manifold $M^n \times \mathbb{R}$ endowed with the Lorentzian warping metric

$$\langle \cdot,\cdot\rangle = \pi^*_{M^n}(\langle \cdot,\cdot\rangle_{M^n}) + (\alpha \circ \pi_{M^n})^2 \pi^*_{\mathbb{R}}(-dt^2), \tag{1.39}$$

where $\alpha = |Y| = \sqrt{-\langle Y, Y\rangle} > 0$ is the warping function.

Remark 1.3 *The importance of standard static spacetimes comes from the fact that they include some classical spacetimes. In what follows we list some of them:*

(a) *A simple example is given by the Lorentz-Minkowski space $\mathbb{L}^{n+1}$, which is isometric to the warped product $(\,\mathbb{R}^n \times \mathbb{R}_1\,,\ \pi^*_{\mathbb{R}^n}(g_{\mathbb{R}^n}) + \pi^*_{\mathbb{R}}(-dt^2)\,)$.*

(b) *The Einstein static universe $(\,\mathbb{S}^n \times \mathbb{R}_1\,,\ \pi^*_{\mathbb{S}^n}(g_{\mathbb{S}^n}) + \pi^*_{\mathbb{R}}(-dt^2)\,)$ is also a standard static space (cf. Example 5.11 of [14]).*

(c) *Another example is given by the exterior Schwarzschild spacetime, which is defined as follows. Let $\mathbb{R}^4$ be given coordinates (t, r, θ, φ), where (r, θ, φ) are the usual spherical coordinates on $\mathbb{R}^3$. Given a positive constant m, the exterior Schwarzschild spacetime is defined on the subset $r > 2m$ of $\mathbb{R}^4$, a subset which is topologically $\mathbb{R}^2 \times \mathbb{S}^2$. The Schwarzschild metric for the region $r > 2m$ is given in (t, r, θ, φ) coordinates by*

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right).$$

Since the metric for this spacetime is invariant under time translations $t \to t + a$, the coordinate vector field $\partial/\partial t$ is a (globally defined) timelike Killing vector field (cf. Section 5.2 of [14] or Chapter 13 of [62]). Consequently, the exterior Schwarzschild spacetime is a standard static spacetime.

(d) *A model that also presents static regions (which appeared shortly after the Schwarzschild spacetime) is the Reissner-Nordström spacetime, whose metric in* (t, r, θ, φ) *coordinates admits the representation*

$$ds^2 = -\left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right) dt^2 + \left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right).$$

This metric has singularities in $r = 0$, $r = r_+$ *and* $r = r_-$, *where* $r_\pm = m \pm (m^2 - e^2)^{1/2}$, *and in regions corresponding to* $+\infty > r > r_+$ *and* $r_- > r > 0$ *we have that the Reissner-Nordström spacetime is static (cf. Section 5.5 of [52]).*

Now, in the configuration described above, let $(M^n \times_\alpha \mathbb{R}_1)_f$ be a *weighted standard static spacetime.* We will consider complete spacelike hypersurfaces

$$x : \Sigma^n \hookrightarrow (M^n \times_\alpha \mathbb{R}_1)_f\,,$$

namely, isometric immersions from a (connected) n-dimensional Riemannian manifold Σ^n into weighted standard static spacetime. As $(M^n \times_\alpha \mathbb{R}_1)_f$ is time-orientable by the timelike vector field Y and $x : \Sigma^n \hookrightarrow (M^n \times_\alpha \mathbb{R}_1)_f$ is a spacelike hypersurface, then Σ^n is orientable (cf. Proposition 5.26 of [62]) and one can choose a globally defined unit normal vector field N on Σ^n having the same time-orientation of $(M^n \times_\alpha \mathbb{R}_1)_f$ (cf. Proposition 5.29 of [62]), that is,

$$\langle Y, N\rangle < 0. \tag{1.40}$$

Such N is said the future-pointing Gauss map of $x : \Sigma^n \hookrightarrow (M^n \times_\alpha \mathbb{R}_1)_f$. We say that $x : \Sigma^n \hookrightarrow (M^n \times_\alpha \mathbb{R})_f$ is *f-maximal* when its f-mean curvature vanishes identically.

Remark 1.4 *Since the timelike Killing vector field* Y *has identically zero conformal factor* ϕ *(more precisely,* $\phi = \frac{1}{n+1}\mathrm{div}Y \equiv 0$, *where* div *stands for the divergence on* $M^n \times_\alpha \mathbb{R}_1$, *it follows from Proposition* 1 *of [59] that* Y *determines in* $M^n \times_\alpha \mathbb{R}_1$ *a codimension one Riemannian foliation by totally geodesic slices* $\Sigma^n_{t_0} = M^n \times \{t_0\}$, $t_0 \in \mathbb{R}$, *with respect to the orientation determined by* $\dfrac{\partial}{\partial t} \equiv Y$. *Moreover, assuming that the weighted function* $f \in C^\infty(M^n \times_\alpha \mathbb{R})$ *is invariant along the flow determinate by* Y, *that is,* $\langle \overline{\nabla} f, Y\rangle = 0$, *from (1.12) we get that each slice* $\Sigma^n_{t_0}$ *is* f*-maximal.*

Remark 1.5 *We observe that the following result is a consequence of a splitting theorem due to Case (see Theorem 1.2 of [22]):*

> *"Let* $\overline{M}^{n+1}_f$ *be a weighted timelike geodesically complete spacetime that contains a timelike line with* $\overline{\mathrm{Ric}}_f(X, X) \geq 0$ *for all timelike vector fields* X, *and whose weighted function* f *is bounded. Then* f *must be constant along timelike line of* $\overline{M}^{n+1}_f$*."*

From Remark 1.5, in any weighted standard static spacetime $(M^n \times_\alpha \mathbb{R}_1)_f$ having nonnegative Bakry-Émery-Ricci tensor for timelike vector fields and with bounded weighted function f, we have that f does not depend on the parameter of the flow associated to the Killing vector field $\frac{\partial}{\partial t} \equiv Y$. Hence, we can see that it is reasonable to consider static spacetimes $M^n \times_\alpha \mathbb{R}_1$ endowed with a weighted function f does not depend on the parameter $t \in \mathbb{R}$. For sake of simplicity, we will denote such an ambient space by

$$M^n_f \times_\alpha \mathbb{R}_1$$

and from now on this notation will be used without further comments.

Aassociated with a spacelike hypersurface $x : \Sigma^n \hookrightarrow M^n_f \times_\alpha \mathbb{R}_1$, we will consider the height function

$$h = (\pi_{\mathbb{R}})\Big|_{\Sigma^n} : \ \Sigma^n \ \rightarrow \ \mathbb{R} \tag{1.41}$$

and the angle function

$$\begin{array}{rccl} \Theta : & \Sigma^n & \rightarrow & \mathbb{R} \\ & p & \mapsto & \Theta(p) = \langle N(p), Y(p) \rangle, \end{array} \tag{1.42}$$

where N is the future-pointing Gauss map of Σ^n and Y is the Killing vector field on $M^n_f \times_\alpha \mathbb{R}_1$. From (1.40), we note that Θ will be always a negative function on Σ^n.

We have that

$$\nabla h = -\frac{1}{\alpha^2} Y^\top, \tag{1.43}$$

where $(\,\cdot\,)^\top$ denote the projection of a smooth vector field in $\mathfrak{X}(M^n \times_\alpha \mathbb{R}_1)$ on $\mathfrak{X}(\Sigma^n)$. Furthermore,

$$N^* = N + \frac{1}{\alpha^2} \Theta Y, \tag{1.44}$$

where $(\,\cdot\,)^*$ denote the projection of a smooth vector field in $\mathfrak{X}(M^n \times_\alpha \mathbb{R}_1)$ on $\mathfrak{X}(M^n)$. From (1.43) and (1.44) it is not difficult to verify that the following relation holds.

$$|\nabla h|^2 = \frac{1}{\alpha^2} |N^*|^2_{M^n}. \tag{1.45}$$

In what follows, until the end of this section, we proceed to describe the inteire Killing graphs in $(M^n_f \times_\alpha \mathbb{R}_1)$. According to [30], we define the *entire Killing graph* $\Sigma(z)$ associated to a smooth function $z \in C^\infty(M)$ as being the hypersurface given by

$$\Sigma(z) = \{\Psi(y, z(y)) : y \in M^n\} \subset M^n \times_\alpha \mathbb{R}_1.$$

The metric induced on M^n from the Lorentzian metric (1.39) via $\Sigma(z)$ is given by

$$\langle \cdot, \cdot \rangle_z = \langle \cdot, \cdot \rangle_M - \alpha^2 dz^2. \tag{1.46}$$

Moreover, $\Sigma(z)$ is spacelike if, and only if, $\alpha^2 |Dz|^2_M < 1$, where Dz denotes the gradient of a function z with respect to the metric $\langle \cdot, \cdot \rangle_M$ of M^n. Indeed, if $\Sigma(z)$ is spacelike, then

$$0 < \langle Dz, Dz \rangle_z = \langle Dz, Dz \rangle_M - \alpha^2 \langle Dz, Dz \rangle^2_M$$

and, hence, we conclude that $\alpha^2|Dz|_M^2 < 1$. Conversely, if $\alpha^2|Dz|_M^2 < 1$ and X is a vector field tangent to $\Sigma(z)$, we obtain, from Cauchy-Schwarz inequality,

$$\langle X, X\rangle_z = \langle X^*, X^*\rangle_M - \alpha^2\langle Dz, X^*\rangle_M^2 \geq \langle X^*, X^*\rangle_M(1-\alpha^2|Dz|_M^2),$$

where X^* is the orthogonal projection of X onto TM^n. Thus, $\langle X, X\rangle_z \geq 0$ and $\langle X, X\rangle_z = 0$ if, and only if, $X = 0$.

The function $G : M^n \times \mathbb{R}_1 \to \mathbb{R}$ given by $G(y,t) = z(y) - t$ is such that $\Sigma(z) = \Psi(G^{-1}(0))$. Thus, for each vector field X tangent to $M^n \times_\alpha \mathbb{R}_1$, we have

$$X(G) = X^*(G) - \frac{1}{\alpha^2}\langle X, \partial_t\rangle\partial_t(G) = \langle \frac{1}{\alpha^2}\partial_t + Dz, X\rangle.$$

Hence,

$$\overline{\nabla} G = \frac{1}{\alpha^2}\partial_t + Dz$$

is a normal vector field on $G^{-1}(0)$ and, consequently,

$$N_0 = \Psi_*(\overline{\nabla} G) = \frac{1}{\alpha^2}Y + \Psi_*(Dz)$$

is a normal timelike vector field on $\Sigma(z)$. Since,

$$|N_0| = \frac{(1-\alpha^2|Dz|_M^2)^{1/2}}{\alpha},$$

it follows that

$$N = \frac{N_0}{|N_0|} = \frac{1}{\alpha(1-\alpha^2|Dz|_M^2)^{1/2}}(Y + \alpha^2\Psi_*(Dz)) \tag{1.47}$$

defines the future-pointing Gauss map of $\Sigma(z)$ such that its angle function $\Theta = \langle N, Y\rangle$ is given by

$$\Theta = -\frac{\alpha}{(1-\alpha^2|Dz|_M^2)^{1/2}} < 0. \tag{1.48}$$

Chapter 2

Conformal Killing graphs in foliated Riemannian spaces whith density: rigidity and stability

In this chapter we investigate the geometry of conformal Killing graphs in a Riemannian manifold $\overline{M}_f^{n+1}$ endowed with a weight function f and having a closed conformal Killing vector field V with conformal factor ψ_V, that is, graphs constructed through the flow generated by V and which are defined over an integral leaf of the foliation $V^{\perp}$ orthogonal to V. For such graphs, we establish some rigidity results under appropriate constraints on the f-mean curvature. Afterwards, we obtain some stability results for f-minimal conformal Killing graphs of $\overline{M}_f^{n+1}$ according to the behavior of ψ_V. Finally, related to conformal Killing graphs immersed in $\overline{M}_f^{n+1}$ with constant f-mean curvature, we study the strong stability. The results presented in this chapter are part of [35].

2.1 Some auxiliary lemmas

This section is devoted to present the analytical machinery that will be used to establish the main results of this chapter.

Let us denote by $\mathcal{L}_f^1(M^n)$ the set of integrable functions on the weighted Riemannian manifold M_f^n with respect to the weighted volume element $d\mu = e^{-f}dM$, where dM stands for the volume element induced by the metric of M_f. Since from (1.13) we have that

$$\mathrm{div}_f X = e^f \mathrm{div}\left(e^{-f}X\right),$$

for all smooth vector field X on M_f, it is not difficult to see that from Proposition 2.1

of [19] we get the following extension of a result due to Yau in [71].

Lemma 2.1 *Let X be a smooth vector field on an oriented n-dimensional complete weighted Riemannian manifold M_f with weight function f such that $\mathrm{div}_f X$ does not change sign on M_f. If $|X| \in \mathcal{L}^1_f(M^n)$, then $\mathrm{div}_f X = 0$.*

The next lemma is due to Wei and Wylie [68] and it extends Theorem 7 of [71].

Lemma 2.2 *All complete noncompact Riemannian manifolds endowed with a bounded weghted function f and with nonnegative Bakry-Émery-Ricci tensor have at least linear f-volume growth.*

In the context of conformal Killing graphs immersed in a weighted Riemannian manifold, following the same ideals of Lemma 4.3 of [37] (see also the proof of Theorem 4.2 of [36]) we obtain the following

Lemma 2.3 *Let $\overline{M}^{n+1}_f$ be a weighted Riemannian manifold endowed with complete closed conformal Killing vector field V an let $\Sigma(z)$ be an entire conformal Killing graph in $\overline{M}^{n+1}_f$, defined on some leaf M^n of the foliation $V^\perp$. If $\Sigma(z)$ lies between two leaves of the foliation $V^\perp$ then $\Sigma(z)$ is complete. Moreover, if $|Dz| \in \mathcal{L}^1_f(M^n)$, then the projection $V^\top$ of V onto $\Sigma(z)$ satisfies $|V^\top| \in \mathcal{L}^1_f(\Sigma(z))$.*

In what follows we assume that the weight function f of $\overline{M}^{n+1}_f$ does not depend on the parameter of the flow associated with the unit vector field $\nu = -V/|V|$, that is, $\langle \overline{\nabla} f, \nu\rangle = 0$. In our next lemma, we present a suitable formula for the drift Laplacian of η_V.

Lemma 2.4 *Let $\overline{M}^{n+1}_f$ be a weighted Riemannian manifold endowed with closed conformal Killing vector field V having conformal factor ψ_V and such that the weight function f does not depend on the parameter of the flow associated to $\nu = -V/|V|$. If $\Sigma(z)$ is a conformal Killing graph in $\overline{M}^{n+1}_f$, with Gauss map N given in (1.28), and η_V is the smooth function on $\Sigma(z)$ defined in (1.29) then*

$$\Delta_f(\eta_V) = -\left\{\overline{\mathrm{Ric}}_f(N,N) + |A|^2\right\}\eta_V - nV^\top(H_f) - n\left\{\psi_V H_f + N(\psi_V)\right\}, \tag{2.1}$$

where A and H_f are the shape operator and the f-mean curvature of $\Sigma(z)$ with respect to N, respectively, and $\overline{\mathrm{Ric}}_f$ denotes the Bakry-Émery-Ricci tensor of $\overline{M}^{n+1}_f$.

Proof. According to the digression presented in Section 1.4.1, we have that (up to isometry) $\overline{M}^{n+1}_f$ can be regarded locally as a weighted warped product of the type (1.24). In this setting, we have that $V = \alpha\,\partial_t$, $\psi_V = \alpha'$, $\nu = -\partial_t$, $|V| = \alpha$, and, consequently, $\langle \overline{\nabla} f, \partial_t\rangle = 0$.

Note that, from (1.12) we get

$$n\langle\partial_t, \nabla H\rangle = n\langle\partial_t^\top, \nabla H\rangle = n\langle\partial_t^\top, \nabla H_f\rangle - \partial_t^\top\langle\overline{\nabla} f, N\rangle, \tag{2.2}$$

where $\partial_t^\top = \partial_t - \langle N, \partial_t\rangle N$ is the projection of ∂_t on the tangent bundle of $\Sigma(z)$.

On the other hand,

$$\begin{aligned}\partial_t^\top\langle\overline{\nabla} f, N\rangle &= \langle\overline{\nabla}_{\partial_t^\top}\overline{\nabla} f, N\rangle + \langle\overline{\nabla} f, \overline{\nabla}_{\partial_t^\top} N\rangle \\ &= \langle\overline{\nabla}_{\partial_t - \langle N, \partial_t\rangle N}\overline{\nabla} f, N\rangle - \langle\overline{\nabla} f, A(\partial_t^\top)\rangle \\ &= \langle\overline{\nabla}_{\partial_t}\overline{\nabla} f, N\rangle - \langle N, \partial_t\rangle\,\overline{\mathrm{Hess}} f(N, N) - \langle\overline{\nabla} f, A(\partial_t^\top)\rangle.\end{aligned} \tag{2.3}$$

Now, taking into account that $\langle\overline{\nabla} f, \partial_t\rangle = 0$ and denoting by $\widetilde{\nabla}$ the Levi-Civita connection on M_p^n, we have $\overline{\nabla} f = \alpha^{-2}\widetilde{\nabla} f$. Then,

$$\begin{aligned}\langle\overline{\nabla}_{\partial_t}\overline{\nabla} f, N\rangle &= \langle\overline{\nabla}_{\partial_t}(\alpha^{-2}\widetilde{\nabla} f), N\rangle \\ &= \langle -2\alpha^{-3}\alpha'\widetilde{\nabla} f + \alpha^{-2}\overline{\nabla}_{\partial_t}\widetilde{\nabla} f, N\rangle.\end{aligned} \tag{2.4}$$

Hence, applying Proposition 7.35 of [62], from (2.4) we get

$$\begin{aligned}\langle\overline{\nabla}_{\partial_t}\overline{\nabla} f, N\rangle &= \langle -2\alpha^{-3}\alpha'\widetilde{\nabla} f + \alpha^{-2}\alpha^{-1}\alpha'\widetilde{\nabla} f, N\rangle \\ &= -\alpha'\alpha^{-3}\langle\widetilde{\nabla} f, N\rangle \;=\; -\alpha'\alpha^{-1}\langle\overline{\nabla} f, N\rangle.\end{aligned} \tag{2.5}$$

Substituting (2.5) in equation (2.3) we get that

$$\partial_t^\top\langle\overline{\nabla} f, N\rangle = -\langle\overline{\nabla} f, N\rangle\alpha^{-1}\alpha' - \langle N, \partial_t\rangle\,\overline{\mathrm{Hess}} f(N, N) - \langle\overline{\nabla} f, A(\partial_t^\top)\rangle. \tag{2.6}$$

From equation (2.2) and (2.6) we conclude that

$$\begin{aligned}-n\alpha\langle\partial_t, \nabla H\rangle &= -n\alpha\langle\partial_t^\top, \nabla H_f\rangle - \alpha'\langle\overline{\nabla} f, N\rangle \\ &\quad - \alpha\,\langle N, \partial_t\rangle\,\overline{\mathrm{Hess}} f(N, N) - \alpha\langle\overline{\nabla} f, A(\partial_t^\top)\rangle.\end{aligned} \tag{2.7}$$

On the other hand, from Proposition 2.1 of [20] we have that

$$\begin{aligned}\Delta\langle N, \alpha\,\partial_t\rangle &= -n\langle\alpha\,\partial_t, \nabla H\rangle - n\left\{\alpha' H + N(\alpha')\right\} \\ &\quad -\langle N, \alpha\,\partial_t\rangle\left\{\overline{\mathrm{Ric}}(N, N) + |A|^2\right\}.\end{aligned} \tag{2.8}$$

So, substituting (2.7) in (2.8) and using (1.11) we obtain

$$\begin{aligned}\Delta\langle N, \alpha\,\partial_t\rangle &= -n\langle\alpha\,\partial_t, \nabla H_f\rangle - \langle N, \alpha\,\partial_t\rangle\left\{\overline{\mathrm{Ric}}_f(N, N) + |A|^2\right\} \\ &\quad -n\left\{\alpha' H_f + N(\alpha')\right\} - \langle\overline{\nabla} f, A(\alpha\,\partial_t^\top)\rangle.\end{aligned} \tag{2.9}$$

Moreover, from (1.30) we verify that

$$\nabla\langle N, \alpha\,\partial_t\rangle = -A(\alpha\,\partial_t^\top). \tag{2.10}$$

We finish the proof using the equations (2.9) and (2.10) into (1.15). ■

We conclude this section by providing an explicit expression for the f-divergence of the tangencial component $V^\top$ of V along a conformal Killing graph.

Lemma 2.5 *Let $\overline{M}^{n+1}_f$ be a weighted Riemannian manifold endowed with closed conformal Killing vector field V having conformal factor ψ_V and such that the weight function f does not depend on the parameter of the flow associated to $\nu = -V/|V|$, and let $\Sigma(z)$ be a conformal Killing graph in $\overline{M}^{n+1}_f$. Then*

$$\mathrm{div}_f V^\top = n\psi_V + n\eta_V H_f, \tag{2.11}$$

where H_f is the f-mean curvature of $\Sigma(z)$ with respect to N and η_V is the smooth function on $\Sigma(z)$ defined in (1.29).

Proof. Since $\langle \overline{\nabla} f, V\rangle = 0$, then, writing $V = V^\top + \eta_V N$, we get

$$\langle \nabla f, V^\top\rangle = -\eta_V \langle \overline{\nabla} f, N\rangle. \tag{2.12}$$

On the other hand, from equation (8.4) of [2] we have

$$\mathrm{div} V^\top = n\psi_V + n\eta_V H, \tag{2.13}$$

where H is the standard mean curvature of $\Sigma(z)$. Hence, from (1.13), (2.13) and (2.12) we obtain (2.11). ■

From Remark 1.1, when a weighted Riemannian manifold $\overline{M}^{n+1}_f$ endowed with complete closed conformal Killing vector field V has bounded weight function f and nonnegative Bakry-Émery-Ricci tensor, we have that f does not depend on the parameter of the flow associated with the unit vector field ν. In this case, we can see that the hypotheses adopted in Lemmas 2.4 and 2.5 on the weight function f are naturally verified.

2.2 Rigidity results for conformal Killing graphs in $\overline{M}^{n+1}_f$

In this section we establish the rigidity results related to conformal Killing graphs in $\overline{M}^{n+1}_f$.

Theorem 2.6 *Let $\overline{M}^{n+1}_f$ be a weighted Riemannian manifold endowed with complete closed conformal Killing vector field V and such that the weight function f does not depend on the parameter of the flow associated to $\nu = -V/|V|$, and let $\Sigma(z)$ be an entire conformal Killing graph in $\overline{M}^{n+1}_f$, defined on some leaf M^n of the foliation $V^\perp$, which lies between two leaves of $V^\perp$. Suppose that the f-mean curvature H_f (not necessarily constant) of $\Sigma(z)$ satisfies the following inequality*

$$0 < H_f \leq \mathcal{H}_f, \tag{2.14}$$

where $\mathcal{H}_f$ is the f-mean curvature of M^n given in (1.22). If $|Dz| \in \mathcal{L}^1_f(M^n)$, then $\Sigma(z)$ is isometric to a leaf of $V^\perp$.

Proof. Let θ be the angle between ν and N. From (2.11) and (2.14), we get

$$\operatorname{div}_f V^\top = n|V|\{\mathcal{H}_f - H_f \cos\theta\} \geq n(1-\cos\theta)H_f|V| \geq 0. \tag{2.15}$$

On the other hand, from Lemma 2.3 we obtain that $\Sigma(z)$ is complete and $|V^\top| \in \mathcal{L}^1_f(\Sigma(z))$. Consequently, we can apply Lemma 2.1 to guarantee that $\operatorname{div}_f V^\top$ vanishes identically on $\Sigma(z)$. Therefore, returning to (2.15) we conclude that $\cos\theta = 1$ on $\Sigma(z)$, that is, the unit vector fields N and ν determine the same direction on $\Sigma(z)$ and, hence, $\Sigma(z)$ must be isometric to a leaf of the foliation $V^\perp$. ■

From the analysis of the sign of $\operatorname{div}_f\left(V^\top\right)$ in the proof of Theorem 2.6, we obtain the following

Theorem 2.7 *Let $\overline{M}^{n+1}_f$ be a weighted Riemannian manifold endowed with complete closed conformal Killing vector field V and such that the weight function f does not depend on the parameter of the flow associated to $\nu = -V/|V|$, and let $\Sigma(z)$ be an entire conformal Killing graph in $\overline{M}^{n+1}_f$, defined on some leaf M^n of the foliation $V^\perp$, which lies between two leaves of $V^\perp$. Suppose that the f-mean curvature H_f of $\Sigma(z)$ is constant and satisfies*

$$0 \leq H_f \leq \mathcal{H}_f,$$

where $\mathcal{H}_f$ is the f-mean curvature of M^n given in (1.22). If $|Dz| \in \mathcal{L}^1_f(M^n)$, then $\Sigma(z)$ is either f-minimal or isometric to a leaf of $V^\perp$.

In the case that the ambient space in the Theorems 2.6 and 2.7 is a weighted warped product of the type (1.25), noting that $\mathcal{H}_f$ admits the expression (1.26), we get the following results:

Corollary 2.8 *Let $\Sigma(z)$ be an entire conformal Killing graph in a weighted warped product $\mathbb{R}\times_\alpha M^n_f$, defined on a slice $M^n_{t_0} = \{t_0\}\times M^n$, $t_0 \in \mathbb{R}$, which lies in a slab of $\mathbb{R}\times_\alpha M^n_f$. Suppose that the f-mean curvature H_f (not necessarily constant) of $\Sigma(z)$ satisfies the following inequality*

$$0 < \alpha H_f \leq \alpha'.$$

If $|Dz| \in \mathcal{L}^1_f(M^n_{t_0})$, then $\Sigma(z)$ is isometric to slice $\{t\}\times M^n$, for some $t \in \mathbb{R}$.

Corollary 2.9 *Let $\Sigma(z)$ be an entire conformal Killing graph in a weighted warped product $\mathbb{R}\times_\alpha M^n_f$, defined on a slice $M^n_{t_0} = \{t_0\}\times M^n$, $t_0 \in \mathbb{R}$, which lies in a slab of $\mathbb{R}\times_\alpha M^n_f$. Suppose that the f-mean curvature H_f of $\Sigma(z)$ is constant and satisfies*

$$0 \leq \alpha H_f \leq \alpha'.$$

If $|Dz| \in \mathcal{L}^1_f(M^n_{t_0})$, then $\Sigma(z)$ is either f-minimal or isometric to slice $\{t\}\times M^n$, for some $t \in \mathbb{R}$.

Continuing with our study, if the f-mean curvature of the conformal Killing graph and the conformal factor of the conformal Killing vector field have opposite signs, we have established the following result.

Theorem 2.10 *Let $\overline{M}_f^{n+1}$ be a weighted Riemannian manifold with nonnegative Bakry-Émery-Ricci tensor $\overline{\mathrm{Ric}}_f$, endowed with complete closed conformal Killing vector field V having conformal factor ψ_V and such that the weight function f is bounded. Let $\Sigma(z)$ be an entire conformal Killing graph in $\overline{M}_f^{n+1}$, defined on some leaf M^n of the foliation $V^\perp$, which lies between two leaves of $V^\perp$, and with Gauss map N given in (1.28). Suppose that ψ_V and the f-mean curvature H_f of $\Sigma(z)$ verify one of the following conditions:*

(a) $H_f \geq 0$ and $\psi_V \leq 0$ on $\Sigma(z)$;

(b) $H_f \leq 0$ and $\psi_V \geq 0$ on $\Sigma(z)$.

If the norm of the second fundamental form $|A|$ of $\Sigma(z)$ is bounded and $|Dz| \in \mathcal{L}_f^1(M^n)$, then $\Sigma(z)$ is totally geodesic and $\overline{\mathrm{Ric}}_f$ in the direction of N vanishes identically. In addition, if $\Sigma(z)$ is noncompact and the Bakry-Émery-Ricci tensor of $\Sigma(z)$ is also nonnegative, then $\Sigma(z)$ is isometric to a totally geodesic leaf of $V^\perp$.

Proof. First of all, we note that f does not depend on the parameter of the flow associated with ν (see Remark 1.5).

Since the support function η_V defined in (1.29) is negative, from either item (a) or (b) jointly with equation (2.11) we obtain that $\mathrm{div}_f\left(V^\top\right)$ does not change sign on $\Sigma(z)$. Since $\Sigma(z)$ lies between two leaves of the foliation $V^\perp$ and $|Dz| \in \mathcal{L}_f^1(M^n)$, from Lemma 2.3 we obtain that $\Sigma(z)$ is complete and $|V^\top| \in \mathcal{L}_f^1(\Sigma(z))$. So, Lemma 2.1 gives $\mathrm{div}_f\left(V^\top\right) = 0$ on $\Sigma(z)$. Therefore, $\psi_V = 0$ and $H_f = 0$ on $\Sigma(z)$.

Now, considering (2.1), we obtain

$$\Delta_f(\eta_V) = -\left\{\overline{\mathrm{Ric}}_f(N,N) + |A|^2\right\}\eta_V \geq 0$$

on $\Sigma(z)$. Moreover, we note that the boundedness of $|A|$ on $\Sigma(z)$ gives

$$|\nabla\eta_V| \leq |A||V^\top| \in \mathcal{L}_f^1(\Sigma(z)).$$

Applying again Lemma 2.1, we get $\Delta_f\left(\eta_V\right) = 0$ on $\Sigma(z)$ and, consequently,

$$\overline{\mathrm{Ric}}_f(N,N) + |A|^2 = 0$$

on $\Sigma(z)$. Since $\overline{\mathrm{Ric}}_f(N,N) \geq 0$, we get $\overline{\mathrm{Ric}}_f(N,N) = 0$ and $A = 0$ on $\Sigma(z)$, that is, $\Sigma(z)$ is totally geodesic.

Proceeding, in view of (1.30), we obtain that $\nabla\eta_V = 0$ on $\Sigma(z)$ and, hence, $\eta_V = \langle V, N\rangle$ is constant and nonzero on $\Sigma(z)$. On the order hand, since V is parallel on $\Sigma(z)$, from (1.19) we have that $\langle V, V\rangle$ is constant on $\overline{M}_f^{n+1}$. Thus,

$$|V^\top|^2 = |V - \langle V, N\rangle N|^2 = \langle V, V\rangle - \langle V, N\rangle^2 \tag{2.16}$$

is also constant on $\Sigma(z)$. Therefore,

$$+\infty > \int_{\Sigma(z)} |V^\top| d\mu = |V^\top| \operatorname{vol}_f(\Sigma(z)), \tag{2.17}$$

where $\operatorname{vol}_f(\Sigma(z))$ is the weighted volume of $\Sigma(z)$. If, in addition, we assume $\Sigma(z)$ is noncompact and that the Bakry-Émery-Ricci tensor of $\Sigma(z)$ is also nonnegative, Lemma 2.2 gives $\operatorname{vol}_f(\Sigma(z)) = +\infty$ and, consequently, the only possibility that we have for validity of (2.17) is that $|V^\top| = 0$ on $\Sigma(z)$. Thus, from (2.16) we get

$$|\langle V, N\rangle| = |V|.$$

Therefore, Cauchy-Schwarz inequality gives that V is parallel to N and, hence, $\Sigma(z)$ must be isometric to a totally geodesic leaf of $V^\perp$. ■

When the f-mean curvature of a conformal Killing graph and the conformal factor of the conformal Killing vector field have the same sign, we have the following

Theorem 2.11 *Let $\overline{M}_f^{n+1}$ be a weighted Riemannian manifold with nonnegative Bakry-Émery-Ricci tensor $\overline{\operatorname{Ric}}_f$, endowed with complete closed conformal Killing vector field V having conformal factor ψ_V and such that the weight function f is bounded. Let $\Sigma(z)$ be an entire conformal Killing graph in $\overline{M}_f^{n+1}$, defined on some leaf M^n of the foliation $V^\perp$, which lies between two leaves of $V^\perp$, with Gauss map N given in (1.28), and with norm of the second fundamental form $|A|$ and f-mean curvature H_f both bounded. Suppose that $|Dz| \in \mathcal{L}_f^1(M^n)$, H_f has the same sign as ψ_V and*

$$\frac{1}{|V|}\frac{\partial\psi_V}{\partial t} \leq -\, n\,(H_f)^2, \tag{2.18}$$

where $t \in \mathbb{R}$ is the parameter of the flow associated with the unit vector field $\nu = -V/|V|$. Then $\Sigma(z)$ is totally geodesic and $\overline{\operatorname{Ric}}_f$ in the direction of N vanishes identically. In addition, if $\Sigma(z)$ is noncompact, $\langle V, V\rangle$ is constant on $\Sigma(z)$ and the Bakry-Émery-Ricci tensor of $\Sigma(z)$ is also nonnegative, then $\Sigma(z)$ is isometric to a totally geodesic leaf of $V^\perp$.

Proof. We have that f does not depend on the parameter of the flow associated with ν (see Remark 1.5). From (1.20) we observe that

$$N(\psi_V) = \langle N, \overline{\nabla}\psi_V\rangle = -\frac{\nu\,(\psi_V)}{|V|}\,\eta_V = -\frac{1}{|V|}\frac{\partial\psi_V}{\partial t}\,\eta_V, \tag{2.19}$$

where η_V is the negative support function defined in (1.29). Thus, in (2.1) we have

$$\Delta_f(\eta_V) = -n\langle\nabla H_f, V\rangle - \left\{\overline{\mathrm{Ric}}_f(N,N) + |A|^2\right\}\eta_V - n\psi_V H_f + \frac{n}{|V|}\frac{\partial\psi_V}{\partial t}\eta_V.$$

From hypothesis (2.18), we get

$$\Delta_f(\eta_V) \geq -n\langle\nabla H_f, V\rangle - \left\{\overline{\mathrm{Ric}}_f(N,N) + |A|^2\right\}\eta_V - n\psi_V H_f - n^2(H_f)^2\eta_V. \tag{2.20}$$

Now, let us consider on $\Sigma(x)$ the smooth vector field

$$X = \nabla\eta_V + nH_f V^\top.$$

Since $\Sigma(z)$ lies between two leaves of the foliation $V^\perp$ and $|Dz| \in \mathcal{L}^1_f(M^n)$, from Lemma 2.3 we obtain that $\Sigma(z)$ is complete and $|V^\top| \in \mathcal{L}^1_f(\Sigma(z))$. Then, from (1.30) we obtain

$$|X| \leq \{|A| + n|H_f|\}|V^\top| \in \mathcal{L}^1_f(\Sigma(z)),$$

since H_f and $|A|$ are bounded on $\Sigma(z)$,

Moreover, from (1.13), (1.14), (2.11) and (2.20) we have

$$\begin{aligned}
\mathrm{div}_f X &= \Delta_f(\eta_V) + n\langle\nabla H_f, V\rangle + nH_f\,\mathrm{div}_f\left(V^\top\right) \\
&\geq -n\langle\nabla H_f, V\rangle - \left\{\overline{\mathrm{Ric}}_f(N,N) + |A|^2\right\}\eta_V \\
&\quad -n\psi_V H_f - n^2(H_f)^2\eta_V + n\langle\nabla H_f, V\rangle \\
&\quad +n^2\psi_V H_f + n^2(H_f)^2\eta_V \\
&= -\left\{\overline{\mathrm{Ric}}_f(N,N) + |A|^2\right\}\eta_V + n(n-1)\psi_V H_f \;\geq\; 0,
\end{aligned} \tag{2.21}$$

where in the last inequality we used that η_V is negative, $\overline{\mathrm{Ric}}_f$ is nonnegative and the assumption that H_f and ψ_V have the same sign on Σ^n. Thus, Lemma 2.1 gives $\mathrm{div}_f X = 0$ on $\Sigma(z)$. Therefore, by returning to (2.21) we obtain that $\overline{\mathrm{Ric}}_f(N,N) = 0$ and $\Sigma(z)$ is totally geodesic.

Finally, if $\Sigma(z)$ is noncompact, $\langle V, V\rangle$ is constant on $\Sigma(z)$ and the Bakry-Émery-Ricci tensor of $\Sigma(z)$ is also nonnegative, then (2.16) holds and we can reason as in the last part of the proof of Theorem 2.10 to conclude that $\Sigma(z)$ is isometric to a totally geodesic leaf of $V^\top$. ■

If the ambient space $\overline{M}^{n+1}_f$ in Theorems 2.10 and 2.11 is a weighted warped product $\mathbb{R}\times_\alpha M^n_f$, we observe that the hypotheses about the Bakry-Émery-Ricci tensor of $\overline{M}^{n+1}_f$ and the weight function f can be disregarded, because in this case we already have to the weigted function f does not depend on the parameter of the flow associated with the unit vector field $-\partial_t$. Hence, when $\overline{M}^{n+1}_f = \mathbb{R}\times_\alpha M^n_f$ we have that Theorems 2.10 and 2.11 can be rescripted, respectively, in the following way.

Corollary 2.12 *Let $\mathbb{R}\times_\alpha M^n_f$ be a weighted warped product with bounded weight function f and let $\Sigma(z)$ be an entire conformal Killing graph in $\mathbb{R}\times_\alpha M^n_f$, defined on a slice*

$M^n_{t_0} = \{t_0\} \times F^n$, $t_0 \in \mathbb{R}$, which lies in a slab of $\mathbb{R} \times_\alpha M^n_f$, and with Gauss map N given in (1.28). Suppose that the warped function α and the f-mean curvature H_f of $\Sigma(z)$ verify one of the following conditions:

(a) $H_f \geq 0$ and $\alpha' \leq 0$ on $\Sigma(z)$;

(b) $H_f \leq 0$ and $\alpha' \geq 0$ on $\Sigma(z)$.

If the norm of the second fundamental form $|A|$ of $\Sigma(z)$ is bounded and $|Dz| \in \mathcal{L}^1_f(M^n_{t_0})$, then $\Sigma(z)$ is totally geodesic and the Bakry-Émery-Ricci tensor of $\mathbb{R} \times_\alpha M^n_f$ in the direction of N vanishes identically. In addition, if $\Sigma(z)$ is noncompact and the Bakry-Émery-Ricci tensor of $\Sigma(z)$ is nonnegative, then $\Sigma(z)$ is isometric to a totally geodesic slice $\{t\} \times M^n$, for same $t \in \mathbb{R}$.

Corollary 2.13 *Let $\mathbb{R} \times_\alpha M^n_f$ be a weighted warped product with bounded weight function f and let $\Sigma(z)$ be an entire conformal Killing graph in $\mathbb{R} \times_\alpha M^n_f$, defined on a slice $M^n_{t_0} = \{t_0\} \times M^n$, $t_0 \in \mathbb{R}$, which lies in a slab of $\mathbb{R} \times_\alpha M^n_f$, with Gauss map N given in (1.28), and with norm of the second fundamental form $|A|$ and f-mean curvature H_f both bounded. Suppose that $|Dz| \in \mathcal{L}^1_f(M^n_{t_0})$, H_f has the same sign as the derivative of the warped function α and*

$$\alpha'' \leq -n\alpha\,(H_f)^2 .$$

Then $\Sigma(z)$ is totally geodesic and the Bakry-Émery-Ricci tensor of $\mathbb{R} \times_\alpha M^n_f$ in the direction of N vanishes identically. In addition, if $\Sigma(z)$ is noncompact, $\langle V, V\rangle$ is constant on $\Sigma(z)$ and the Bakry-Émery-Ricci tensor of $\Sigma(z)$ is nonnegative, then $\Sigma(z)$ is isometric to a totally geodesic slice $\{t\} \times M^n$, $t \in \mathbb{R}$.

2.3 Stability of f-minimal conformal Killing graphs

Let $\overline{M}^{n+1}_f$ be a weighted Riemannian manifold, with weight function f and endowed with closed conformal Killing vector field V, and let $x : \Sigma(z) \looparrowright \overline{M}^{n+1}_f$ be an conformal Killing graph with Gauss map N defined in (1.28). In this setting, we denote by $d\Sigma(z)$ the volume element with respect to the metric induced by $x : \Sigma(z) \looparrowright \overline{M}^{n+1}_f$ and we mean by $C^\infty_0(\Sigma(z))$ the set of all functions of class C^∞ on $\Sigma(z)$ supported compactly.

It is well known that, given a function $\varphi \in C^\infty_0(\Sigma(z))$ there exists a *normal variation with compact support an fixed boundary*

$$x_s : \Sigma(z) \to \overline{M}^{n+1}_f, \quad \text{for } s \in (-\epsilon, \epsilon), \tag{2.22}$$

of $x : \Sigma(z) \looparrowright \overline{M}^{n+1}_f$, that is,

(i) $x_s = \mathrm{Id}$ outside a compact subset of $\Sigma(z)$;

(*ii*) for $s \in (-\epsilon, \epsilon)$, the map $x_s : \Sigma(z) \to \overline{M}_f^{n+1}$ is a immersion such that $x_0(p) = x(p)$ for all $p \in \Sigma(z)$;

(*iii*) $x_s(p) = p$ for all $p \in \partial\Sigma(z)$.

Moreover, associated with $x_s : \Sigma(z) \to \overline{M}_f^{n+1}$ we have that *the variational normal field* is φN and the first variation of the *weighted area functional*

$$\begin{array}{rccl} \mathcal{A}_f : & (-\epsilon, \epsilon) & \to & \mathbb{R} \\ & s & \mapsto & \mathcal{A}_f(s) = \mathrm{Area}_f\Big(x_s(\Sigma(z))\Big) = \displaystyle\int_{\Sigma(z)} d\mu_s\,, \end{array} \tag{2.23}$$

where $d\mu_s = e^{-f} d\Sigma(z)_s$ and $d\Sigma(z)_s$ denotes the volume element of $\Sigma(z)$ with respect to the metric induced by $x_s : \Sigma(z) \to \overline{M}_f^{n+1}$, is given by (see, for instance, [23], Lemma 3.2)

$$\delta_\varphi\left(\mathcal{A}_f\right) \;=\; \frac{d\mathcal{A}}{ds}(0) \;=\; n\int_{\Sigma(z)} \varphi H_f \, d\mu. \tag{2.24}$$

As a consequence, $x : \Sigma(z) \looparrowright \overline{M}_f^{n+1}$ is a f-minimal if and only if $\delta_\varphi\left(\mathcal{A}_f\right) = 0$ for every smooth function $\varphi \in C_0^\infty(\Sigma(z))$. In other words, f-minimal conformal Killing graphs in $\overline{M}_f^{n+1}$ are characterized as critical points of $\mathcal{A}_f$.

The stability operator of this variational problem is given by the second variation formula for the f-area, which in our case is written as follows (see Proposition 3.5 of [23] for $H_f = 0$)

$$\delta_\varphi^2\left(\mathcal{A}_f\right) \;=\; \frac{d^2\mathcal{A}}{ds^2}(0) \;=\; -\int_{\Sigma(z)} \varphi L_f(\varphi) \, d\mu \tag{2.25}$$

with

$$L_f = \Delta_f + |A|^2 + \overline{\mathrm{Ric}}_f(N, N),$$

where Δ_f is the drift Laplacian operator on $\Sigma(z)$, N is the Gauss map of $\Sigma(z)$, $|A|$ denotes the length of the shape operator A of $\Sigma(z)$ and $\overline{\mathrm{Ric}}_f$ is the Bakry-Émery-Ricci tensor of $\overline{M}_f^{n+1}$.

For f-minimal conformal Killing graphs in $\overline{M}_f^{n+1}$, the above discussion motivates the following notion of stability.

Definition 2.14 *Let $\overline{M}_f^{n+1}$ be a weighted Riemannian manifold, with weight function f and endowed with closed conformal Killing vector field V, and let $x : \Sigma(z) \looparrowright \overline{M}_f^{n+1}$ be a f-minimal conformal Killing graph. We say that $x : \Sigma(z) \looparrowright \overline{M}_f^{n+1}$ is L_f-stable if $\delta_\varphi^2\left(\mathcal{A}_f\right) \geq 0$ for every $\varphi \in C_0^\infty(\Sigma(z))$.*

In order to proof our main theorem in this section, we will need to use the following auxiliary result, which gives a sufficient condition for a f-minimal hypersurfaces be L_f-stable.

Lemma 2.15 *Let $\overline{M}_f^{n+1}$ be a weighted Riemannian manifold, with weight function f and endowed with closed conformal Killing vector field V, and let $x : \Sigma(z) \looparrowright \overline{M}_f^{n+1}$ be a f-minimal conformal Killing graph. If there exists a positive smooth function $u \in C^\infty(\Sigma(z))$ such that $L_f(u) \leq 0$, then $x : \Sigma(z) \looparrowright \overline{M}_f^{n+1}$ is L_f-stable.*

Proof. Let us assume that there exists such a function u and take $\varphi \in C_0^\infty(\Sigma(z))$. Then, we can choose $\varrho \in C_0^\infty(\Sigma(z))$ satisfying $\varphi = \varrho u$. Hence, from (1.16) and (2.25) we have

$$\begin{aligned}
\delta_\varphi^2(\mathcal{A}_f) &= -\int_{\Sigma(z)} \varphi L_f(\varphi)\, d\mu \;=\; -\int_{\Sigma(z)} \varrho\, u L_f(\varrho\, u)\, d\mu \qquad (2.26)\\
&= -\int_{\Sigma(z)} \left(\varrho^2 u L_f(u) + \varrho\, u^2 \Delta_f(\varrho) + 2\varrho\, u \langle \nabla\varrho, \nabla u\rangle\right) d\mu \\
&\geq -\int_{\Sigma(z)} \left(\varrho\, u^2 \Delta(\varrho) + 2\varrho\, u \langle \nabla\varrho, \nabla u\rangle - \varrho\, u^2 \langle \nabla\varrho, \nabla f\rangle\right) d\mu. \\
&= -\int_{\Sigma(z)} \left(\varrho\, u^2 \Delta(\varrho) + \frac{1}{2}\langle \nabla\varrho^2, \nabla u^2\rangle - \varrho\, u^2 \langle \nabla\varrho, \nabla f\rangle\right) d\mu.
\end{aligned}$$

On the other hand, we can see that

$$\operatorname{div}(u^2\nabla\varrho^2) = \langle \nabla u^2, \nabla\varrho^2\rangle + u^2\Delta(\varrho^2) = \langle \nabla u^2, \nabla\varrho^2\rangle + 2\varrho\, u^2\Delta(\varrho) + 2u^2|\nabla\varrho|^2.$$

Therefore, from the weighted version of divergence theorem (see Lemma 2.2 of [21]), we get from last equation together with (2.26) that

$$\begin{aligned}
\delta_\varphi^2(\mathcal{A}_f) &\geq -\int_{\Sigma(z)} \left(\frac{1}{2}\operatorname{div}(u^2\nabla\varrho^2) - u^2|\nabla\varrho|^2 - \varrho\, u^2\langle \nabla\varrho, \nabla f\rangle\right) d\mu \\
&= -\int_{\Sigma(z)} \left(\frac{1}{2}\operatorname{div}_f(u^2\nabla\varrho^2) - u^2|\nabla\varrho|^2\right) d\mu \;=\; \int_{\Sigma(z)} u^2|\nabla\varrho|^2\, d\mu \;\geq\; 0
\end{aligned}$$

and, therefore, $x : \Sigma(z) \looparrowright \overline{M}_f^{n+1}$ is L_f-stable. ■

Now, analyzing the behavior of the conformal factor ψ_V along a conformal Killing graph, we will state and prove our main result concerning L_f-stability. In what follows, $t \in \mathbb{R}$ denotes the parameter of the flow associated with the unit vector field $\nu = -V/|V|$.

Theorem 2.16 *Let $\overline{M}_f^{n+1}$ be a weighted Riemannian manifold nonnegative Bakry-Émery-Ricci tensor, endowed with complete closed conformal Killing vector field V having conformal factor ψ_V and whose weight function f is bounded, and let $x : \Sigma(z) \looparrowright \overline{M}_f^{n+1}$ be a f-minimal conformal Killing graph.*

(a) If $\dfrac{\partial\psi_V}{\partial t} \leq 0$ on $\Sigma(z)$ then $x : \Sigma(z) \looparrowright \overline{M}_f^{n+1}$ is L_f-stable.

(b) If $\Sigma(z)$ is compact and $\dfrac{\partial\psi_V}{\partial t} \geq 0$ on $\Sigma(z)$ then $x : \Sigma(z) \looparrowright \overline{M}_f^{n+1}$ is L_f-stable if and only if ψ_V is constant on $\Sigma(z)$.

(c) *If $\Sigma(z)$ is compact and* $\dfrac{\partial \psi_V}{\partial t} > 0$ *on $\Sigma(z)$ then $x : \Sigma(z) \looparrowright \overline{M}_f^{n+1}$ cannot be L_f-stable.*

Proof. We have that f does not depend on the parameter of the flow associated with ν (see Remark 1.5). On $\Sigma(z)$, we consider the smooth positive function $u = -\eta_V$, where η_V is defined in (1.29). Then, from (2.1) and (2.19) we obtain

$$L_f(u) = \frac{n}{|V|}\frac{\partial \psi_V}{\partial t}u, \tag{2.27}$$

and, with a direct application of Lemma 2.15, the result of item (a) is obtained immediately.

Now, let us consider (b). Note that in this case $C_0^\infty(\Sigma(z)) = C^\infty(\Sigma(z))$. So, if $x : \Sigma(z) \looparrowright \overline{M}_f^{n+1}$ is L_f-stable, from Definition 2.14 and equation (2.27) we get

$$0 \;\le\; \delta_u^2(\mathcal{A}_f) \;=\; -\int_{\Sigma(z)} uL_f(u)\,d\mu, \;=\; -n\int_{\Sigma(z)} \frac{u^2}{|V|}\frac{\partial \psi_V}{\partial t}\,d\mu \;\le\; 0, \tag{2.28}$$

which guarantees us $\dfrac{\partial \psi_V}{\partial t} = 0$ on $\Sigma(z)$. The converse follows from item (a).

Finally, we prove (c). Assuming the opposite, if we would have $x : \Sigma(z) \looparrowright \overline{M}_f^{n+1}$ L_f-stable then, from the analysis of signals studied in (2.28), we obtain

$$0 \;\le\; -n\int_{\Sigma(z)} \frac{u^2}{|V|}\frac{\partial \psi_V}{\partial t}\,d\mu \;<\; 0,$$

which is absurd. ■

When the ambient space is a weighted warped product of the type (1.25), we can apply Theorem 2.16 to obtain the following result.

Corollary 2.17 *Let $x : \Sigma(z) \looparrowright \mathbb{R} \times_\alpha M_f^n$ be a f-minimal conformal Killing graph.*

(a) *If the warping function α satisfies $\alpha'' \le 0$ on $\Sigma(z)$ then $x : \Sigma(z) \looparrowright \mathbb{R} \times_\alpha M_f^n$ is L_f-stable.*

(b) *If $\Sigma(z)$ compact and the warping function α satisfies $\alpha'' \ge 0$ on $\Sigma(z)$ then $x : \Sigma(z) \looparrowright \mathbb{R} \times_\alpha M_f^n$ is L_f-stable if and only if $\alpha = at + b$ on $\Sigma(z)$, for some $a, b \in \mathbb{R}$.*

(c) *If $\Sigma(z)$ compact and the warping function α satisfies $\alpha'' > 0$ on $\Sigma(z)$ then $x : \Sigma(z) \looparrowright \overline{M}_f^{n+1}$ cannot be L_f-stable.*

2.4 Stability of constant f-mean curvature conformal Killing graphs

Let $\overline{M}_f^{n+1}$ be a weighted Riemannian manifold, with weight function f and endowed with closed conformal Killing vector field V, and let $x : \Sigma(z) \looparrowright \overline{M}_f^{n+1}$ be an *closed* (that is, compact and without boundary) conformal Killing graph with Gauss map N defined in given in (1.28).

In what follows we consider the set

$$\mathcal{G} = \left\{ \varphi \in C^\infty(\Sigma(z)) \ : \ \int_{\Sigma(z)} \varphi \, d\mu = 0 \right\},$$

formed by all the smooth functions on $\Sigma(z)$ with weighted integral mean equal to zero, where $d\mu = e^{-f} d\Sigma(z)$ and $d\Sigma(z)$ is the volume element with respect to the metric induced by $x : \Sigma(z) \looparrowright \overline{M}_f^{n+1}$.

According the ideas established in the Lemmas 2.1 and 2.2 of [8] (see also Lemma 3.2 of [23]), every smooth function $\varphi \in \mathcal{G}$ induces a normal variation (namely, a smooth function of form (2.22) checking only item (ii)) of $x : \Sigma(z) \looparrowright \overline{M}_f^{n+1}$, with variational normal field φN and with first variation $\delta_\varphi(\mathcal{A}_f)$ of the weighted area functional $\mathcal{A}_f : (-\epsilon, \epsilon) \to \mathbb{R}$, defined in (2.23), given by the expression (2.24). As a consequence of (2.24), any closed conformal Killing graph $x : \Sigma(z) \looparrowright \overline{M}_f^{n+1}$ with constant f-mean curvature H_f is a critical point of $\mathcal{A}_f$ restricted to all functions φ belonging to $\mathcal{G}$. Geometrically, this condition means that the variations under consideration preserve a certain weighted volume function (for more details, see Section 3 of [23]). For these critical points, Proposition 3.5 of [23] (see also Proposition 2.5 of [8]) asserts that the stability of the corresponding variational problem is given by the second variation

$$\delta^2_\varphi(\mathcal{A}_f) \ = \ -\int_{\Sigma(z)} \left\{ \Delta_f(\varphi) + \left(|A|^2 + \overline{\mathrm{Ric}}_f(N,N) \right)(\varphi) \right\} \varphi \, d\mu \tag{2.29}$$

where Δ_f is the drift Laplacian operator on $\Sigma(z)$, N is the Gauss map of $\Sigma(z)$, $|A|$ denotes the length of the shape operator A of $\Sigma(z)$ and $\overline{\mathrm{Ric}}_f$ is the Bakry-Émery-Ricci tensor of $\overline{M}_f^{n+1}$.

From (2.29), let us now note that $\delta^2_\varphi(\mathcal{A}_f)$ depends only on $\varphi \in C^\infty(\Sigma(z))$. The following notion of stability now makes sense.

Definition 2.18 *Let $\overline{M}_f^{n+1}$ be a weighted Riemannian manifold, with weight function f and endowed with closed conformal Killing vector field V, and let $x : \Sigma(z) \looparrowright \overline{M}_f^{n+1}$ be a closed conformal Killing graph with constant f-mean curvature H_f. We say that $x : \Sigma(z) \looparrowright \overline{M}_f^{n+1}$ is strongly f-stable when $\delta^2_u(\mathcal{A}_f) \geq 0$ for every $\varphi \in C^\infty(\Sigma(z))$.*

We are now in position to state and prove the following rigidity result for strongly f-stable conformal Killing graphs.

Theorem 2.19 *Let $\overline{M}_f^{n+1}$ be a weighted Riemannian manifold nonnegative Bakry-Émery-Ricci tensor, endowed with complete closed conformal Killing vector field V having conformal factor ψ_V and whose weight function f is bounded. Let $x : \Sigma(z) \looparrowright \overline{M}_f^{n+1}$ be a strongly f-stable closed conformal Killing graph. Suppose that*

$$\frac{\partial \psi_V}{\partial t} \geq \max\{\psi_V H_f, 0\}, \tag{2.30}$$

where $t \in \mathbb{R}$ is the parameter of the flow associated with the unit vector field $\nu = -V/|V|$. If the set where $\psi_V = 0$ has empty interior in $\Sigma(z)$, then $\Sigma(z)$ is either f-minimal or isometric to a leaf of the foliation $V^\perp$.

Proof. As seen in Remark 1.5, we have that f not depend of $t \in \mathbb{R}$. Let us consider in $\overline{M}_f^{n+1}$ the global parametrization (1.23). Since $x : \Sigma(z) \looparrowright \overline{M}_f^{n+1}$ is strongly f-stable, it follows from Definition 2.18 and (2.29) that

$$-\int_{\Sigma(z)} \{\Delta_f(\varphi) + \{\overline{\mathrm{Ric}}_f(N,N) + |A|^2\}\varphi\}\,\varphi\, d\mu \geq 0, \tag{2.31}$$

for all $\varphi \in C^\infty(\Sigma(z))$. In particular, since H_f is constant on $\Sigma(z)$, taking the negative function η_V defined in (1.29) we get from (2.1) that

$$\Delta_f(\eta_V) + \{\overline{\mathrm{Ric}}_f(N,N) + |A|^2\}\eta_V = -n\{\psi_V H_f + N(\psi_V)\}.$$

Thus, from (2.31) we have that

$$\int_{\Sigma(z)} \{\psi_V H_f + N(\psi_V)\}\,\eta_V\, d\mu \geq 0. \tag{2.32}$$

On the other hand, it follows from (1.20) that

$$N(\psi_V) = \langle N, \overline{\nabla}\psi_V\rangle = \nu(\psi_V)\langle N, \nu\rangle = -\frac{\partial \psi_V}{\partial t}\cos\theta,$$

where θ is the angle between N and $-\nu$. Substituting the above into (2.32), we finally arrive at

$$\int_{\Sigma(z)} \left(\psi_V H_f - \frac{\partial \psi_V}{\partial t}\cos\theta\right)|V|\cos\theta\, d\mu \geq 0.$$

Now, from (2.30) we obtain

$$\begin{aligned} 0 &\leq \int_{\Sigma(z)} \left\{\psi_V H_f - \frac{\partial \psi_V}{\partial t}\cos\theta\right\}|V|\cos\theta\, d\mu \\ &\leq \int_{\Sigma(z)} (1-\cos\theta)\frac{\partial \psi_V}{\partial t}|V|\cos\theta\, d\mu \leq 0. \end{aligned}$$

Hence,

$$(1-\cos\theta)\frac{\partial \psi_V}{\partial t} = 0 \qquad \text{and} \qquad \frac{\partial \psi_V}{\partial t} = -\psi_V H_f$$

on $\Sigma(z)$. But, since H_f is constant on $\Sigma(z)$, $\Sigma(z)$ is either f-minimal or $H_f \neq 0$ on $\Sigma(z)$. If this last case occurs, the condition on the zero set of ψ_V on $\Sigma(z)$ together with the above give $\dfrac{\partial \psi_V}{\partial t} \neq 0$ on a dense subset of $\Sigma(z)$ and, hence, $\cos\theta = 1$ on this set. By continuity, $\cos\theta = 1$ on $\Sigma(z)$. Therefore, in this case, $\Sigma(z)$ must be a leaf of the foliation $V^\perp$. ■

We close this chapter observing that, when the ambient space is a weighted warped product of the type (1.25), we can apply Theorem 2.19 to obtain the following result.

Corollary 2.20 *Let $x : \Sigma(z) \looparrowright \mathbb{R} \times_\alpha M^n_f$ be a strongly f-stable closed conformal Killing graph. Suppose that the warped function α satisfies*

$$\alpha'' \geq \max\{\alpha' H_f, 0\}.$$

If the set where $\alpha' = 0$ has empty interior in $\Sigma(z)$, then $\Sigma(z)$ is either f-minimal or isometric to the slice $\{t_0\} \times M^n$, for some $t_0 \in \mathbb{R}$.

Chapter 3

Uniqueness for the weighted mean curvature equation in weighted Killing warped products

In this chapter, our purpose is to obtain uniqueness results related to the mean curvature equation for entire Killing graphs constructed over the base M^n of a weighted warped product of the type $M^n_f \times_\alpha \mathbb{R}$ with warping function α and density f. For this, we establish a suitable f-parabolicity criterion and, under appropriate constraints on the Bakry-Émery-Ricci tensor and on the f-mean curvature, we prove some rigidity results concerning two-sided hypersurfaces immersed in $M^n_f \times_\alpha \mathbb{R}$. The results presented in this chapter are part of [32].

3.1 A f-parabolicity criterion for two-sided hypersurfaces in $(M^n \times_\alpha \mathbb{R})_f$

Following the ideas of [40, Subsection 4.3], our aim in this section is just to obtain a f-parabolicity criterion for two-sided hypersurfaces immersed in a Killing warped product.

Given a weighted manifold $\overline{M}^{n+1}_f$, we define, for any compact subset $K \subset \Sigma^n$, the *f-capacity* of K as being

$$\mathrm{cap}_f(K) \;=\; \inf\left\{ \int_{\overline{M}} |\nabla u|^2 d\mu \;\;:\;\; u \in \mathrm{Lip}_0(\overline{M}) \;\;\text{and}\;\; u|_K \equiv 1 \right\},$$

where $d\mu$ is the volume element in $\overline{M}^{n+1}$ given in (1.10) and $\mathrm{Lip}_0(\overline{M})$ is the set of all compactly supported Lipschitz functions on $\overline{M}^{n+1}$. The following statement relates the notion of f-capacity to the concept of f-parabolicity (cf. [50, Proposition 2.1]).

Lemma 3.1 *The weighted manifold $\overline{M}^{n+1}_f$ is f-parabolic if and only if $\mathrm{cap}_f(K) = 0$ for any compact set $K \subset \overline{M}^{n+1}$.*

Let us recall that given two Riemannian manifolds $(M^{n+1}, \langle\cdot,\cdot\rangle_M)$ and $(\overline{M}^{n+1}, \langle\cdot,\cdot\rangle_{\overline{M}})$, a diffeomorphism φ from M^{n+1} onto $\overline{M}^{n+1}$ is called a *quasi-isometry* if there exists a constant $c \geq 1$ such that

$$c^{-1}|v|_{\langle\cdot,\cdot\rangle_M} \;\leq\; |d\varphi_p(v)|_{\langle\cdot,\cdot\rangle_{\overline{M}}} \;\leq\; c|v|_{\langle\cdot,\cdot\rangle_M}$$

for all $v \in T_pM$ and any $p \in M^{n+1}$ (see [55] for more details). In this case, given a smooth function $f : \overline{M}^{n+1} \to \mathbb{R}$, we can reason as in [49, Corollary 5.3] to verify that the $(f\circ\varphi)$-capacity of compact subsets in M^{n+1} changes under a quasi-isometry at most by a constant factor of the f-capacity of compact subsets in $\overline{M}^{n+1}$. From Lemma 3.1, it is not difficult to see that we obtain the following result (for a proof, see [38, Lemma 2]).

Lemma 3.2 *Keeping the same notation above, we have:*

- (a) *Given a quasi-isometry $\varphi : M^{n+1} \to \overline{M}^{n+1}$ and a smooth function $f : \overline{M}^{n+1} \to \mathbb{R}$, $\overline{M}^{n+1}$ is f-parabolic if and only if M^{n+1} is $(f\circ\varphi)$-parabolic;*
- (b) *Let $\widetilde{M}$ be the universal Riemannian covering of M^{n+1} with canonical projection $\pi_M : \widetilde{M} \to M^{n+1}$. If $\widetilde{M}$ is $(f\circ\pi_M)$-parabolic, then M^{n+1} is f-parabolic.*

Recall that every connected manifold M^{n+1} has universal covering, that is, there exist a simply connected manifold $\widetilde{M}$ (called the universal covering of M^{n+1}) and a smooth map $\pi_M : \widetilde{M} \to M^{n+1}$ (called the covering map) such that each point $p \in M^{n+1}$ has a connected neighborhood U that is evenly covered by π_M, that is, π_M maps each component of $\pi_M^{-1}(U)$ diffeomorphically onto U (for more details, see [62, Appendix A]). Moreover, if M^{n+1} is a Riemannian manifold, then it is possible to give $\widetilde{M}$ a Riemannian structure such that the covering map $\pi_M : \widetilde{M} \to M^{n+1}$ is a local isometry. In this case, $\widetilde{M}$ is said the universal Riemannian covering of M^{n+1} (cf. [43, page 152]).

From now on, we will denote by $\widetilde{M}$ the universal Riemannian covering of the base M^n, with projection $\widetilde{\pi} : \widetilde{M} \to M^n$, and $\widetilde{f}$ will denote the composition $f\circ\widetilde{\pi}$. In this setting, we have the following f-parabolicity criterion for complete two-sided hypersurfaces into weighted Killing warped products.

Proposition 3.3 *Let $(M^n \times_\alpha \mathbb{R})_f$ be a weighted Killing warped product and let $x : \Sigma^n \looparrowright (M^n \times_\alpha \mathbb{R})_f$ be a complete two-sided hypersurface such that the function*

$$\eta := \frac{\alpha}{\Theta} \tag{3.1}$$

is bounded and strictly positive on Σ^n, where Θ is the angle function of $x : \Sigma^n \looparrowright (M^n \times_\alpha \mathbb{R})_f$ defined in (1.33). If M^n has $\tilde{f}$-parabolic universal Riemannian covering, then Σ^n is f-parabolic.

Proof. From Lemma 3.2 we have that

(i) f-parabolicity is invariant under a quasi-isometry;

(ii) if the universal Riemannian covering $\tilde{\Sigma}$ of Σ^n is $(f \circ \pi_\Sigma)$-parabolic, then Σ^n is also f-parabolic.

Denoting $\pi = \pi_M \circ x$, $\pi_* = d\pi$ and $h_* = dh$, for any tangent vector $v \in T_p\Sigma$ and some $p \in \Sigma^n$, from Cauchy-Schwartz inequality we have that

$$\langle v, v\rangle \;=\; \langle \pi_* v, \pi_* v\rangle_M + \alpha^2 \langle h_* v, h_* v\rangle_{\mathbb{R}} \;\leq\; \langle \pi_* v, \pi_* v\rangle_M + \alpha^2 |\nabla h|^2 \langle v, v\rangle,$$

and then

$$\left(1 - \alpha^2 |\nabla h|^2\right) \langle v, v\rangle \;\leq\; \langle \pi_* v, \pi_* v\rangle_M.$$

By definition of the function η and from (1.36) we get

$$\frac{1}{\eta^2}\langle v, v\rangle \;\leq\; \langle \pi_* v, \pi_* v\rangle_M.$$

Taking into account our hypothesis, we have that

$$c^{-1}\langle v, v\rangle \;\leq\; \langle \pi_* v, \pi_* v\rangle_M, \tag{3.2}$$

where

$$c \;:=\; \sup_{\Sigma^n} \eta^2 \;\geq\; 1.$$

Consequently, π is a local diffeomorphism and we can reason as in the proof of [43, Lemma 7.3.3] (see also [56, Lemma 8.8.1]) to conclude that π is a covering map.

On the other hand, we see that

$$\langle v, v\rangle \;=\; \langle \pi_* v, \pi_* v\rangle_M + \alpha^2 \langle h_* v, h_* v\rangle_{\mathbb{R}} \;\geq\; \langle \pi_* v, \pi_* v\rangle_M.$$

Since $c \geq 1$, we obtain that

$$\langle \pi_* v, \pi_* v\rangle_M \;\leq\; c\langle v, v\rangle. \tag{3.3}$$

It follows from (3.2) and (3.3) that

$$c^{-1}\langle \pi_* v, \pi_* v\rangle \;\leq\; \langle v, v\rangle \;\leq\; c\langle \pi_* v, \pi_* v\rangle. \tag{3.4}$$

So, let $\tilde{\Sigma}$ be the universal Riemannian covering of Σ^n with projection $\pi_\Sigma : \tilde{\Sigma} \to \Sigma^n$. Then, the map $\pi_0 = \pi \circ \pi_\Sigma : \tilde{\Sigma} \to M^n$ is a covering map. Now, if $\widetilde{M}$ is the universal Riemannian covering of M^n with projection $\tilde{\pi} : \widetilde{M} \to M^n$, then there exists a diffeomorphism $\varphi : \widetilde{\Sigma} \to \widetilde{M}$ such that $\widetilde{\pi} \circ \varphi = \pi_0$. Moreover, φ is a quasi-isometry. Indeed, if v is a tangent vector at some point of $\widetilde{\Sigma}$, from (3.4) we have that

$$\begin{aligned}\langle \varphi_* v, \varphi_* v\rangle_{\widetilde{M}} &= \langle \widetilde{\pi}_*(\varphi_* v), \widetilde{\pi}_*(\varphi_* v)\rangle_M = \langle (\pi_0)_* v, (\pi_0)_* v\rangle_M \\ &= \langle \pi_*((\pi_\Sigma)_* v), \pi_*((\pi_\Sigma)_* v)\rangle_M \le c\langle (\pi_\Sigma)_* v, (\pi_\Sigma)_* v\rangle_\Sigma = c\langle v, v\rangle_{\widetilde{\Sigma}}.\end{aligned}$$

Analogously, we obtain

$$\langle \varphi_* v, \varphi_* v\rangle_{\widetilde{M}} \ge c^{-1}\langle v, v\rangle_{\widetilde{\Sigma}}.$$

Therefore, since the universal Riemannian covering of M^n is $\widetilde{f}$-parabolic, it follows that the universal Riemannian covering of Σ^n is $(f \circ \pi_\Sigma)$-parabolic and, hence, Σ^n must be also f-parabolic. ■

When the ambient space is just a weighted product space $(M^n \times \mathbb{R})_f$, from Proposition 3.3 we get the following f-parabolicity criterion.

Corollary 3.4 *Let $(M^n \times \mathbb{R})_f$ be a weighted product space and let $x : \Sigma^n \looparrowright (M^n \times \mathbb{R})_f$ be a complete two-sided hypersurface such that the angle function Θ given in (1.33) is bounded away from zero. If M^n has $\widetilde{f}$-parabolic universal Riemannian covering, then Σ^n is f-parabolic.*

3.2 Rigidity results for two-sided hypersurfaces in $M^n_f \times_\alpha \mathbb{R}$

In our first rigidity theorem for two-sided hypersurfaces immersed in $M^n_f \times_\alpha \mathbb{R}$, we deal with a specific weighted function $f = \log \alpha^2$. We note that it will not be assumed the constancy of the $\log \alpha^2$-mean curvature $H_{\log \alpha^2}$ of the hypersurface.

Theorem 3.5 *Let $M^n_{\log \alpha^2} \times_\alpha \mathbb{R}$ be a weighted Killing warped product whose base M^n has $\log \widetilde{\alpha}^{\,2}$-parabolic universal Riemannian covering. Let $x : \Sigma^n \looparrowright M^n_{\log \alpha^2} \times_\alpha \mathbb{R}$ be a complete two-sided hypersurface such that the function $\eta = \alpha/\Theta$ defined in (3.1) is bounded and strictly positive. Suppose that the $\log \alpha^2$-mean curvature $H_{\log \alpha^2}$ and the function $\langle \nabla \alpha, \nabla h\rangle$ have opposite signs on Σ^n, where h is the height function of $x : \Sigma^n \looparrowright M^n_{\log \alpha^2} \times_\alpha \mathbb{R}$ given in (1.32). If Σ^n is contained in a slab of $M^n_{\log \alpha^2} \times_\alpha \mathbb{R}$, then $x(\Sigma^n)$ is a slice $M^n \times \{t_0\}$, for some $t_0 \in \mathbb{R}$.*

Proof. From [39, Lemma 2], we have

$$\Delta h = n\alpha^{-2}\Theta\, H_{\log \alpha^2}. \tag{3.5}$$

Then, from (1.15) and (3.5),

$$\Delta_{\log \alpha^2} h = n\alpha^{-2}\Theta\, H_{\log \alpha^2} - \langle \nabla \log \alpha^2, \nabla h\rangle = n\alpha^{-2}\Theta\, H_{\log \alpha^2} - \frac{2}{\alpha}\langle \nabla \alpha, \nabla h\rangle.$$

Taking into account that $H_{\log\alpha^2}$ and the function $\langle\nabla\alpha,\nabla h\rangle$ have opposite signs on Σ^n, we conclude that $\Delta_{\log\alpha^2}h$ does not change sing on Σ^n. But, since Σ^n is contained in a slab of $M^n_{\log\alpha^2}\times_\alpha\mathbb{R}$, Proposition 3.3 guarantees the $\log\alpha^2$-parabolicity of Σ^n. Therefore, h must be constant and, consequently, $x(\Sigma^n)$ is a slice $M^n\times\{t_0\}$, for some $t_0\in\mathbb{R}$. ■

From Theorem 3.5 we obtain the following rigidity result for $\log\alpha^2$-minimal complete hypersurfaces immersed into $M^n_{\log\alpha^2}\times_\alpha\mathbb{R}$.

Corollary 3.6 *Let $M^n_{\log\alpha^2}\times_\alpha\mathbb{R}$ be a weighted Killing warped product whose base M^n has $\log\widetilde{\alpha}^{\,2}$-parabolic universal Riemannian covering. Let $x:\Sigma^n\looparrowright M^n_{\log\alpha^2}\times_\alpha\mathbb{R}$ be a $\log\alpha^2$-minimal complete two-sided hypersurface contained in a slab and such that the function $\eta=\alpha/\Theta$ defined in (3.1) is bounded and strictly positive. If the function $\langle\nabla\alpha,\nabla h\rangle$ does not change sign on Σ^n, where h is the height function of $x:\Sigma^n\looparrowright M^n_{\log\alpha^2}\times_\alpha\mathbb{R}$ given in (1.32), then $x(\Sigma^n)$ is a slice $M^n\times\{t_0\}$, for some $t_0\in\mathbb{R}$.*

We can obtain an equation similar to (3.5) involving the $\log\alpha^{-2}$-Laplacian and the standard mean curvature of $x:\Sigma^n\looparrowright M^n_{\log\alpha^{-2}}\times_\alpha\mathbb{R}$ to get the following result for minimal complete hypersurfaces.

Theorem 3.7 *Let $M^n_{\log\alpha^{-2}}\times_\alpha\mathbb{R}$ be a weighted Killing warped product whose base M^n has $\log\widetilde{\alpha}^{\,-2}$-parabolic universal Riemannian covering. Let $x:\Sigma^n\looparrowright M^n_{\log\alpha^{-2}}\times_\alpha\mathbb{R}$ be a minimal complete two-sided hypersurface such that the angle function Θ defined in (1.33) is strictly positive, the function $\eta=\alpha/\Theta$ given in (3.1) is bounded and*

$$\inf_{\Sigma^n}\alpha\ >\ 0. \tag{3.6}$$

If the Bakry-Émery-Ricci tensor $\mathrm{Ric}_{\log\alpha^{-2}}$ of $x:\Sigma^n\looparrowright M^n_{\log\alpha^{-2}}\times_\alpha\mathbb{R}$ satisfies $\mathrm{Ric}_{\log\alpha^{-2}}\geq\kappa$, for some positive constant $\kappa\in\mathbb{R}$, then $x(\Sigma^n)$ is a slice $M^n\times\{t_0\}$, for some $t_0\in\mathbb{R}$.

Proof. Firstly, observe that

$$\Delta h\ =\ \mathrm{Div}\,(\nabla h)\ =\ \alpha^2\Big\langle\nabla\alpha^{-2},\nabla h\Big\rangle+n\alpha^{-2}H\Theta\ =\ \Big\langle\nabla\log\alpha^{-2},\nabla h\Big\rangle+n\alpha^{-2}H\Theta.$$

So, from this last equation and from (1.15) we get

$$\Delta_{\log\alpha^{-2}}h\ =\ n\alpha^{-2}H\Theta \tag{3.7}$$

On the other hand, from Bochner's formula applied in Σ^n with density $\log\alpha^{-2}$ (see [68, page 378]) we have that

$$\frac{1}{2}\Delta_{\log\alpha^{-2}}|\nabla h|^2\ =\ |\mathrm{Hess}\,h|^2+\mathrm{Ric}_{\log\alpha^{-2}}(\nabla h,\nabla h)+\langle\nabla\Delta_{\log\alpha^{-2}}h,\nabla h\rangle. \tag{3.8}$$

Now, taking into account our restriction on $\mathrm{Ric}_{\log\alpha^{-2}}$ and the the minimality of $x : \Sigma^n \looparrowright M^n_{\log\alpha^{-2}} \times_\alpha \mathbb{R}$, from (3.7) and (3.8) we obtain

$$\frac{1}{2}\Delta_{\log\alpha^{-2}}|\nabla h|^2 \;\geq\; \mathrm{Ric}_{\log\alpha^{-2}}(\nabla h, \nabla h) \;\geq\; \kappa|\nabla h|^2 \;\geq\; 0. \tag{3.9}$$

Moreover, from (1.36), the condition (3.6) implies in the boundedness of $|\nabla h|^2$ on Σ^n. Thus, since Proposition 3.3 assures us that Σ^n is $\log\alpha^{-2}$-parabolic, from (3.9) we must have that $|\nabla h|^2$ is constant on Σ^n. Returning to (3.9), we obtain that $|\nabla h| = 0$ on Σ^n and, therefore, there exists $t_0 \in \mathbb{R}$ such that $x(\Sigma^n) = M^n \times \{t_0\}$. ■

When the ambient space is a weighted product space $M^n_f \times \mathbb{R}$, we obtain the following rigidity result which can be regarded as an extension of [40, Theorem 7].

Theorem 3.8 *Let $M^n_f \times \mathbb{R}$ be a weighted product space whose base M^n has $\tilde{f}$-parabolic universal Riemannian covering and Bakry-Émery-Ricci tensor $\widetilde{\mathrm{Ric}}_f$ satisfying $\widetilde{\mathrm{Ric}}_f \geq -\kappa$, for some positive constant $\kappa \in \mathbb{R}$. Let $x : \Sigma^n \looparrowright M^n_f \times \mathbb{R}$ be a complete two-sided hypersurface with constant f-mean curvature, such that the angle function Θ defined in (1.33) is bounded away from zero. If the height function h of $x : \Sigma^n \looparrowright M^n_f \times \mathbb{R}$ satisfies*

$$|\nabla h|^2 \;\leq\; \frac{c}{\kappa}|A|^2, \tag{3.10}$$

for some constant $c \in (0,1)$, then $x(\Sigma^n)$ is a slice $M^n \times \{t_0\}$, for some $t_0 \in \mathbb{R}$.

Proof. Since H_f is assumed to be constant, from [39, Lemma 1] we get

$$\Delta_f\Theta \;=\; -\left(\widetilde{\mathrm{Ric}}_f(N^*, N^*) + |A|^2\right)\Theta. \tag{3.11}$$

Moreover, since we are assuming that Θ is bounded away from zero, for an appropriate choice of Gauss map N of $x : \Sigma^n \looparrowright M^n_f \times \mathbb{R}$ we get that $\Theta > 0$ on Σ^n.

Thus, taking into account our constraint on $\widetilde{\mathrm{Ric}}$, from (1.36) and (3.11) we obtain

$$\Delta_f\Theta \;\leq\; -\left(|A|^2 - \kappa|\nabla h|^2\right)\Theta. \tag{3.12}$$

Using hypothesis (3.10), from (3.12) we have that

$$\Delta_f\Theta \;\leq\; -(1-c)|A|^2\Theta \;\leq\; 0. \tag{3.13}$$

Consequently, from (3.13) we get that the angle function Θ is a positive f-superharmonic function on Σ^n. Hence, we can apply Corollary 3.4 to guarantee that Θ must be constant on Σ^n. So, returning to (3.13) we see that Σ^n is totally geodesic. Therefore, hypothesis (3.10) assures that h is constant on Σ^n and, consequently, there exists $t_0 \in \mathbb{R}$ such that $x(\Sigma^n) = M^n \times \{t_0\}$. ■

3.3 Entire Killing graphs and the f-mean curvature equation in $M_f^n \times_\alpha \mathbb{R}$

In this section we present the main results of this chapter, namely, uniqueness results for the f-mean curvature equation of entire Killing graphs constructed over the base M^n of a weighted Killing warped product $M_f^n \times_\alpha \mathbb{R}$.

The shape operator $A : \mathfrak{X}(\Sigma^n(z)) \to \mathfrak{X}(\Sigma^n(z))$ of the entire Killing graphs $\Sigma^n(z)$, presented in section 1.4.3, with respect to Gauss map N given in (1.37) is

$$\begin{aligned} AX &= \frac{\alpha}{(1+\alpha^2|Dz|_M^2)^{1/2}} D_X Dz - \frac{\alpha^3\langle D_X Dz, Dz\rangle}{(1+\alpha^2|Dz|_M^2)^{3/2}} Dz - \frac{\alpha^2\langle D\alpha, X\rangle |Dz|_M^2}{(1+\alpha^2|Dz|_M^2)^{3/2}} Dz \\ &+ \frac{\langle D\alpha, X\rangle}{(1+\alpha^2|Dz|_M^2)^{1/2}} Dz + \frac{\langle Dz, X\rangle}{(1+\alpha^2|Dz|_M^2)^{1/2}} D\alpha, \end{aligned} \tag{3.14}$$

for every $X \in \mathfrak{X}(\Sigma^n(z))$. It follows from (3.14) that the mean curvature $H(z)$ of an entire Killing graph $\Sigma^n(z)$ is given by

$$nH(z) = \operatorname{div}\left(\frac{\alpha Dz}{(1+\alpha^2|Dz|_M^2)^{1/2}}\right) + \frac{\langle Dz, D\alpha\rangle}{(1+\alpha^2|Dz|_M^2)^{1/2}}, \tag{3.15}$$

where $\operatorname{div}(\,\cdot\,)$ stands for the divergence on M^n with respect to the metric $\langle\cdot,\cdot\rangle_M$. A direct computation shows that the f-mean curvature $H_f(z)$ of $\Sigma^n(z)$ is given by

$$nH_f(z) = \operatorname{div}_f\left(\frac{\alpha Dz}{(1+\alpha^2|Dz|_M^2)^{1/2}}\right) + \frac{\langle Dz, D\alpha\rangle}{(1+\alpha^2|Dz|_M^2)^{1/2}},$$

where $\operatorname{div}_f(\,\cdot\,)$ is the f-divergence on M^n.

In particular, an entire Killing graph $\Sigma(z)$ have constant f-mean curvature if and only if the function $z \in C^\infty(M)$ satisfies the following elliptic partial differential equation:

$$\operatorname{div}_f\left(\frac{\alpha Dz}{(1+\alpha^2|Dz|_M^2)^{1/2}}\right) + \frac{\langle Dz, D\alpha\rangle}{(1+\alpha^2|Dz|_M^2)^{1/2}} = C, \tag{3.16}$$

for some constant $C \in \mathbb{R}$.

In what follows, we will use the results of Section 3.2 to obtain uniqueness results for equation (3.16). We start by applying a consequence of Theorem 3.5 to get the following uniqueness result for $\log\alpha^2$-minimal entire Killing graph in a weighted Killing warped product $M^n_{\log\alpha^2} \times_\alpha \mathbb{R}$.

Theorem 3.9 *Let $M^n_{\log\alpha^2} \times_\alpha \mathbb{R}$ be a weighted Killing warped product whose base M^n has $\log\widetilde{\alpha}^{\,2}$-parabolic universal Riemannian covering. Suppose that the entire Killing graph $\Sigma^n(z)$ associated to $z \in C^\infty(M^n)$ is such that $\langle\nabla\alpha, x_*(Dz)\rangle$ does not change* *sign. Then, for all positive constant $c \in \mathbb{R}$, the only bounded solutions of the problem*

$$\begin{cases} \operatorname{div}_{\log \alpha^2}\left(\dfrac{\alpha Dz}{(1+\alpha^2|Dz|_M^2)^{1/2}}\right) + \dfrac{\langle Dz, D\alpha\rangle}{(1+\alpha^2|Dz|_M^2)^{1/2}} \;=\; 0, \quad \textit{in} \quad M^n \\ \\ \alpha^2|Dz|_M^2 \;\leq\; c, \end{cases}$$

are the constant ones.

Proof.

We observe that the boundness of $\alpha^2|Dz|_M^2$ is equivalent to the boundenes of function η defined in (3.1). Indeed, from (1.38),

$$\eta \;=\; \left(1+\alpha^2|Dz|_M^2\right)^{1/2}. \tag{3.17}$$

Furthermore, from (1.37) we have that

$$N^* \;=\; N - N^\perp \;=\; -\frac{\alpha\, x_*(Dz)}{(1+\alpha^2|Dz|_M^2)^{1/2}}. \tag{3.18}$$

On the other hand, we observe that

$$\begin{aligned} \langle \nabla\alpha, \nabla h\rangle \;&=\; \nabla h(\alpha) \;=\; \frac{1}{\alpha^2}\, Y^\top(\alpha) \;=\; \frac{1}{\alpha^2}\, Y^\top\left(\langle Y, Y\rangle^{\frac{1}{2}}\right) \\ &=\; \frac{1}{\alpha^2}\left(\frac{1}{2}\,\langle Y, Y\rangle^{-1/2}\, Y^\top\langle Y, Y\rangle\right) \;=\; \frac{1}{2\alpha^3}\, Y^\top\langle Y, Y\rangle \\ &=\; \frac{1}{\alpha^3}\,\langle\overline{\nabla}_{Y^\top} Y, Y\rangle \;=\; \frac{1}{\alpha^3}\,\langle\overline{\nabla}_{Y-\Theta N} Y, Y\rangle \\ &=\; \frac{1}{\alpha^3}\left(\underbrace{\langle\overline{\nabla}_{Y} Y, Y\rangle}_{0} - \langle\overline{\nabla}_{\Theta N} Y, Y\rangle\right) \;=\; -\frac{1}{\alpha^3}\,\langle\overline{\nabla}_{\Theta N} Y, Y\rangle \\ &=\; -\frac{\Theta}{\alpha^3}\,\langle\overline{\nabla}_{N} Y, Y\rangle \;=\; -\frac{\Theta}{2\alpha^3}\, N\langle Y, Y\rangle \;=\; -\frac{\Theta}{2\alpha^3}\, N(\alpha^2) \\ &=\; -\frac{\Theta}{2\alpha^3}\, 2\alpha N^*(\alpha) \;=\; -\frac{\Theta}{\alpha^2}\langle\nabla\alpha, N^*\rangle. \end{aligned} \tag{3.19}$$

Hence, from (3.19) and (3.18) follows that

$$\langle\nabla\alpha, \nabla h\rangle \;=\; \frac{\Theta}{\alpha}\left\langle\overline{\nabla}\alpha, \frac{x_*(Dz)}{(1+\alpha^2|Dz|_M^2)^{1/2}}\right\rangle \;=\; \frac{\Theta}{\alpha\,(1+\alpha^2|Dz|_M^2)^{1/2}}\left\langle\overline{\nabla}\alpha, x_*(Dz)\right\rangle.$$

Therefore, $\langle\nabla\alpha, \nabla h\rangle$ does not change of sign on $\Sigma^n(z)$ if and only if $\langle\overline{\nabla}\alpha, x_*(Dz)\rangle$ does not change of sign on $\Sigma^n(z)$, and the result follows from Corollary 3.6. ∎

In our next result, we will apply Theorem 3.7 to study a problem related to the usual mean curvature equation for minimal entire Killing graphics immersed into $M^n_{\log\alpha^2} \times_\alpha \mathbb{R}$.

Theorem 3.10 *Let $M^n_{\log\alpha^2} \times_\alpha \mathbb{R}$ be a weighted Killing warped product whose base M^n has $\log\widetilde{\alpha}^{-2}$-parabolic universal Riemannian covering. Suppose that the entire Killing graph $\Sigma^n(z)$ associated to $z \in C^\infty(M^n)$ is such that $\inf_{\Sigma^n(z)} \alpha > 0$ and that the Bakry-Émery-Ricci tensor $\mathrm{Ric}_{\log\alpha^2}$ of $\Sigma^n(z)$ satisfies $\mathrm{Ric}_{\log\alpha^2} \geq \kappa$, for some positive constant κ. Then, for all positive constant $c \in \mathbb{R}$, the only solutions of the problem*

$$\begin{cases} \operatorname{div}\left(\dfrac{\alpha Dz}{(1+\alpha^2|Dz|^2_M)^{1/2}}\right) + \dfrac{\langle Dz, D\alpha\rangle}{(1+\alpha^2|Dz|^2_M)^{1/2}} = 0, & \text{in} \quad M^n \\ \\ \alpha^2|Dz|^2_M \leq c, & \end{cases}$$

are the constant ones.

Proof. By the previous digression, we can conclude that an entire weighted Killing graph $\Sigma^n(z)$ is minimal if and only if $z \in C^\infty(M^n)$ satisfies the equation (3.15) for $H(z)$ identically zero. From equation (3.17), the condition $\alpha^2|Dz|^2_M \leq c$ ensures that the function η defined in (3.1) is bounded and our choice of the Gauss map (1.37) guarantees that η has strict sign. Hence, the result follows from Theorem 3.7. ∎

For a weighted product space $M^n_f \times \mathbb{R}$, we establish the following uniqueness result for f-minimal entire Killing graphs.

Theorem 3.11 *Let $M^n_f \times \mathbb{R}$ be a weighted product space whose base M^n has $\widetilde{f}$-parabolic universal Riemannian covering and with its Bakry-Émery-Ricci tensor $\widetilde{\mathrm{Ric}}_f$ satisfying $\widetilde{\mathrm{Ric}}_f \geq -\kappa$, for some positive constant κ. Let $\Sigma^n(z)$ be the entire Killing graph associated to $z \in C^\infty(M^n)$ such that $|A|^2 \leq k$. For any $c \in (0,1)$, the only solutions of the problem*

$$\begin{cases} \operatorname{div}_f\left(\dfrac{Dz}{(1+|Dz|^2_M)^{1/2}}\right) = C, & \text{in} \quad M^n \\ \\ |Dz|^2_M \leq \dfrac{c|A|^2}{k - c|A|^2}, & \end{cases}$$

are the constant ones.

Proof. We note that from (1.36) and (3.18) we obtain

$$|\nabla h|^2 = \frac{|Dz|^2_M}{1+|Dz|^2_M}. \tag{3.20}$$

Hence, using (3.20), we conclude that the hypothesis (3.10) is equivalent to

$$|Dz|^2_M \leq \frac{c|A|^2}{k - c|A|^2}.$$

Moreover, it is not difficult to verify that this previous inequality jointly with our constraint in $|A|^2$ and (1.38) imply that the angle function Θ of $\Sigma^n(z)$ is bounded away from zero. Therefore, the result follows from Theorem 3.8. ∎

An important example of weighted manifold is the so-called *Gaussian space* $\mathbb{G}^n$, which corresponds to the Euclidean space $\mathbb{R}^n$ endowed with the Gaussian probability measure

$$d\mu = (2\pi)^{-\frac{n}{2}} e^{-\frac{|y|^2}{2}} dy^2. \tag{3.21}$$

Concerning the weighted product space $\mathbb{G}^n \times \mathbb{R}$, Hieu and Nam extended in [53, Theorem 4] the classical Bernstein's theorem showing that the only weighted minimal graphs $\Sigma^n(z)$ of functions $z(y_1, \dots, y_n) = y_{n+1}$ over $\mathbb{G}^n$ are the hyperplanes $y_{n+1} = \text{constant}$.

Finally, taking into account this previous digression, from Theorem 3.11 we obtain the following uniqueness result for f-minimal entire Killing graph in the weighted product space $\mathbb{G}^n \times \mathbb{R}$.

Corollary 3.12 *Consider the weighted product space $\mathbb{G}^n \times \mathbb{R}$, where $\mathbb{G}^n$ is the Gaussian space which is endowed with Gaussian density f defined implicitly by (3.21). Let $\Sigma^n(z)$ be the entire Killing graph associated to $z \in C^\infty(\mathbb{G}^n)$ such that $|A|$ is bounded. For any positive constants $\sup_{\Sigma^n(z)} |A|^2 \leq k$ and $c \in (0,1)$, the only solutions of the problem*

$$\begin{cases} \operatorname{div}_f \left(\dfrac{Dz}{(1 + |Dz|_M^2)^{1/2}} \right) = C, \quad \textit{in} \quad \mathbb{G}^n \\ \\ |Dz|_M^2 \leq \dfrac{c|A|^2}{k - c|A|^2}, \end{cases}$$

are the constant ones.

Proof. We have that the f-volume of $\mathbb{G}^n$ is equal to 1 (see, for instance, the last equation of the proof of [53, Theorem 4]). Then, [54, Remark 3] guarantees that $\mathbb{G}^n$ is f-parabolic. Moreover, with a straightforward computation we get that the Bakry-Émery-Ricci tensor $\widetilde{\mathrm{Ric}}_f$ of $\mathbb{G}^n$ satisfies the equality $\widetilde{\mathrm{Ric}}_f = 1$. Therefore, since $\mathbb{G}^n$ is also simply connected, the result follows from Theorem 3.11. ■

Chapter 4

Spacelike hypersurfaces immersed in weighted standard static spacetimes: uniqueness, nonexistence and stability

Along this chapter, in weighted standard static spacetimes, we study some aspects of the geometry of spacelike hypersurfaces through of drift Laplacian of two functions support naturally related to them. For such hypersurfaces, with some restrictions on density function and the geometry of the ambient spacetime, we begin by stating and showing some results of uniqueness and nonexistence, several of them not assuming that the hypersurface to be of constant weighted mean curvature. Versions of these results are given for entire Killing graphs, that is, graphs constructed over an integral leaf of the distribution of smooth vector fields orthogonal to timelike Killing vector field. Finally, for closed spacelike hypersurface immersed in a weighted standard static spacetime with constant weighted mean curvature, we study a notion of stability via the first eigenvalue of the drift Laplacian. The results presented in this chapter are part of [33].

4.1 Uniqueness and nonexistence results in standard static spacetimes

We begin this section by providing a formula for the classical Laplacian of the height function of a spacelike hypersurface immersed in a standard static space $M^n \times_\alpha \mathbb{R}$ in terms of a certain weighted mean curvature. More precisely, we have the following

Proposition 4.1 *Let $x : \Sigma^n \looparrowright M^n \times_\alpha \mathbb{R}_1$ be an immersed spacelike hypersurface and let $h \in C^\infty(\Sigma^n)$ be the height function defined in (1.41). Then*

$$\Delta h = -n\alpha^{-2}\Theta\, H_{\log\alpha^2}, \tag{4.1}$$

where Θ is the angle function defined in (1.42) and $H_{\log\alpha^2}$ is the $\log\alpha^2$-mean curvature of Σ^n.

Proof. Let $\{E_1, \ldots, E_n\}$ be an orthonormal frame defined in a neighborhood of some point of Σ^n. From (1.43) we note that

$$\begin{aligned}
\alpha^{-2}\mathrm{div}\,(\nabla h) &= \alpha^{-2}\mathrm{div}\left(-\alpha^{-2}Y^\top\right) \\
&= -\alpha^{-2}\Big\langle\nabla\alpha^{-2}, Y^\top\Big\rangle - \alpha^{-4}\mathrm{div}\left(Y^\top\right) \\
&= \Big\langle\nabla\alpha^{-2}, \nabla h\Big\rangle - \alpha^{-4}\mathrm{div}\,(Y + \Theta N) \\
&= \Big\langle\nabla\alpha^{-2}, \nabla h\Big\rangle - \alpha^{-4}\sum_{i=1}^n \langle\nabla_{E_i}(Y + \Theta N), E_i\rangle \\
&= \Big\langle\nabla\alpha^{-2}, \nabla h\Big\rangle - \alpha^{-4}\sum_{i=1}^n \Big\langle\overline{\nabla}_{E_i}(Y + \Theta N), E_i\Big\rangle \\
&= \Big\langle\nabla\alpha^{-2}, \nabla h\Big\rangle - \alpha^{-4}\sum_{i=1}^n \underbrace{\Big\langle\overline{\nabla}_{E_i}Y, E_i\Big\rangle}_{0} - \alpha^{-4}\sum_{i=1}^n \Big\langle\overline{\nabla}_{E_i}(\Theta N), E_i\Big\rangle \\
&= \Big\langle\nabla\alpha^{-2}, \nabla h\Big\rangle - \alpha^{-4}\sum_{i=1}^n \Big[E_i(\Theta)\underbrace{\langle N, E_i\rangle}_{0} + \Theta\langle\overline{\nabla}_{E_i}N, E_i\rangle\Big] \\
&= \Big\langle\nabla\alpha^{-2}, \nabla h\Big\rangle + \alpha^{-4}\Theta\,\mathrm{tr}(A) \;=\; \Big\langle\nabla\alpha^{-2}, \nabla h\Big\rangle - n\alpha^{-4}H\Theta.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\Delta h &= \mathrm{div}\,(\nabla h) \;=\; \alpha^2\Big\langle\nabla\alpha^{-2}, \nabla h\Big\rangle - n\alpha^{-2}H\Theta \\
&= \Big\langle\nabla\log\alpha^{-2}, -\alpha^2\, Y^\top\Big\rangle - n\alpha^{-2}H\Theta \\
&= -\alpha^{-2}\Big\langle\overline{\nabla}\log\alpha^{-2}, Y^\top\Big\rangle - n\alpha^{-2}H\Theta \\
&= -\alpha^{-2}\Big\langle\overline{\nabla}\log\alpha^{-2}, Y + \Theta N\Big\rangle - n\alpha^{-2}H\Theta \\
&= -\alpha^{-2}\underbrace{\Big\langle\overline{\nabla}\log\alpha^{-2}, Y\Big\rangle}_{0} - \alpha^{-2}\Big\langle\overline{\nabla}\log\alpha^{-2}, N\Big\rangle\Theta - n\alpha^{-2}H\Theta \\
&= -\alpha^{-2}\Theta\Big\{nH + \Big\langle\overline{\nabla}(\log\alpha^{-2}), N\Big\rangle\Big\} \;=\; -n\alpha^{-2}\Theta\, H_{\log\alpha^2},
\end{aligned}$$

where in the last equality we use (1.12). ■

In order to obtain the first rigidity result of this chapter, we will need another key lemma. The next one corresponds to Theorem 3 of [71]. In what follows, given a n-dimensional Riemannian manifold Σ^n, we use the notation

$$\mathcal{L}^q(\Sigma^n) = \left\{ u : \Sigma^n \to \mathbb{R} \ : \ \int_{\Sigma^n} |u|^q d\Sigma \ \ll +\infty \right\},$$

where $d\Sigma$ denotes the standard volume element of Σ^n.

Lemma 4.2 *Let u be a nonnegative smooth subharmonic function on a complete Riemannian manifold Σ^n. If $u \in \mathcal{L}^q(\Sigma^n)$, for some $q > 1$, then u is constant on Σ^n.*

We will apply the previous lemma to get the following result

Theorem 4.3 *The only complete spacelike hypersurfaces immersed into standard static spacetime $M^n \times_\alpha \mathbb{R}_1$ with nonnegative $\log \alpha^2$-mean curvature and whose height function h is nonnegative and satisfies the condition $h \in \mathcal{L}^q(\Sigma^n)$, for some $q > 1$, are the slices $M^n \times \{t\}$, $t \in \mathbb{R}$.*

Proof. In fact, let $x : \Sigma^n \looparrowright M^n \times_\alpha \mathbb{R}_1$ be such a spacelike hypersurface. Since $\Theta < 0$ and $H_{\log \alpha^2} \geq 0$ on Σ^n, from (4.1) we have that $\Delta h \geq 0$ on Σ^n. From Lemma 4.2, we conclude that h is constant on Σ^n and, hence, there is $t_0 \in \mathbb{R}$ such that $x(\Sigma^n) = M^n \times \{t_0\}$. ■

From the proof of Theorem 4.3, we get the following

Corollary 4.4 *The only parabolic complete spacelike hypersurfaces immersed into standard static spacetime $M^n \times_\alpha \mathbb{R}_1$ with nonnegative $\log \alpha^2$-mean curvature and lying in a slab of $M^n \times_\alpha \mathbb{R}_1$ are the slices $M^n \times \{t\}$, $t \in \mathbb{R}$.*

A Riemannian manifold Σ^n is said to be *stochastically complete* if, for some (and, hence, for any) $(x, t) \in \Sigma^n \times (0, +\infty)$, the heat kernel $p(x, y, t)$ of the Laplace-Beltrami operator Δ (that is, the minimal, positive fundamental solution of the heat operator $\Delta - \partial/\partial_t$; for more details concerning the heat kernel of the Laplace-Beltrami operator, see [51]) satisfies the conservation property

$$\int_{\Sigma^n} p(x, y, t) d\xi(y) = 1. \tag{4.2}$$

From the probabilistic viewpoint, stochastically completeness is the property of a stochastic process to have infinite life time. For the Brownian motion on a manifold, the conservation property (4.2) means that the total probability of the particle to be found in the state space is constantly equal to one (cf. [45, 49, 67]).

Any parabolic manifold is stochastically complete but the opposite implication is not true. For example, all Euclidean spaces $\mathbb{R}^n$ (with Euclidean measure) are stochastically complete, whereas $\mathbb{R}^n$ is parabolic if and only if $n \in \{1, 2\}$. On the other hand, Pigola, Rigoli and Setti showed that stochastic completeness turns out to be equivalent to the validity of a weak form of the Omori-Yau maximum principle (see Theorem 1.1 of [63] or Theorem 3.1 of [64]), as can be expressed below.

Lemma 4.5 *A Riemannian manifold Σ^n is stochastically complete if and only if, for every $u \in C^2(\Sigma^n)$ satisfying $\sup_{\Sigma^n} u \ll +\infty$, there exists a sequence of points $\{p_j\}_{j=1}^{+\infty} \subset \Sigma^n$ such that*

$$\lim_{j\to+\infty} u(p_j) = \sup_{\Sigma^n} u \qquad \textit{and} \qquad \limsup_{j\to+\infty} \Delta u(p_j) \leq 0.$$

We will also need of the next lemma, which is just a consequence of a more general extension of Liouville's theorem due to Yau in [70].

Lemma 4.6 *The only harmonic semi-bounded functions defined on an n-dimensional complete Riemannian manifold whose Ricci curvature is nonnegative are the constant ones.*

Applying these previous lemmas, we obtain the following result.

Theorem 4.7 *Let $x : \Sigma^n \looparrowright M^n \times_\alpha \mathbb{R}_1$ be a stochastically complete spacelike hypersurface which lies in a slab of $M^n \times_\alpha \mathbb{R}_1$. If the $\log \alpha^2$-mean curvature $H_{\log \alpha^2}$ of $x : \Sigma^n \looparrowright M^n \times_\alpha \mathbb{R}_1$ is a nonnegative constant, then $x : \Sigma^n \looparrowright M^n \times_\alpha \mathbb{R}_1$ is $\log \alpha^2$-maximal. Moreover, if $x : \Sigma^n \looparrowright M^n \times_\alpha \mathbb{R}_1$ is complete with nonnegative Ricci curvature, then $x(\Sigma^n)$ is a slice $M^n \times \{t_0\}$, for some $t_0 \in \mathbb{R}$.*

Proof. From Proposition 4.1 we have that $\alpha^2 \Delta h = -nH_{\log \alpha^2}\,\Theta$ on Σ^n. So, taking into account that the height function h of Σ^n is bounded, from Lemma 4.5 we get a sequence $\{p_j\}_{j=1}^{+\infty} \subset \Sigma^n$ such that

$$0 \geq \limsup_{j\to+\infty} \alpha^2 \Delta h(p_j) = n \limsup_{j\to+\infty}(-H_{\log \alpha^2}\,\Theta(p_j)) = -nH_{\log \alpha^2} \liminf_{j\to+\infty} \Theta(p_j) \geq 0. \tag{4.3}$$

Then, we have that $H_{\log \alpha^2} = 0$ on Σ^n and, hence, h is harmonic on Σ^n.

On the other hand, since Σ^n lies in a slab then there exists a constant β such that $h - \beta > 0$. Thus, if $\mathrm{Ric} \geq 0$, then from Lemma 4.6 we can conclude that h is constant on Σ^n. Therefore, we conclude that there is $t_0 \in \mathbb{R}$ such that $x(\Sigma^n) = M^n \times \{t_0\}$. ■

In particular, from the analysis of signals realized in (4.3) we can established the following nonexistence result.

Corollary 4.8 *There do not exist stochastically complete spacelike hypersurface immersed into standard static spacetime $M^n \times_\alpha \mathbb{R}_1$ which lies in a slab of $M^n \times_\alpha \mathbb{R}_1$ and whose $\log \alpha^2$-mean curvature is a positive constant.*

In order to establish our next result, we will need of an extension of Hopf's theorem on a complete noncompact Riemannian manifold due to Yau in [71].

Lemma 4.9 *Let u be a smooth function on a complete Riemannian manifold Σ^n, such that Δu does not change sign on Σ^n. If $|\nabla u| \in \mathcal{L}^1(\Sigma^n)$, then Δu vanishes identically on Σ^n.*

Now, we are in a position to present the following result:

Theorem 4.10 *Let $M^n \times_\alpha \mathbb{R}_1$ be a standard static spacetime and $x : \Sigma^n \looparrowright M^n \times_\alpha \mathbb{R}_1$ be a complete spacelike hypersurface whose $\log \alpha^2$-mean curvature $H_{\log \alpha^2}$ does not change sign. If the gradient ∇h of the height function h of $x : \Sigma^n \looparrowright M^n \times_\alpha \mathbb{R}_1$ has integrable norm on Σ^n then $x : \Sigma^n \looparrowright M^n \times_\alpha \mathbb{R}_1$ is $\log \alpha^2$-maximal. Moreover, if $x(\Sigma^n)$ lies in a slab of $M^n \times_\alpha \mathbb{R}$ then $x(\Sigma^n)$ is a slice $M^n \times \{t_0\}$, for some $t_0 \in \mathbb{R}$.*

Proof. Taking into account our restrictions on $H_{\log \alpha^2}$ and Θ, from (4.1) we get that Δh does not change sign on Σ^n. Moreover, since $|\nabla h| \in \mathcal{L}^1(\Sigma^n)$, from Lemma 4.9 we get that $\Delta h = 0$ and, returning again in (4.1) we have that Σ^n is $\log \alpha^2$-maximal.

On the other hand, from (1.45) we also note that

$$\Delta h^2 \;=\; 2h\Delta h + 2|\nabla h|^2 \;=\; 2\alpha^{-2}|N^*|^2 \;\geq\; 0. \tag{4.4}$$

If we assume that $x(\Sigma^n)$ lies in a slab of $M^n \times_\alpha \mathbb{R}$ then h is bounded on Σ^n. So, since h is bounded on Σ^n and using once more that $|\nabla h| \in \mathcal{L}^1(\Sigma^n)$, Lemma 4.9 guarantees also that $\Delta h^2 = 0$. Therefore, from (4.4) we obtain that N^* vanishes identically on Σ^n, which means that N and the Killing vector field Y are collinear. Since Y determines in $M^n \times_\alpha \mathbb{R}_1$ a codimension one Riemannian foliation by totally geodesic slices $M^n \times \{t\}$, $t \in \mathbb{R}$, we conclude that there is $t_0 \in \mathbb{R}$ such that $x(\Sigma^n) = M^n \times \{t_0\}$. ■

As a consequence of Theorem 4.10, we will obtain the following non-parametric result concerning entire Killing graphs in $M^n \times_\alpha \mathbb{R}_1$ (Cf. Subsecction 1.4.4).

Corollary 4.11 *Let $\Sigma^n(z)$ be an entire Killing graph which lies in a slab of the standard static spacetime $M^n \times_\alpha \mathbb{R}_1$ whose base M^n is complete. Suppose there is a positive constant $c < 1$ such that the gradient Dz of the function $z \in C^\infty(M^n)$ satisfies*

$$\sup_{\Sigma^n(z)} \alpha^2 |Dz|^2_{M^n} \leq c. \tag{4.5}$$

If the $\log \alpha^2$-mean curvature $H_{\log \alpha^2}$ of $\Sigma^n(z)$ does not change sign and $|Dz| \in \mathcal{L}^1(M^n)$, then $\Sigma^n(z)$ is a slice $M^n \times \{t_0\}$, for some $t_0 \in \mathbb{R}$.

Proof. First, from (4.5) we observe that $\Sigma^n(z)$ is spacelike. Now, we claim that $\Sigma^n(z)$ is complete. Indeed, let X be any vector field tangent to $\Sigma^n(z)$. From (1.46) and from the Cauchy-Schwarz inequality we get

$$\langle X, X\rangle_z \;=\; \langle X^*, X^*\rangle_{M^n} - \alpha^2\langle Dz, X^*\rangle_{M^n} \;\geq\; (1 - \alpha^2|Dz|^2_{M^n})\langle X^*, X^*\rangle_{M^n}.$$

Then, from (4.5) we obtain

$$\ell_u(\gamma) \geq (1-c)^{1/2}\ell_{M^n}(\gamma^*),$$

where $\ell_z(\gamma)$ stands for the length of a curve γ on $\Sigma^n(z)$ with respect to the induced metric (1.46) and $\ell_{M^n}(\gamma^*)$ denotes the length of the projection γ^* of γ onto M^n with respect to its metric $\langle\cdot,\cdot\rangle_{M^n}$. Consequently, since projections onto M^n of divergent curves on $\Sigma^n(z)$ give divergent curves on M^n and as we are assume that the metric $\langle\cdot,\cdot\rangle_{M^n}$ is complete, we can apply Hopf-Rinow theorem to conclude that the induced metric (1.46) is also complete.

On the other hand, from (1.47) we obtain

$$N^* = \frac{1}{(1-\alpha^2|Dz|^2_{M^n})^{1/2}}\,\alpha\Psi_*(Dz),$$

So, from (1.45) and (4.5) we have that the height function h of $\Sigma^n(z)$ satisfies

$$|\nabla h|^2 = \frac{1}{1-\alpha^2|Dz|^2_{M^n}}\,|Dz|^2_{M^n} \leq \frac{1}{1-c}\,|Dz|^2_{M^n}. \tag{4.6}$$

Therefore, from Theorem 4.10 we get that $\Sigma^n(z)$ is a slice $M^n\times\{t_0\}$, for some $t_0\in\mathbb{R}$. ■

4.2 Uniqueness and nonexistence results in weighted standard static spacetimes

Motivated by Remark 1.5, in this section we will consider standard static spacetimes $M^n_f\times_\alpha\mathbb{R}_1$ endowed with a weight function f not depending on the parameter $t\in\mathbb{R}$, that is, $\langle\overline{\nabla} f, Y\rangle = 0$.The following key proposition provides an explicit formula for the drift Laplacian of the angle function Θ defined in (1.42).

Proposition 4.12 *Let $x:\Sigma^n\looparrowright M^n_f\times_\alpha\mathbb{R}_1$ be a spacelike hypersurface and let $\Theta\in C^\infty(\Sigma^n)$ be the angle function defined in (1.42). Then*

$$\Delta_f\Theta = nY^\top(H_f) + \left(\widetilde{\mathrm{Ric}}_f(N^*,N^*) - \frac{1}{\alpha}\,\widetilde{\mathrm{Hess}}\,\alpha(N^*,N^*) + \Theta^2\frac{\widetilde{\Delta}_f(\alpha)}{\alpha^3} + |A|^2\right)\Theta. \tag{4.7}$$

Here, Y is the Killing vector field on $M^n_f\times_\alpha\mathbb{R}$, $\alpha=|Y|>0$, N is the unit normal vector field on Σ^n, Δ_f and $\widetilde{\Delta}_f$ represent the f-Laplacians on Σ^n and M^n, respectively, $\widetilde{\mathrm{Ric}}_f$ and $\widetilde{\mathrm{Hess}}$ are the Bakry-Émery-Ricci tensor and the Hessian operator on M^n, $|A|^2$ represent the square of the norm of the shape operator A of Σ^n with respect to the orientation given by N and N^* is the projection of N on the tangent bundle of M^n.

Proof. Firstly, since Y is a Killing vector field for any $X \in \mathfrak{X}(\Sigma^n)$ we have

$$\left\langle \nabla\Theta, X \right\rangle = X(\Theta) = X(\langle N, Y\rangle) = \left\langle \overline{\nabla}_X N, Y \right\rangle + \left\langle N, \overline{\nabla}_X Y \right\rangle = \left\langle -A(Y^\top) - \overline{\nabla}_N Y, X \right\rangle,$$

which assures us that

$$\nabla\Theta = -A(Y^\top) - (\overline{\nabla}_N Y)^\top. \tag{4.8}$$

On the other hand, from (1.12) we note that

$$\begin{aligned} nY^\top(H) &= Y^\top\left(nH_f + \left\langle \overline{\nabla} f, N \right\rangle\right) \\ &= nY^\top(H_f) + Y^\top\left(\left\langle \overline{\nabla} f, N \right\rangle\right) \\ &= nY^\top(H_f) + \left\langle Y, \overline{\mathrm{Hess}}\, f(N) \right\rangle + \Theta\, \overline{\mathrm{Hess}}\, f(N,N) - \left\langle A(Y^\top), \nabla f \right\rangle, \end{aligned} \tag{4.9}$$

where we used the decomposition $Y = Y^\top - \Theta N$.

Moreover, since f is supposed to be invariant along the flow determinate by Y, from (4.8) we get that

$$\begin{aligned} \left\langle \nabla\Theta, \nabla f \right\rangle &= -\left\langle A(Y^\top) + (\overline{\nabla}_N Y)^\top, \nabla f \right\rangle \\ &= -\left\langle A(Y^\top), \nabla f \right\rangle - \left\langle \overline{\nabla}_N Y, \overline{\nabla} f \right\rangle \\ &= -\left\langle A(Y^\top), \nabla f \right\rangle + \left\langle Y, \overline{\nabla}_N \overline{\nabla} f \right\rangle \\ &= -\left\langle A(Y^\top), \nabla f \right\rangle + \left\langle Y, \overline{\mathrm{Hess}}\, f(N) \right\rangle. \end{aligned} \tag{4.10}$$

Substituting (4.10) into (4.9) we get

$$nY^\top(H) = nY^\top(H_f) + \Theta\, \overline{\mathrm{Hess}}\, f(N,N) + \left\langle \nabla\Theta, \nabla f \right\rangle. \tag{4.11}$$

From Proposition 2.12 of [8] we have

$$\Delta\Theta = nY^\top(H) + \Theta\left(\mathrm{Ric}(N,N) + |A|^2\right), \tag{4.12}$$

Thus, from (1.15), (4.12) and (4.11) we obtain that

$$\Delta_f\, \Theta = nY^\top(H_f) + \left(\overline{\mathrm{Ric}}_f(N,N) + |A|^2\right)\Theta. \tag{4.13}$$

Now, if we consider the decomposition $N = N^* + N^\perp$ of N, where $(\,\cdot\,)^\perp$ denote the projection of a vector field in $\mathfrak{X}(M^n \times_\alpha \mathbb{R}_1)$ on $\mathfrak{X}(\mathbb{R}_1)$, we have

$$\begin{aligned} \overline{\mathrm{Hess}} f(N,N) &= \left\langle \overline{\nabla}_N \overline{\nabla} f, N \right\rangle \\ &= \left\langle \overline{\nabla}_N \widetilde{\nabla} f, N^* + N^\perp \right\rangle \\ &= \widetilde{\mathrm{Hess}} f(N^*, N^*) + \frac{1}{\alpha}\left\langle \widetilde{\nabla} f, \widetilde{\nabla}\alpha \right\rangle |N^\perp|^2 \\ &= \widetilde{\mathrm{Hess}} f(N^*, N^*) - \frac{1}{\alpha^3}\left\langle \widetilde{\nabla} f, \widetilde{\nabla}\alpha \right\rangle \Theta^2. \end{aligned} \tag{4.14}$$

From Corollary 7.43 of [62] we get that

$$\overline{\mathrm{Ric}}(N,N) = \widetilde{\mathrm{Ric}}(N^*,N^*) - \frac{1}{\alpha}\widetilde{\mathrm{Hess}}\,\alpha(N^*,N^*) + \Theta^2\frac{\widetilde{\Delta}(\alpha)}{\alpha^3} \tag{4.15}$$

Now, from equations (1.11), (4.14) and (4.15), we have that

$$\overline{\mathrm{Ric}}_f(N,N) = \widetilde{\mathrm{Ric}}_f(N^*,N^*) - \frac{1}{\alpha}\widetilde{\mathrm{Hess}}\,\alpha(N^*,N^*) + \Theta^2\frac{\widetilde{\Delta}_f(\alpha)}{\alpha^3} \tag{4.16}$$

Therefore, from equations (4.16) and (4.13) we obtain (4.7). ■

Next, we obtain the following result concerning f-parabolic spacelike hypersurfaces immersed in a weighted static spacetime. As usual, expressions that have $\widetilde{(\,\cdot\,)}$ correspond to objects defined on M^n.

Theorem 4.13 *Let $M^n_f \times_\alpha \mathbb{R}_1$ be a weighted standard static spacetime with $\widetilde{\mathrm{Ric}}_f \geq -\kappa$, for some constant $\kappa > 0$, and α being a convex warping function such that $\langle\widetilde{\nabla} f, \widetilde{\nabla}\alpha\rangle \leq 0$. Let $x : \Sigma^n \looparrowright M^n_f \times_\alpha \mathbb{R}_1$ be a f-parabolic spacelike hypersurface with constant f-mean curvature and angle function Θ bounded from below. If the height function h and the shape operator A of Σ^n satisfy*

$$|\nabla h|^2 \leq \frac{c}{\kappa\alpha^2}|A|^2, \tag{4.17}$$

for some constant $c \in (0,1)$, then $x(\Sigma^n)$ is contained in a slice $M^n \times \{t_0\}$, for some $t_0 \in \mathbb{R}$.

Proof. Let us first observe that at points where N^* is different from zero we have

$$\frac{1}{\alpha}\widetilde{\mathrm{Hess}}\,\alpha(N^*,N^*) = \frac{|N^*|^2}{\alpha}\widetilde{\mathrm{Hess}}\,\alpha\left(\frac{N^*}{|N^*|},\frac{N^*}{|N^*|}\right) = \frac{\Theta^2-\alpha^2}{\alpha^3}\widetilde{\mathrm{Hess}}\,\alpha\left(\frac{N^*}{|N^*|},\frac{N^*}{|N^*|}\right)$$

and, taking a local orthonormal frame $\left\{E_1 = \frac{N^*}{|N^*|}, E_2, \ldots, E_n\right\}$ tangent to M^n, we also have

$$\frac{\Theta^2}{\alpha^3}\widetilde{\Delta}(\alpha) = \frac{\Theta^2}{\alpha^3}\widetilde{\mathrm{Hess}}\,\alpha\left(\frac{N^*}{|N^*|},\frac{N^*}{|N^*|}\right) + \frac{\Theta^2}{\alpha^3}\sum_{i=2}^{n}\widetilde{\mathrm{Hess}}\,\alpha(E_i,E_i).$$

Then,

$$-\frac{1}{\alpha}\widetilde{\mathrm{Hess}}\,\alpha(N^*,N^*) + \frac{\Theta^2}{\alpha^3}\widetilde{\Delta}(\alpha) = \frac{1}{\alpha}\widetilde{\mathrm{Hess}}\,\alpha\left(\frac{N^*}{|N^*|},\frac{N^*}{|N^*|}\right) + \frac{\Theta^2}{\alpha^3}\sum_{i=2}^{n}\widetilde{\mathrm{Hess}}\,\alpha(E_i,E_i)$$

and, from (1.15), we get

$$\begin{aligned}-\frac{1}{\alpha}\widetilde{\mathrm{Hess}}\,\alpha(N^*,N^*) + \frac{\Theta^2}{\alpha^3}\widetilde{\Delta}_f(\alpha) &= \frac{1}{\alpha}\widetilde{\mathrm{Hess}}\,\alpha\left(\frac{N^*}{|N^*|},\frac{N^*}{|N^*|}\right) \\ &\quad + \frac{\Theta^2}{\alpha^3}\sum_{i=2}^{n}\widetilde{\mathrm{Hess}}\,\alpha(E_i,E_i) - \frac{\Theta^2}{\alpha^3}\langle\widetilde{\nabla} f,\widetilde{\nabla}\alpha\rangle \geq 0,\end{aligned} \tag{4.18}$$

where in the last step we use the convexity of α and the hypothesis $\langle \widetilde{\nabla} f, \widetilde{\nabla}\alpha\rangle \leq 0$.

Now, noting that H_f is constant, $\Theta < 0$ on Σ^n and taking into account our constraint on $\widetilde{\mathrm{Ric}}$, from (1.45) and (4.18) jointly with Proposition 4.12 we obtain

$$\Delta_f \Theta \;\leq\; \left(-\kappa\alpha^2|\nabla h|^2 + |A|^2\right)\Theta. \tag{4.19}$$

Using hypothesis (4.17), from (4.19) we obtain that

$$\Delta_f(-\Theta) \;\geq\; (1-c)|A|^2(-\Theta)\,. \tag{4.20}$$

Hence, from (4.20) we have that $-\Theta$ is a bounded positive subharmonic function on Σ^n and, since we are assuming that Σ^n is f-parabolic, $-\Theta$ must be constant on Σ^n. So, returning to (4.20), we see that Σ^n is totally geodesic. Therefore, hypothesis (4.17) assures that h is constant on Σ^n, that is, there exists $t_0 \in \mathbb{R}$ such that $\Sigma^n \subset M^n \times \{t_0\}$.
■

As a direct consequence of Theorem 4.13, we get the following

Corollary 4.14 *Let $M^n_f \times_\alpha \mathbb{R}_1$ be a weighted standard static spacetime with $\widetilde{\mathrm{Ric}}_f \geq -\kappa$, for some constant $\kappa > 0$, and α being a convex warping function such that $\langle \widetilde{\nabla} f, \widetilde{\nabla}\alpha\rangle \leq 0$. There is not nonzero constant f-mean curvature f-parabolic spacelike hypersurface immersed into $M^n_f \times_\alpha \mathbb{R}_1$ with angle function bounded from below and such that the height function and the its shape operator satisfy the condition (4.17), for some constant $c \in (0,1)$.*

Remark 4.15 *We note that there is a large family of weighted standard static spacetimes $M^n_f \times_\alpha \mathbb{R}_1$ that satisfy the conditions of Theorem 4.13. For example, if we define on M^n the smooth function $f = a\alpha + b$, with $a < 0$ and $b \in \mathbb{R}$, then we obtain that $\langle \widetilde{\nabla} f, \widetilde{\nabla}\alpha\rangle = a|\widetilde{\nabla}\alpha|^2 \leq 0$ and the Bakry-Émery-Ricci tensor $\widetilde{\mathrm{Ric}}_f$ of M^n is given by*

$$\widetilde{\mathrm{Ric}}_f \;=\; \widetilde{\mathrm{Ric}} + a\,\widetilde{\mathrm{Hess}}\,\alpha.$$

In addition, if $\widetilde{\mathrm{Ric}} \geq -\kappa$, for some positive constant κ, and α is chosen such that $0 \leq \widetilde{\mathrm{Hess}}\,\alpha \leq \beta$ for some constant β, then $\widetilde{\mathrm{Ric}}_f \geq -(k + |a|\beta)$. Hence, $M^n_f \times_\alpha \mathbb{R}_1$ verifies the requested conditions of Theorem 4.13.

Another situation happens when we define on M^n the smooth function $f = e^{a\alpha} + b$, with $a < 0$ and $b \in \mathbb{R}$. In this other case, with the same constraints on α and $\widetilde{\mathrm{Ric}}$ assumed in the previous case, we have that $\langle \widetilde{\nabla} f, \widetilde{\nabla}\alpha\rangle = ae^{a\alpha}|\widetilde{\nabla}\alpha|^2 \leq 0$ and

$$\widetilde{\mathrm{Ric}}_f \;=\; \widetilde{\mathrm{Ric}} + a^2 e^{a\alpha}\langle \widetilde{\nabla}\alpha, \cdot\,\rangle^2 + ae^{a\alpha}\,\widetilde{\mathrm{Hess}}\,\alpha \;\geq\; -(k + |a|\beta).$$

Therefore, this second ambient space also contemplates the hypothesis of Theorem 4.13.

In the context of Killing graphs, from Theorem 4.13 we obtain the following

Corollary 4.16 *Let $M^n_f \times_\alpha \mathbb{R}_1$ be a weighted standard static spacetime with $\widetilde{\mathrm{Ric}}_f \geq -\kappa$, for some constant $\kappa > 0$, and α being a convex warping function such that $\langle \widetilde{\nabla} f, \widetilde{\nabla}\alpha\rangle \leq 0$. Let $\Sigma^n(z)$ be a f-parabolic entire Killing graph in $M^n_f \times_\alpha \mathbb{R}_1$ with constant f-mean curvature, angle function Θ bounded from below and whose norm of its shape operator A satisfy*

$$|A|^2 \leq \frac{\kappa}{1-c} \tag{4.21}$$

for some constant $c \in (0,1)$. If the gradient Dz of the function z satisfy

$$|Dz|^2_{M^n} \leq \frac{(1-c)c}{\kappa\alpha^2}|A|^2 \tag{4.22}$$

then $\Sigma^n(z) = M^n \times \{t_0\}$, for some $t_0 \in \mathbb{R}$.

Proof. From (4.21) and (4.22), we get $\sup_{\Sigma^n(z)} \alpha^2 |Dz|^2_{M^n} \leq c$. So, from the first part of the proof of the Corollay 4.11 we obtain that $\Sigma^n(z)$ is spacelike and complete. Now, from (4.6) and (4.22) we obtain (4.17). Finally, the result is obtained as a direct application of the Theorem 4.13. ■

In order to characterize slices of weighted standard static spacetimes $M^n_f \times_\alpha \mathbb{R}_1$, we observe that one of the hypotheses of the Theorem 4.13 is exactly the inferior limitation of the Bakry-Émery-Ricci tensor $\widetilde{\mathrm{Ric}}_f$ of M^n by some constant. When this limitation is given by zero, we have the following result that establishes other sufficient conditions for an spacelike hypersurface to be a slice.

Theorem 4.17 *Let $M^n_f \times_\alpha \mathbb{R}_1$ be a weighted standard static spacetime with $\widetilde{\mathrm{Ric}}_f \geq 0$ and α being a convex warping function such that $\langle \widetilde{\nabla} f, \widetilde{\nabla}\alpha\rangle \leq 0$. Let $x : \Sigma^n \looparrowright M^n_f \times_\alpha \mathbb{R}_1$ be a f-parabolic spacelike hypersurface with constant f-mean curvature and angle function Θ bounded from below. Then, Σ^n is totally geodesic. Moreover, if $\widetilde{\mathrm{Ric}}_f$ is strictly positive at some point p_0 of Σ^n, then $x(\Sigma^n)$ is contained in a slice $M^n \times \{t_0\}$, for some $t_0 \in \mathbb{R}$.*

Proof. Since the f-mean curvature of $x : \Sigma^n \looparrowright M^n_f \times_\alpha \mathbb{R}_1$ is constant, α is a concave function such that $\langle \widetilde{\nabla} f, \widetilde{\nabla}\alpha\rangle \leq 0$ and $\widetilde{\mathrm{Ric}}_f$ is nonnegative, from Proposition 4.12 and (4.18) we obtain that

$$\Delta_f \Theta \leq \left(\widetilde{\mathrm{Ric}}_f(N^*, N^*) + |A|^2\right)\Theta \leq 0. \tag{4.23}$$

Thus, the weighted parabolicity of Σ^n assures that Θ is constant on it. So, returning to (4.23) we have that $|A| = 0$, that is, Σ^n is totally geodesic.

We claim that α is constant. Indeed, first we note that all $X \in \mathfrak{X}(\Sigma^n)$ can be written as

$$X = X^* - \frac{\langle X, Y\rangle}{\alpha^2} Y,$$

where $(\,\cdot\,)^*$ denote the projection on $\mathfrak{X}(M^n)$. Since Σ^n is totally geodesic, from Proposition 7.35 of[62], we have that

$$\begin{aligned}\langle\nabla\Theta, X\rangle &= X(\langle N, Y\rangle) = \langle N, \overline{\nabla}_X Y\rangle\\ &= \langle N, \overline{\nabla}_{X^*} Y\rangle - \frac{\langle X, Y\rangle}{\alpha^2}\langle N, \overline{\nabla}_Y Y\rangle\\ &= \frac{1}{\alpha}\langle X, \overline{\nabla}\alpha\rangle\langle N, Y\rangle - \frac{1}{\alpha}\langle X, Y\rangle\langle N, \overline{\nabla}\alpha\rangle,\end{aligned}$$

which implies

$$\nabla\Theta = \frac{1}{\alpha}\left(\Theta\overline{\nabla}\alpha - \langle N, \overline{\nabla}\alpha\rangle Y\right).$$

Given that Θ is constant on Σ^n, since the vector fields $\overline{\nabla}\alpha$ and Y are linearly independent, from the last equation, we obtain that α is also constant on Σ^n. So, our affirmation stay showed.

On the other hand, from (1.44) it is not difficult to see that

$$|N^*|^2_{M^n} = \left(\frac{\Theta^2}{\alpha^2} - 1\right),$$

which implies that $|N^*|_{M^n}$ is also constant. But, supposing that $\widetilde{\mathrm{Ric}}_f$ is strictly positive at some point p_0 of Σ^n, since (4.23) give us that $\widetilde{\mathrm{Ric}}_f(N^*, N^*)(p_0) = 0$, it follows that $N^*(p_0) = 0$. Therefore, N^* must be vanishes on Σ^n and, consequently, $x(\Sigma^n)$ must be contained in a slice $M^n \times \{t_0\}$, for some $t_0 \in \mathbb{R}$. ■

In particular, Theorem 4.17 gives us the following result of nonexistence.

Corollary 4.18 *Let $M^n_f \times_\alpha \mathbb{R}_1$ be a weighted standard static spacetime with $\widetilde{\mathrm{Ric}}_f \geq 0$ and α being a convex warping function such that $\langle\widetilde{\nabla} f, \widetilde{\nabla}\alpha\rangle \leq 0$. There is not nonzero constant f-mean curvature f-parabolic spacelike hypersurface immersed into $M^n_f \times_\alpha \mathbb{R}_1$ with angle function bounded from below.*

From Theorem 4.17, we can reason as in the proof of Corollary 4.11 in order to obtain the following result:

Corollary 4.19 *Let $M^n_f \times_\alpha \mathbb{R}_1$ be a weighted standard static spacetime with $\widetilde{\mathrm{Ric}}_f \geq 0$ and α being a convex warping function such that $\langle\widetilde{\nabla} f, \widetilde{\nabla}\alpha\rangle \leq 0$. Let $\Sigma^n(z)$ be a f-parabolic entire graph with constant f-mean curvature and angle function Θ bounded from below. If the norm of the gradient Dz of the function $z \in C^\infty(M^n)$ satisfies (4.5), then $\Sigma^n(z)$ is totally geodesic. Moreover, if $\widetilde{\mathrm{Ric}}_f$ is strictly positive at some point p_0 of $\Sigma^n(z)$, then $\Sigma^n(z)$ is a slice $M^n \times \{t_0\}$, for some $t_0 \in \mathbb{R}$.*

4.3 A notion of stability in weighted standard static spacetimes

For a compact spacelike hypersurface $x : \Sigma^n \looparrowright M_f \times_\alpha \mathbb{R}_1$ with boundary $\partial\Sigma$ (possibly empty), we define a *variation* of it as being the smooth mapping

$$\begin{array}{rcl} X : (-\epsilon, \epsilon) \times \Sigma^n & \to & M_f^n \times_\alpha \mathbb{R}_1 \\ (s, p) & \mapsto & X(s, p) \end{array}$$

satisfying the following two conditions:

(i) for all $s \in (-\epsilon, \epsilon)$, the map $X_s : \Sigma^n \looparrowright M_f^n \times_\alpha \mathbb{R}_1$ given by $X_s(p) = X(s, p)$ is a Riemannian immersion such that $X_0 = x$;

(ii) $X_s\Big|_{\partial\Sigma} = x\Big|_{\partial\Sigma}$, for all $s \in (-\epsilon, \epsilon)$.

In all that follows, we let $d\Sigma_s$ for denote the volume element of the warping metric (1.39) induced on $\Sigma_s^n = X_s(\Sigma^n)$ and N_s will be the future-pointing Gauss map along of Σ_s^n. Moreover, we also consider in Σ_s^n the weighted volume form given by $d\sigma_s = e^{-f} d\Sigma_s$. When $s = 0$ all these objects coincide with ones defined in Σ^n, respectively. Moreover for any open subset Ω of $M_f^n \times_\alpha \mathbb{R}_1$ with compact closure, $\mathrm{Vol}_f(\Omega)$ and $\mathrm{Area}_f(\Omega)$ will denote the *weighted volume* and *weighted area* of Ω, respectively.

The *variational field* associated to the variation X is the smooth vector field $\dfrac{\partial X}{\partial s}\Big|_{s=0}$. Letting

$$u_s = -\Big\langle \frac{\partial X}{\partial s}, N_s \Big\rangle, \tag{4.24}$$

we get

$$\frac{\partial X}{\partial s}\Big|_{s=0} = u_0 N + \left(\frac{\partial X}{\partial s}\Big|_{s=0} \right)^{\top}.$$

The *balance of weighted volume* and the *weighted area functional* associated to the variation X are the functionals

$$\begin{array}{rl} \mathcal{V}_f : (-\epsilon, \epsilon) \to & \mathbb{R} \\ s & \mapsto \mathcal{V}_f(s) = \mathrm{Vol}_f\left(X\left([0, s] \times \Sigma^n\right)\right) = \displaystyle\int_{[0,s]\times\Sigma^n} X^*(d\overline{\sigma}) \end{array}$$

and

$$\begin{array}{rl} \mathcal{A}_f : (-\epsilon, \epsilon) \to & \mathbb{R} \\ s & \mapsto \mathcal{A}_f(s) = \mathrm{Area}_f\left(\Sigma_s^n\right) = \displaystyle\int_{\Sigma_s^n} d\sigma_s, \end{array}$$

respectively, where $d\overline{\sigma}$ is the volume element on induced by the warping metric (1.39). We say that the variation X is *weighted volume-preserving* of Σ^n if $\mathcal{V}_f(s) = \mathcal{V}_f(0) = 0$, for all $s \in (-\epsilon, \epsilon)$.

The following result is well known and, in the context of weighted Lorentzian manifolds, it can be found in Lemmas 1 and 2 of [42].

Lemma 4.20 *Let $X : (-\epsilon, \epsilon) \times \Sigma^n \to M^n_f \times_\alpha \mathbb{R}_1$ be a variation of the closed (that is, compact and without boundary) spacelike hypersurface $x : \Sigma^n \looparrowright M^n_f \times_\alpha \mathbb{R}_1$. If u_s is the smooth function given in (4.24) then*

$$\frac{d}{ds}\mathcal{V}_f(s) = \int_{\Sigma^n_s} u_s\, d\sigma_s \qquad \text{and} \qquad \frac{d}{ds}\mathcal{A}_f(s) = n\int_{\Sigma^n_s} (H_f)_s\, u_s\, d\sigma_s$$

where $(H_f)_s = H_f(s,\cdot)$ denotes the f-mean curvature of Σ^n_s. In particular, X is weighted volume-preserving of Σ^n if and only if $\int_{\Sigma^n_s} u_s\, d\sigma_s = 0$ for all $s \in (-\epsilon, \epsilon)$.

Remark 4.21 *Applying the same topological arguments used to prove Proposition 3.2 of [4], we conclude that a closed spacelike hypersurface Σ^n immersed in a standard static spacetime $M^n \times_\alpha \mathbb{R}_1$ can only exist when the Riemannian base M^n is also compact. On the other hand, it is not difficult to verify that Lemma 2.2 of [8] still remains valid for the context of weighted standard static spacetimes. More specifically, given a closed spacelike hypersurface $x : \Sigma^n \looparrowright M^n_f \times_\alpha \mathbb{R}_1$, if $u \in C^\infty(\Sigma^n)$ is such that*

$$\int_{\Sigma^n} u d\sigma = 0,$$

then there exists a weighted volume-preserving variation $X : (-\epsilon, \epsilon) \times \Sigma^n \to M^n_f \times_\alpha \mathbb{R}_1$ of $x : \Sigma^n \looparrowright M^n_f \times_\alpha \mathbb{R}_1$ whose variational field is $\left.\frac{\partial X}{\partial s}\right|_{s=0} = uN$.

In order to characterize closed spacelike hypersurfaces $x : \Sigma^n \looparrowright M^n_f \times_\alpha \mathbb{R}_1$ with constant f-mean curvature, we consider the variational problem of maximizing the weighted area functional $\mathcal{A}_f$ for all variations $X : (-\epsilon, \epsilon) \times \Sigma^n \to M^n_f \times_\alpha \mathbb{R}_1$ of $x : \Sigma^n \looparrowright M^n_f \times_\alpha \mathbb{R}_1$ that keeps the balance of weighted volume $\mathcal{V}_f$ equal to zero. The Lagrange multiplier method leads us then to the associated weighted Jacobi functional

$$\begin{array}{rcl} \mathcal{J}_f : (-\epsilon, \epsilon) & \to & \mathbb{R} \\ s & \mapsto & \mathcal{J}_f(s) = \mathcal{A}_f(s) - \lambda \mathcal{V}_f(s), \end{array} \tag{4.25}$$

where λ is a constant to be determined. As an immediate consequence of Lemma 4.20 we get that the first variation of $\mathcal{J}_f$ takes the following form

$$\frac{d}{ds}\mathcal{J}_f(s) = \int_{\Sigma^n_s} \left\{ n\,(H_f)_s - \lambda \right\} u_s\, d\sigma_s, \tag{4.26}$$

where u_s is the smooth function given in (4.24). Thinking about making the best possible choice of λ, let

$$\overline{\mathcal{H}} = \frac{1}{\mathrm{Area}_f(\Sigma^n)} \int_{\Sigma^n} H_f\, d\sigma \tag{4.27}$$

be an integral mean of the f-mean curvature H_f on Σ^n. We call the attention to the fact that, in case H_f is constant, we have

$$\overline{\mathcal{H}} = H_f, \tag{4.28}$$

and this notation will be used in what follows without further comments. Therefore, if we choose $\lambda = n\overline{\mathcal{H}}$, from (4.26) we arrive at

$$\frac{d}{ds}\mathcal{J}_f(s) = n\int_{\Sigma_s^n}\left\{(H_f)_s - \overline{\mathcal{H}}\right\} u_s\, d\sigma_s. \tag{4.29}$$

Reasoning as in the proof of Proposition 2.7 of [9] we get

Proposition 4.22 *The following statements are equivalent:*

(a) $x : \Sigma^n \looparrowright M_f^n \times_\alpha \mathbb{R}_1$ *is a closed spacelike hypersurface with constant f-mean curvature H_f;*

(b) $\dfrac{d}{ds}\mathcal{A}_f(0) = 0$ *for all weighted volume-preserving variation* $X : (-\epsilon,\epsilon)\times\Sigma^n \to M_f^n \times_\alpha \mathbb{R}_1$ *of* $x : \Sigma^n \looparrowright M_f^n \times_\alpha \mathbb{R}_1$*;*

(c) $\dfrac{d}{ds}\mathcal{J}_f(0) = 0$ *for every variation* $X : (-\epsilon,\epsilon)\times\Sigma^n \to M_f^n \times_\alpha \mathbb{R}_1$ *of* $x : \Sigma^n \looparrowright M_f^n \times_\alpha \mathbb{R}_1$.

Proof. We will show the result making the sequence $(a) \Rightarrow (c)$, $(c) \Rightarrow (b)$, $(b) \Rightarrow (a)$.

$(a) \Rightarrow (c)$: The result follows directly from (4.28) and (4.29).

$(c) \Rightarrow (b)$: We have

$$0 = \frac{d}{ds}\mathcal{J}_f(0) = \frac{d}{ds}\mathcal{A}_f(0) + n\overline{\mathcal{H}}\frac{d}{ds}\mathcal{V}_f(0)$$

or all variation $X : (-\epsilon,\epsilon)\times\Sigma^n \to M_f^n \times_\alpha \mathbb{R}_1$ of $x : \Sigma^n \looparrowright M_f^n \times_\alpha \mathbb{R}_1$. But if the variation preserves the volume of $x : \Sigma^n \looparrowright M_f^n \times_\alpha \mathbb{R}_1$ then $\dfrac{d}{ds}\mathcal{V}_f(0) = 0$. Hence, $\dfrac{d}{ds}\mathcal{A}_f(0) = 0$ for all weighted volume-preserving variation $X : (-\epsilon,\epsilon)\times\Sigma^n \to M_f^n\times_\alpha\mathbb{R}_1$ of $x : \Sigma^n \looparrowright M_f^n \times_\alpha \mathbb{R}_1$.

$(b) \Rightarrow (a)$: Suppose there is p_0 in Σ^n such that $(H_f - \overline{\mathcal{H}})(p_0) \neq 0$. We can assume that $(H_f - \overline{\mathcal{H}})(p_0) > 0$. From the definition of $\overline{\mathcal{H}}$ in (4.27) we can obtain another point $q_0 \in \Sigma^n$ such that $(H_f - \overline{\mathcal{H}})(q_0) < 0$. Indeed, from (4.28) we have

$$\begin{aligned}\int_{\Sigma^n}(H_f - \overline{\mathcal{H}})\, d\sigma &= \int_{\Sigma^n} H_f\, d\sigma - \overline{\mathcal{H}}\,\mathrm{Area}_f(\Sigma^n) \qquad (4.30)\\ &= \int_{\Sigma^n} H_f\, d\sigma - \left(\frac{1}{\mathrm{Area}_f(\Sigma^n)}\int_{\Sigma^n} H_f\, d\sigma\right)\mathrm{Area}_f(\Sigma^n) = 0.\end{aligned}$$

So, if $(H_f - \overline{\mathcal{H}})(q) > 0$ for every $q \in \Sigma^n$, since there is $p_0 \in \Sigma^n$ such that $(H_f - \overline{\mathcal{H}})(p_0) > 0$, then

$$\int_{\Sigma^n}(H_f - \overline{\mathcal{H}})\, d\sigma > 0,$$

inequality that is in contradiction with (4.30).

Thus, the sets

$$\Sigma^+ = \{\, q \in \Sigma^n \,:\, (H_f - \overline{\mathcal{H}})(q) > 0 \,\} \qquad \text{and} \qquad \Sigma^- = \{\, q \in \Sigma^n \,:\, (H_f - \overline{\mathcal{H}})(q) < 0 \,\}$$

are well defined.

Now, consider nonnegative smooth functions φ and ψ such that $p_0 \in \operatorname{supp}\varphi \subset \Sigma^+$, $\operatorname{supp}\psi \subset \Sigma^-$ and

$$\int_{\Sigma^n} (\varphi + \psi)(H_f - \overline{\mathcal{H}})\, d\sigma \;=\; 0.$$

where $\operatorname{supp}\varphi$ and $\operatorname{supp}\psi$ denote the support of φ and the support of ψ, respectively. If we consider the smooth function $u = (\varphi+\psi)(H_f - \overline{\mathcal{H}})$ then, according to Remark 4.21, there is a weighted volume-preserving variation $X : (-\epsilon, \epsilon) \times \Sigma^n \to M_f^n \times_\alpha \mathbb{R}_1$ of $x : \Sigma^n \looparrowright M_f^n \times_\alpha \mathbb{R}_1$ whose variational field is $\left.\dfrac{\partial X}{\partial s}\right|_{s=0} = uN$. By hypothesis and Lemma 4.20,

$$0 \;=\; \frac{d}{ds}\mathcal{A}_f(0) \;=\; n\int_{\Sigma^n} H_f\, u\, d\sigma.$$

Since $\displaystyle\int_{\Sigma^n} u\, d\sigma = 0$, we obtain

$$0 \;=\; n\int_{\Sigma^n} H_f\, u\, d\sigma - n\overline{\mathcal{H}}\int_{\Sigma^n} u\, d\sigma \;=\; n\int_{\Sigma^n}(H_f - \overline{\mathcal{H}})u\, d\sigma \;=\; n\int_{\Sigma^n}(\varphi+\psi)(H_f-\overline{\mathcal{H}})^2\, d\sigma \;>\; 0,$$

which is a contradiction. Therefore, $H_f = \overline{\mathcal{H}}$ on Σ^n. ∎

In particular, Proposition 4.22 guarantees that a closed spacelike hypersurface $x : \Sigma^n \looparrowright M_f^n \times_\alpha \mathbb{R}_1$ is a critical point of the variational problem described above if and only if its f-mean curvature H_f is constant. Motivated by this fact, we establish the following

Definition 4.23 *Let $x : \Sigma^n \looparrowright M_f^n \times_\alpha \mathbb{R}_1$ be a closed spacelike hypersurface having constant f-mean curvature. We say that $x : \Sigma^n \looparrowright M_f^n \times_\alpha \mathbb{R}_1$ is f-stable if $\dfrac{d^2}{ds^2}\mathcal{A}_f(0) \leq 0$, for all weighted volume-preserving variation $X : \Sigma^n \times (-\epsilon, \epsilon) \to M_f^n \times_\alpha \mathbb{R}_1$ of $x : \Sigma^n \looparrowright M_f^n \times_\alpha \mathbb{R}_1$.*

Let $x : \Sigma^n \looparrowright M_f^n \times_\alpha \mathbb{R}_1$ be a closed spacelike hypersurface as described in Definition 4.23. We consider the set

$$\mathcal{G} \;=\; \left\{\, u \in C^\infty(\Sigma^n) \;:\; \int_{\Sigma^n} u\, d\sigma = 0 \,\right\}. \tag{4.31}$$

Just as [9], we can establish the following criterion of f-stability.

Proposition 4.24 *With the notations considered above, $x : \Sigma^n \looparrowright M_f^n \times_\alpha \mathbb{R}_1$ is f-stable if and only if $\dfrac{d^2}{ds^2}\mathcal{J}_f(0)(u) \leq 0$ for all $u \in \mathcal{G}$.*

Proof. Suppose that $x : \Sigma^n \looparrowright M^n_f \times_\alpha \mathbb{R}_1$ is f-stable and consider $u \in \mathcal{G}$. From Remark 4.21, there is a weighted volume-preserving variation $X : (-\epsilon, \epsilon) \times \Sigma^n \to M^n_f \times_\alpha \mathbb{R}_1$ of $x : \Sigma^n \looparrowright M^n_f \times_\alpha \mathbb{R}_1$ whose variational field is $\left.\dfrac{\partial X}{\partial s}\right|_{s=0} = uN$. Then, $\dfrac{d^2}{ds^2}\mathcal{V}_f(0)(u) = 0$. Hence, from (4.25) and Definition 4.23 we obtain

$$\frac{d^2}{ds^2}\mathcal{J}_f(0)(u) \;=\; \frac{d^2}{ds^2}\mathcal{A}_f(0)(u) - \lambda\frac{d^2}{ds^2}\mathcal{V}_f(0)(u) \;=\; \frac{d^2}{ds^2}\mathcal{A}_f(0)(u) \;\leq\; 0$$

Conversely, suppose that $\dfrac{d^2}{ds^2}\mathcal{J}_f(0)(u) \leq 0$ for all $u \in \mathcal{G}$. Let $X : (-\epsilon, \epsilon) \times \Sigma^n \to M^n_f \times_\alpha \mathbb{R}_1$ be an weighted volume-preserving variation of $x : \Sigma^n \looparrowright M^n_f \times_\alpha \mathbb{R}_1$, and let uN be the normal component of the variation vector $\left.\dfrac{\partial X}{\partial s}\right|_{s=0}$. From Lemma 4.20,

$$\int_{\Sigma^n} u\, d\sigma \;=\; \frac{d}{ds}\mathcal{V}_f(0) \;=\; 0,$$

which implies that $u \in \mathcal{G}$. Therefore, from hypotheses,

$$0 \;\geq\; \frac{d^2}{ds^2}\mathcal{J}_f(0)(u) \;=\; \frac{d^2}{ds^2}\mathcal{A}_f(0)(u) - \lambda\underbrace{\frac{d^2}{ds^2}\mathcal{V}_f(0)(u)}_{0} \;=\; \frac{d^2}{ds^2}\mathcal{A}_f(0)(u),$$

which according to Definition 4.23 tells us that $x : \Sigma^n \looparrowright M^n_f \times_\alpha \mathbb{R}_1$ is f-stable. ■

The sought formula for the second variation of Jacobi functional $\mathcal{J}_f$ is given in the following

Proposition 4.25 *Let $x : \Sigma^n \looparrowright M^n_f \times_\alpha \mathbb{R}_1$ be a closed spacelike hypersurface having constant f-mean curvature H_f. If $X : (-\epsilon, \epsilon) \times \Sigma^n \to M^n_f \times_\alpha \mathbb{R}_1$ is a variation of $x : \Sigma^n \looparrowright M^n_f \times_\alpha \mathbb{R}_1$ then the second variation $\dfrac{d^2}{ds^2}\mathcal{J}_f(0)$ of the weighted Jacobi functional $\mathcal{J}_f$ is given by*

$$\frac{d^2}{ds^2}\mathcal{J}_f(0)(u) \;=\; \int_{\Sigma^n} u\,\mathfrak{L}_f(u)\, d\sigma, \tag{4.32}$$

for any $u \in C^\infty(\partial\Omega)$, where $\mathcal{L}_f : C^\infty(\Sigma^n) \to C^\infty(\Sigma^n)$ is the weighted Jacobi operator given by

$$\mathfrak{L}_f \;=\; \Delta_f - \left\{\widetilde{\mathrm{Ric}}_f(N^*, N^*) - \frac{1}{\alpha}\widetilde{\mathrm{Hess}}\,\alpha(N^*, N^*) + \Theta^2\frac{\widetilde{\Delta}_f(\alpha)}{\alpha^3} + |A|^2\right\}. \tag{4.33}$$

Here, Δ_f and $\widetilde{\Delta}_f$ represent the f-Laplacians on Σ^n and M^n, respectively, Θ be the angle function defined in (1.42), N is the future-pointing Gauss map of $x : \Sigma^n \looparrowright M^n_f \times_\alpha \mathbb{R}_1$, $\widetilde{\mathrm{Ric}}_f$ and $\widetilde{\mathrm{Hess}}$ are the Bakry-Émery-Ricci tensor and the Hessian operator on M^n, $|A|^2$ represent the square of the norm of the shape operator A of $x : \Sigma^n \looparrowright M^n_f \times_\alpha \mathbb{R}_1$ and N^* is the projection of N on the tangent bundle of M^n.

Proof. Since H_f is constant, from (4.29) and (4.28) we have that

$$\frac{d^2}{ds^2}\mathcal{J}_f(0)(u_0) = n\int_{\Sigma^n}\left(\frac{\partial\,(H_f)_s}{\partial s}\Big|_{s=0}\right)u_0\,d\sigma + n\int_{\Sigma^n}\left(\underbrace{H_f-\overline{\mathcal{H}}}_{0}\right)\frac{\partial}{\partial s}\left(u_s\,d\sigma_s\right)\Big|_{s=0},$$

where u_s is the smooth function given in (4.24).

On the other hand, reasoning as in the proof of equation (3.5) of [23], we obtain

$$n\,\frac{\partial\,(H_f)_s}{\partial s}\Big|_{s=0} = \Delta_f\,(u_0) - \left\{\overline{\mathrm{Ric}}_f(N,N) + |A^2|\right\}u_0.$$

Hence,

$$\frac{d^2}{ds^2}\mathcal{J}_f(0)(u_0) = \int_{\Sigma^n}\left\{\Delta_f(u_0) - \left\{\overline{\mathrm{Ric}}_f(N,N) + |A|^2\right\}u_0\right\}u_0\,d\sigma. \tag{4.34}$$

Now, from equations (4.16) and (4.34) we obtain

$$\frac{d^2}{ds^2}\mathcal{J}_f(0)(u_0) = \int_{\Sigma^n} u_0\,\mathfrak{L}_f(u_0)\,d\sigma, \tag{4.35}$$

where $\mathfrak{L}_f$ is given in (4.33). To finish the proof, we observe that the expression (4.35) depends only on the hypersurface Σ^n and on the function $u_0 \in C^\infty(\Sigma^n)$. ■

To show our next result, let us remember that the *eigenvalue problem* for the drift Laplacian Δ_f on a closed Riemannian manifold Σ^n is the determination of the existence or not of nontrivial solutions (that is, not identically zero) $u \in C^\infty(\Sigma)$ for the partial differential equation

$$\Delta_f(u) + \xi\,u = 0$$

on Σ^n. In this case, the corresponding function u is an *eigenfunction* associated with the *eigenvalue* ξ. By the spectral theorem we know that all the eigenvalues of Δ_f are determined by a sequence of eigenvalues $\{\xi_j\}_{j=0}^{+\infty}$ satisfying

$$0 = \xi_0 < \xi_1 \le < \xi_2 \le \cdots \le \xi_j \le \xi_{j+1} \le \cdots,$$

repeated according to their multiplicity, and

$$\lim_{j\to+\infty}\xi_j = +\infty$$

(see, for instance, Section 1 of [12]). Moreover, the variational characterization of ξ_1 gives

$$\xi_1 = \min_{u\in\mathcal{G}\setminus\{0\}}\frac{-\displaystyle\int_{\Sigma^n} u\Delta_f(u)\,d\sigma}{\displaystyle\int_{\Sigma^n} u^2\,d\sigma}, \tag{4.36}$$

where $\mathcal{G}$ is defined in (4.31).

We can now present our characterization of f-stability concerning closed spacelike hypersurfaces immersed in a weighted standard static spacetime.

Theorem 4.26 *Let $x : \Sigma^n \looparrowright M^n_f \times_\alpha \mathbb{R}_1$ be a closed spacelike hypersurface with constant f-mean curvature. Suppose that*

$$\xi \;=\; -\widetilde{\mathrm{Ric}}_f(N^*, N^*) + \frac{1}{\alpha}\,\widetilde{\mathrm{Hess}}\,\alpha(N^*, N^*) - \Theta^2\,\frac{\widetilde{\Delta}_f(\alpha)}{\alpha^3} - |A|^2$$

is a nonzero constant on Σ^n, where Δ_f and $\widetilde{\Delta}_f$ represent the f-Laplacians on Σ^n and M^n, respectively, Θ be the angle function defined in (1.42), N is the future-pointing Gauss map of $x : \Sigma^n \looparrowright M^n_f \times_\alpha \mathbb{R}_1$, $\widetilde{\mathrm{Ric}}_f$ and $\widetilde{\mathrm{Hess}}$ are the Bakry-Émery-Ricci tensor and the Hessian operator on M^n, $|A|^2$ represent the square of the norm of the shape operator A of $x : \Sigma^n \looparrowright M^n_f \times_\alpha \mathbb{R}_1$ and N^* is the projection of N on the tangent bundle of M^n. Then $x : \Sigma^n \looparrowright M^n_f \times_\alpha \mathbb{R}_1$ is f-stable if and only if ξ is the first nonzero eigenvalue of drift Laplacian Δ_f on Σ^n.

Proof. Initially, since the f-mean curvature of $x : \Sigma^n \looparrowright M^n_f \times_\alpha \mathbb{R}_1$ and ξ are constant on Σ^n, from Proposition 4.12 we can see that ξ belongs to the sequence of eigenvalues $\{\xi_j\}_{j=0}^{+\infty}$ of the drift Laplacian Δ_f on Σ^n.

If $\xi = \xi_1$, then from (4.32), (4.33) and (4.36) we obtain

$$\frac{d^2}{ds^2}\,\mathcal{J}_f(0)(u) \;=\; \int_{\Sigma^n} \{u\Delta_f(u) + \xi\, u^2\}\, d\sigma \;\le\; (-\xi + \xi)\int_{\Sigma^n} u^2\, d\sigma \;=\; 0$$

for any $u \in \mathcal{G}$ and, according to Proposition 4.24, $x : \Sigma^n \looparrowright M^n_f \times_\alpha \mathbb{R}_1$ is f-stable.

Conversely, suppose that $x : \Sigma^n \looparrowright M^n_f \times_\alpha \mathbb{R}_1$ is f-stable, so that $\dfrac{d^2}{ds^2}\,\mathcal{J}_f(0)(u) \le 0$ for all $u \in \mathcal{G}$. Let u be an eigenfunction associated to the first nonzero eigenvalue ξ_1 of Δ_f. Consequently, by (4.32) and (4.33) we get

$$0 \;\ge\; \frac{d^2}{ds^2}\,\mathcal{J}_f(0)(u) \;=\; (-\xi_1 + \xi)\int_{\Sigma^n} u^2\, d\sigma.$$

Therefore, since $\xi_1 \le \xi$, we must have $\xi_1 = \xi$. ■

Chapter 5

Uniqueness for the weighted mean curvature equation in weighted standad static spacetimes

Our aim here is to obtain uniqueness results concerning the mean curvature equation in a weighted standard static spacetime $M^n_f \times_\alpha \mathbb{R}_1$ having warping function α and whose weight function f does not depend on the parameter $t \in \mathbb{R}$. For this, we establish a f-parabolicity criterion in order to study the rigidity of spacelike hypersurfaces immersed in $M^n_f \times_\alpha \mathbb{R}_1$ and, in particular, entire Killing graphs constructed over the Riemannian base M^n. Applications to weighted standard static spacetimes of the type $\mathbb{G}^n \times_\alpha \mathbb{R}_1$, where $\mathbb{G}^n$ denotes for the so-called Gaussian space, are also given. The results presented in this chapter are part of [34].

5.1 A f-parabolicity criterion for spacelike hypersurfaces in $(M^n \times_\alpha \mathbb{R}_1)_f$

In [65], Romero, Rubio and Salamanca investigated the parabolicity of complete spacelike hypersurfaces in GRW spacetimes whose Riemannian fiber has a parabolic universal Riemannian covering. In this setting, they were able to guarantee the parabolicity of complete spacelike hypersurfaces, under suitable boundedness assumptions on the warping function and on the hyperbolic angle function of these hypersurfaces. Our aim in this section is just, following the ideas of [41], to obtain an extension of this parabolicity criterion to the context of standard static spacetimes.

Taking into account the digression presented at the beginning of Section 3.1, from now on we will denote by $\widetilde{M}$ the universal Riemannian covering of base M^n with projection $\widetilde{\pi} : \widetilde{M} \to M^n$ and $\widetilde{f}$ will denote the composition $f \circ \widetilde{\pi}$. In this setting, a standard static spacetime $(M^n \times_\alpha \mathbb{R}_1)_f$ will be said *spattialy f-parabolic* if the universal Riemannian covering $\widetilde{M}$ of its base M^n is $\widetilde{f}$-parabolic.

Proposition 5.1 *Let $(M^n \times_\alpha \mathbb{R}_1)_f$ be a weighted standard static spacetimes which is spatially $\widetilde{f}$-parabolic. If $x : \Sigma^n \looparrowright \overline{M}^{n+1}$ is a spacelike hypersurface such that the function $\eta := \frac{\Theta}{\alpha}$ is bounded on it, then Σ^n is f-parabolic.*

Proof. From Lemma 3.2 we have that

(i) f-parabolicity is invariant under a quasi-isometry;

(ii) if the universal Riemannian covering $\widetilde{\Sigma}$ of Σ^n is $(f \circ \pi_\Sigma)$-parabolic, then Σ^n is also f-parabolic.

Denoting $\pi = \pi_M \circ x : \Sigma^n \to M^n$, for any tangent vector $v \in T\Sigma$ we have

$$\langle v, v \rangle = \langle \pi_* v, \pi_* v \rangle_M - \alpha^2 \langle h_* v, h_* v \rangle_{\mathbb{R}} \leq c \langle \pi_* v, \pi_* v \rangle_M,$$

where $c = \sup_\Sigma \eta^2 \geq 1$. In particular, by previous inequality we see that $\pi_{*,p} : T_p\Sigma \to T_{\pi(p)}M$ is a isomorphism for every $p \in \Sigma^n$. Then, from inverse function theorem we get that π is a local diffeomorphism and applying Lemma 7.3.3 of [43] (see also Lemma 8.8.1 of [56]) we can to conclude that π is a covering map and that M^n is complete.

On the other hand, using the Cauchy-Schwartz inequality we see that

$$\langle \nabla h, v \rangle^2 \leq \langle \nabla h, \nabla h \rangle \langle v, v \rangle$$

and, consequently, since $h_* v = dh(v) = \langle \nabla h, v \rangle$, we have

$$\begin{aligned} \langle v, v \rangle &= \langle \pi_* v, \pi_* v \rangle_M - \alpha^2 \langle h_* v, h_* v \rangle_{\mathbb{R}} \\ &= \langle \pi_* v, \pi_* v \rangle_M - \alpha^2 \langle \nabla h, v \rangle^2 \\ &\geq \langle \pi_* v, \pi_* v \rangle_M - \alpha^2 |\nabla h|^2 \langle v, v \rangle, \end{aligned}$$

that is,

$$\langle v, v \rangle \left(1 + \alpha^2 |\nabla h|^2\right) \geq \langle \pi_* v, \pi_* v \rangle_M.$$

By definition of the function η and from (1.45) we get

$$\langle v, v \rangle \geq \frac{1}{\eta^2} \langle \pi_* v, \pi_* v \rangle_M$$

From our hypothesis we conclude that

$$c^{-1} \langle \pi_* v, \pi_* v \rangle_M \leq \langle v, v \rangle \leq c \langle \pi_* v, \pi_* v \rangle_M. \tag{5.1}$$

So, let $\widetilde{\Sigma}$ be the universal Riemannian covering of Σ^n with projection $\pi_\Sigma : \widetilde{\Sigma} \to \Sigma^n$. Then, the map $\pi_0 = \pi \circ \pi_\Sigma : \widetilde{\Sigma} \to M^n$ is a covering map. Now, if $\widetilde{M}$ is the universal Riemannian covering of M^n with projection $\widetilde{\pi} : \widetilde{M} \to M^n$, then there exists a diffeomorphism $\varphi : \widetilde{\Sigma} \to \widetilde{M}$ such that $\widetilde{\pi} \circ \varphi = \pi_0$. Moreover, φ is a quasi-isometry. Indeed, if $v \in T\widetilde{\Sigma}$, we have from (5.1) that

$$\begin{aligned}
\langle \varphi_* v, \varphi_* v \rangle_{\widetilde{M}} &= \langle \widetilde{\pi}_*(\varphi_* v), \widetilde{\pi}_*(\varphi_* v) \rangle_M \\
&= \langle (\pi_0)_* v, (\pi_0)_* v \rangle_M \\
&= \langle \pi_*((\pi_\Sigma)_* v), \pi_*((\pi_\Sigma)_* v) \rangle_M \\
&\leq c \langle (\pi_\Sigma)_* v, (\pi_\Sigma)_* v \rangle_\Sigma \\
&= c \langle v, v \rangle_{\widetilde{\Sigma}}.
\end{aligned}$$

Analogously, we obtain

$$\langle \varphi_* v, \varphi_* v \rangle_{\widetilde{M}} \geq c^{-1} \langle v, v \rangle_{\widetilde{\Sigma}}.$$

Therefore, since the universal Riemannian covering of M^n is $\widetilde{f}$-parabolic, it follows that the universal Riemannian covering of Σ^n is $(f \circ \pi_\Sigma)$-parabolic and, hence, Σ^n must be also f-parabolic. ■

5.2 Rigidity results for spacelike hypersurfaces in $M_f^n \times_\alpha \mathbb{R}_1$

In this section, we will apply the Proposition 5.1 in order to obtain rigidity results for spacelike hypersurfaces in $M_f^n \times_\alpha \mathbb{R}_1$. Some of these results are rereadings of theorems presented in Chapter 4, for which the hypothesis of f-parabolicity is replaced in part by restrictions on the angle function and the warped function, which, in addition to having value in itself, will be important to establish the uniqueness results for the weighted mean curvature equation in the next section.

Theorem 5.2 *Let $M_f^n \times_\alpha \mathbb{R}_1$ be a weighted standard static spacetimes which is spatially $\widetilde{f}$-parabolic. Suppose that $\widetilde{\mathrm{Ric}}_f \geq 0$, the warping function α is convex and $\langle \widetilde{\nabla} f, \widetilde{\nabla} \alpha \rangle \leq 0$. Let $x : \Sigma^n \looparrowright \overline{M}^{n+1}$ be an immersed spacelike hypersurface with constant f-mean curvature H_f such that its angle function Θ is bounded and $\inf_\Sigma \alpha > 0$. Then, Σ^n is totally geodesic and α is a positive constant. In addition, if $\widetilde{\mathrm{Ric}}_f$ is positive at some point $p_0 \in \Sigma^n$, then Σ^n is contained in a slice $M^n \times \{t_0\}$, for some $t_0 \in \mathbb{R}$.*

Proof. It follows from Theorem 4.17 and Proposition 5.1. ■

In the next result, we treat the case where $\widetilde{\mathrm{Ric}}_f$ is not necessarily nonnegative.

Theorem 5.3 *Let $M_f^n \times_\alpha \mathbb{R}_1$ be a weighted standard static spacetimes which is spatially $\widetilde{f}$-parabolic. Suppose that $\widetilde{\mathrm{Ric}}_f \geq -\kappa$, for some constant $\kappa > 0$, and that α is a convex* *warping function such that* $\langle \widetilde{\nabla} f, \widetilde{\nabla} \alpha \rangle \leq 0$. *Let* $x : \Sigma^n \hookrightarrow M^n_f \times_\alpha \mathbb{R}_1$ *be an immersed spacelike hypersurface with constant* f*-mean curvature, bounded angle function* Θ *and such that* $\inf_\Sigma \alpha > 0$. *If the height function* h *satisfies*

$$|\nabla h|^2 \;\leq\; \frac{c}{\kappa \alpha^2}\,|A|^2, \tag{5.2}$$

for some constant $c \in (0,1)$, *then* Σ^n *is contained in a slice* $M^n \times \{t_0\}$, *for some* $t_0 \in \mathbb{R}$.

Proof. It follows from Theorem 4.13 and Proposition 5.1. ∎

In what follows, we will deal with specific weight functions that will be defined in terms of the warping function α. In the next theorem, the weighted mean curvature $H_{\log \alpha^2}$ of the spacelike hypersurface is not supposed to be constant. Indeed, we just assume a certain control on the sign of $H_{\log \alpha^2}$.

Theorem 5.4 *Let* $M^n_{\log \alpha^2} \times_\alpha \mathbb{R}_1$ *be a weighted standard static spacetimes which is spatially* $\log \widetilde{\alpha}^{\,2}$*-parabolic. Let* $x : \Sigma^n \hookrightarrow M^n_{\log \alpha^2} \times_\alpha \mathbb{R}_1$ *be an immersed spacelike hypersurface such that* η *is bounded. Suppose that the* $\log \alpha^2$*-mean curvature* $H_{\log \alpha^2}$ *and the function* $\langle \nabla \alpha, \nabla h \rangle$ *have opposite signs. If* Σ^n *lies in a slab, then* Σ^n *is contained in a slice* $M^n \times \{t_0\}$, *for some* $t_0 \in \mathbb{R}$.

Proof.

By (1.15) and from Proposition 4.1, we have that

$$\begin{aligned}
\Delta_{\log \alpha^2} h &= -n\alpha^{-2}\Theta\, H_{\log \alpha^2} - \langle \nabla \log \alpha^2, \nabla h \rangle \\
&= -n\alpha^{-2}\Theta\, H_{\log \alpha^2} - \frac{2}{\alpha}\langle \nabla \alpha, \nabla h \rangle.
\end{aligned}$$

Taking into account that $H_{\log \alpha^2}$ and $\langle \nabla \alpha, \nabla h \rangle$ have opposite signs, we conclude that $\Delta_{\log \alpha^2} h$ does not change sing. Therefore, since Proposition 5.1 guarantees the $\log \alpha^2$-parabolicity of Σ^n, h must be constant and, consequently, Σ^n is contained in a slice $M^n \times \{t_0\}$, for some $t_0 \in \mathbb{R}$. ∎

From Theorem 5.4 we also have the following

Corollary 5.5 *Let* $M^n_{\log \alpha^2} \times_\alpha \mathbb{R}_1$ *be a weighted standard static spacetimes which is spatially* $\log \widetilde{\alpha}^{\,2}$*-parabolic. Let* $x : \Sigma^n \hookrightarrow \overline{M}^{n+1}$ *be a* $\log \alpha^2$*-maximal spacelike hypersurface, contained in a slab, such that* η *is bounded. If the function* $\langle \nabla \alpha, \nabla h \rangle$ *does not change sign, then* Σ^n *is contained in a slice* $M^n \times \{t_0\}$, *for some* $t_0 \in \mathbb{R}$.

Proceeding, we also get the following rigidity result:

Theorem 5.6 *Let* $M^n_{\log \alpha^{-2}} \times_\alpha \mathbb{R}_1$ *be a weighted standard static spacetimes which is spatially* $\log \widetilde{\alpha}^{\,-2}$*-parabolic. Let* $x : \Sigma^n \hookrightarrow M^n_{\log \alpha^{-2}} \times_\alpha \mathbb{R}_1$ *be a maximal spacelike hypersurface such that* η *is bounded and* $\inf_\Sigma \alpha > 0$. *If* $\mathrm{Ric}_{\log \alpha^{-2}} \geq \kappa$, *for some constant* $\kappa > 0$, *then* Σ^n *is contained in a slice* $M^n \times \{t_0\}$, *for some* $t_0 \in \mathbb{R}$.

Proof. Firstly, observe that, reasoning as in the proof of Proposition 4.1, we obtain

$$\begin{aligned}\Delta h &= \operatorname{div}(\nabla h) = \alpha^2\Big\langle \nabla\alpha^{-2}, \nabla h\Big\rangle - n\alpha^{-2}H\Theta \\ &= \Big\langle \nabla \log\alpha^{-2}, \nabla h\Big\rangle - n\alpha^{-2}H\Theta.\end{aligned}$$

Therefore, using (1.15), we get

$$\Delta_{\log\alpha^{-2}} h = -n\alpha^{-2}H\Theta. \tag{5.3}$$

Now, from Bochner's formula (see page 378 of [68]) we have that

$$\frac{1}{2}\Delta_{\log\alpha^{-2}}|\nabla h|^2 = |\mathrm{Hess}\, h|^2 + \mathrm{Ric}_{\log\alpha^{-2}}(\nabla h, \nabla h) + \langle \nabla\Delta_{\log\alpha^{-2}} h, \nabla h\rangle. \tag{5.4}$$

Consequently, taking into account our restriction on $\mathrm{Ric}_{\log\alpha^{-2}}$ and the assumption that Σ^n is maximal, from (5.3) and (5.4), we obtain that

$$\frac{1}{2}\Delta_{\log\alpha^{-2}}|\nabla h|^2 \geq \mathrm{Ric}_{\log\alpha^{-2}}(\nabla h, \nabla h) \geq \kappa|\nabla h|^2 \geq 0. \tag{5.5}$$

On the other hand, Proposition 5.1 guarantees that Σ^n is $\log\alpha^{-2}$-parabolic. Since, from (1.45), $\inf_\Sigma \alpha > 0$ implies in the boundedness of $|\nabla h|$ and, consequently, in the boundedness of $|\nabla h|^2$, we conclude from $\log\alpha^{-2}$- parabolicity of Σ^nthat $|\nabla h|^2$ is constant, and then $\Delta_{\log\alpha^2}|\nabla h|^2 = 0$. Returning to (5.5), we obtain that $|\nabla h| = 0$ and Σ^n is contained in a slice. ■

5.3 Entire Killing graphs and the mean curvature equation in $M^n_f \times_\alpha \mathbb{R}_1$

Let $\Sigma(z)$ be a entire Killing graph as decribed in the Subsection 1.4.4 . For each vector field X tangent to M^n, the shape operator A of $\Sigma(z)$ with respect to N is given by

$$\begin{aligned}AX &= -\frac{\alpha}{(1-\alpha^2|Dz|^2_M)^{1/2}}D_X Dz - \frac{\alpha^3\langle D_X Dz, Dz\rangle}{(1-\alpha^2|Dz|^2_M)^{3/2}}Dz - \frac{\alpha^2\langle D\alpha, X\rangle|Dz|^2_M}{(1-\alpha^2|Dz|^2_M)^{3/2}}Dz \\ &- \frac{\langle D\alpha, X\rangle}{(1-\alpha^2|Dz|^2_M)^{1/2}}Dz - \frac{\langle Dz, X\rangle}{(1-\alpha^2|Dz|^2_M)^{1/2}}D\alpha,\end{aligned} \tag{5.6}$$

where D denotes the Levi-Civita connections in M^n.

So, it follows from (5.6) that the mean curvature H_z of a spacelike entire Killing graph $\Sigma(z)$ is given by

$$nH(z) = \mathrm{Div}\left(\frac{\alpha Dz}{(1+\alpha^2|Dz|^2_M)^{1/2}}\right) + \frac{\langle Dz, D\alpha\rangle}{(1+\alpha^2|Dz|^2_M)^{1/2}},$$

where Div stands for the divergence operator on M^n with respect to the metric $\langle\,\cdot,\cdot\rangle_M$. A direct computation shows that the f-mean curvature is given by

$$n(H_z)_f = \mathrm{Div}_f\left(\frac{\alpha Dz}{(1-\alpha^2|Dz|_M^2)^{1/2}}\right) + \frac{\langle Dz, D\alpha\rangle}{(1-\alpha^2|Dz|_M^2)^{1/2}}.$$

From the previous digression, an entire Killing graph $\Sigma(z)$ is spacelike with constant f-mean curvature C if, and only if, the function $z \in C^\infty(M)$ satisfies the following elliptic partial differential equation of f-divergence form

$$\begin{cases} \mathrm{Div}_f\left(\dfrac{\alpha Dz}{(1-\alpha^2|Dz|_M^2)^{1/2}}\right) + \dfrac{\langle Dz, D\alpha\rangle}{(1-\alpha^2|Dz|_M^2)^{1/2}} = C, \quad \text{in} \quad M^n \\ \\ \alpha^2|Dz|_M^2 < 1. \end{cases} \tag{5.7}$$

In what follows, we will use the theorems obtained in the previous section, on entire Killing graph context, to obtain uniqueness results for equations of the type (5.7).We start applying the Theorem 5.2 to get the following:

Theorem 5.7 *Let $M_f^n \times_\alpha \mathbb{R}_1$ be a weighted standard static spacetimes which is spatially $\widetilde{f}$-parabolic with convex warping function α, $\langle \widetilde{\nabla} f, \widetilde{\nabla}\alpha\rangle \le 0$ and $\widetilde{\mathrm{Ric}}_f \ge 0$. If the entire Killing graph $\Sigma(z)$ associated to $z \in C^\infty(M)$ is such that $\alpha|_{\Sigma(z)}$ is bounded and $\widetilde{\mathrm{Ric}}_f$ is positive at some point $p_0 \in \Sigma(z)$, the only solutions of the problem*

$$\begin{cases} \mathrm{Div}_f\left(\dfrac{\alpha Dz}{(1-\alpha^2|Dz|_M^2)^{1/2}}\right) + \dfrac{\langle Dz, D\alpha\rangle}{(1-\alpha^2|Dz|_M^2)^{1/2}} = C, \qquad z \in C^\infty(M) \\ \\ \sup_{\Sigma(z)} (\alpha^2|Dz|_M^2) < 1, \end{cases}$$

are the constant ones.

Proof. Since we are supposing that $\sup \alpha^2|Dz|_M^2 < 1$, from (1.48), the boundness of $\alpha|_{\Sigma(z)}$ is equivalent to the boundness of Θ. Furthermore, we observe that the condition $\sup \alpha^2|Dz|_M^2 < 1$ also implies the boundness of η. Indeed, using (1.48) again, we have that

$$\eta = \frac{1}{(1-\alpha^2|Dz|_M^2)^{1/2}}.$$

Hence, we can disregard the hypothesis $\inf_{\Sigma(z)} \alpha > 0$ in the Theorem 5.2 to obtain the present result. ■

Concerned with the weighted product space $\mathbb{G}^n \times \mathbb{R}_1$, where $\mathbb{G}^n$ is the Gaussian space, An et al extended the classical Bernstein's theorem [15] showing that the only weighted minimal graphs $\Sigma^n(z)$ of functions $z(y_2, \cdots, y_{n+1}) = y_1$ over $\mathbb{G}^n$, with $\sup_{\Sigma(z)} |Dz|_{\mathbb{G}} < 1$, are the hyperplanes $y_1 = \text{constant}$ (see Theorem 4 of [5]).

Taking into account this previous digression, from Theorem 5.7 we obtain an extension of Theorem 4 of [5].

Corollary 5.8 *Consider the weighted standard static spacetime $\mathbb{G}^n \times_\alpha \mathbb{R}_1$, where $\mathbb{G}^n$ is the Gaussian space and the warping function α is convex with $\langle \widetilde{\nabla} f, \widetilde{\nabla}\alpha\rangle \leq 0$. If the entire Killing graph $\Sigma(z)$ associated to $z \in C^\infty(\mathbb{G})$ is such that $\alpha|_{\Sigma(z)}$ is bounded, the only solutions of the problem*

$$\begin{cases} \mathrm{Div}_f\left(\dfrac{\alpha Dz}{(1-\alpha^2|Dz|^2_{\mathbb{G}})^{1/2}}\right) + \dfrac{\langle Dz, D\alpha\rangle}{(1-\alpha^2|Dz|^2_{\mathbb{G}})^{1/2}} = C, \qquad z \in C^\infty(\mathbb{G}) \\ \\ \sup_{\Sigma(z)} (\alpha^2|Dz|^2_{\mathbb{G}}) < 1, \end{cases}$$

are the constant ones.

Proof. We note that, since $\mathrm{Vol}_f(\mathbb{G}^n) = 1$, Remark 3 of [54] guarantees that $\mathbb{G}^n$ is f-parabolic. Moreover, with a straightforward computation, we get that $\widetilde{\mathrm{Ric}}_f = 1$. Therefore, since $\mathbb{G}^n$ is also simply connected, the result follows from Theorem 5.7. ■

The next result is an application of Theorem 5.3.

Theorem 5.9 *Let $M^n_f \times_\alpha \mathbb{R}_1$ be a weighted standard static spacetime which is spatially $\widetilde{f}$-parabolic with convex warping function α, $\langle \widetilde{\nabla} f, \widetilde{\nabla}\alpha\rangle \leq 0$ and $\widetilde{\mathrm{Ric}}_f \geq -\kappa$, for some constant $\kappa > 0$. If the entire Killing graph $\Sigma(z)$ associated to z is such that $\alpha|_{\Sigma(z)}$ is bounded and $c \in (0,1)$ is a constant, the only solutions of the problem*

$$\begin{cases} \mathrm{Div}_f\left(\dfrac{\alpha Dz}{(1-\alpha^2|Dz|^2_{M})^{1/2}}\right) + \dfrac{\langle Dz, D\alpha\rangle}{(1-\alpha^2|Dz|^2_{M})^{1/2}} = C, \qquad z \in C^\infty(M) \\ \\ \sup_{\Sigma(z)} (\alpha^2|Dz|^2_{M}) < \dfrac{c|A|^2}{c|A|^2+\kappa}, \end{cases}$$

are the constant ones.

Proof. From (1.47), we have that

$$N^* = N - N^\perp = \frac{\alpha\Psi_*(Dz)}{(1-\alpha^2|Dz|^2_M)^{1/2}}, \tag{5.8}$$

and this equation give us that

$$|N^*|^2_M = \frac{\alpha^2|Dz|^2_M}{1-\alpha^2|Dz|^2_M}. \tag{5.9}$$

The equations (1.45) and (5.9) give us the following relation:

$$|\nabla h|^2 = \frac{|Dz|^2_M}{1-\alpha^2|Dz|^2_M}. \tag{5.10}$$

Now, using (5.10) we conclude that the hypothesis (5.2) is equivalent to

$$\alpha^2|Dz|^2_M \leq \frac{c|A|^2}{c|A|^2+\kappa}.$$

Futhermore, since $\kappa > 0$, we have that that $\dfrac{c|A|^2}{c|A|^2+\kappa} \leq 1$. Hence, the result follows from Theorem 5.3. ■

Reasoning as in the Corollary 5.8, we have:

Corollary 5.10 *Consider the weighted standard static spacetime $\mathbb{G}^n \times_\alpha \mathbb{R}_1$, where $\mathbb{G}^n$ is the Gaussian space and the warping function α is convex with $\langle \widetilde{\nabla} f, \widetilde{\nabla}\alpha\rangle \leq 0$. If the entire Killing graph $\Sigma(z)$ associated to $z \in C^\infty(\mathbb{G})$ is such that $\alpha|_{\Sigma(z)}$ is bounded, then, for any constants $k > 0$ and $c \in (0,1)$, the only solutions of the problem*

$$\begin{cases} \mathrm{Div}_f\left(\dfrac{\alpha Dz}{(1-\alpha^2|Dz|^2_{\mathbb{G}})^{1/2}}\right) + \dfrac{\langle Dz, D\alpha\rangle}{(1-\alpha^2|Dz|^2_{\mathbb{G}})^{1/2}} = C, \qquad z \in C^\infty(\mathbb{G}) \\ \\ \sup_{\Sigma(z)}(\alpha^2|Dz|^2_{\mathbb{G}}) < \dfrac{c|A|^2}{c|A|^2+\kappa}, \end{cases}$$

are the constant ones.

From Theorem 5.4, we obtain the following:

Theorem 5.11 *Let $M^n_{\log\alpha^2} \times_\alpha \mathbb{R}_1$ be a weighted standard static spacetimes which is spatially $\log \widetilde{\alpha}^{\,2}$-parabolic. If the entire Killing graph associate to z is such that $\langle \nabla\alpha, \Psi_*(Dz)\rangle$ does not change sign, then the only bounded solutions of the problem*

$$\begin{cases} \mathrm{Div}_{\log\alpha^2}\left(\dfrac{\alpha Dz}{(1-\alpha^2|Dz|^2_{M})^{1/2}}\right) + \dfrac{\langle Dz, D\alpha\rangle}{(1-\alpha^2|Dz|^2_{M})^{1/2}} = C, \qquad z \in C^\infty(M) \\ \\ \sup_{\Sigma(z)}(\alpha^2|Dz|^2_{M}) < 1. \end{cases}$$

are the constant ones.

Proof. Firstly, observe that

$$\begin{aligned} \langle \nabla\alpha, \nabla N\rangle &= \nabla h(\alpha) = -\frac{1}{\alpha^2}Y^\top(\alpha) = -\frac{1}{\alpha^2}Y^\top\big((-\langle Y,Y\rangle)^{\frac{1}{2}}\big) \\ &= -\frac{1}{\alpha^2}\Big(\frac{1}{2}(-\langle Y,Y\rangle)^{\frac{1}{2}}Y^\top\langle Y,Y\rangle\Big) = -\frac{1}{2\alpha^3}Y^\top\langle Y,Y\rangle\big) \\ &= -\frac{1}{\alpha^3}\langle \overline{\nabla}_{Y^\top}Y, Y\rangle = -\frac{1}{\alpha^3}\langle \overline{\nabla}_{Y+\Theta N}Y, Y\rangle \\ &= -\frac{1}{\alpha^3}\big(\underbrace{\langle \overline{\nabla}_Y Y, Y\rangle}_{0} + \langle \overline{\nabla}_{\Theta N}Y, Y\rangle\big) = -\frac{1}{\alpha^3}\langle \overline{\nabla}_{\Theta N}Y, Y\rangle \qquad (5.11) \\ &= -\frac{\Theta}{\alpha^3}\langle \overline{\nabla}_N Y, Y\rangle = -\frac{\Theta}{2\alpha^3}N\langle Y,Y\rangle = -\frac{\Theta}{2\alpha^3}N(\alpha^2) \\ &= -\frac{\Theta}{2\alpha^3} - 2\alpha N^*(\alpha) = \frac{\Theta}{\alpha^2}\langle \overline{\nabla}\alpha, N^*\rangle. \end{aligned}$$

On the other hand, in (5.8), we have that

$$N^* = N - N^\perp = \frac{\alpha \Psi_*(Dz)}{(1 - \alpha^2 |Dz|^2_M)^{1/2}}.$$

Hence, from (5.11) and (5.8) we obtain

$$\langle \nabla \alpha, \nabla N \rangle = \frac{\Theta}{\alpha} \langle \overline{\nabla} \alpha, \frac{\alpha \Psi_*(Dz)}{(1 - \alpha^2 |Dz|^2_M)^{1/2}} \rangle = \frac{\Theta}{\alpha (1 - \alpha^2 |Dz|^2_M)^{1/2}} \langle \overline{\nabla} \alpha, \Psi_*(Dz) \rangle.$$

Therefore, $\langle \nabla \alpha, \nabla N \rangle$ do not change of sign if and only if $\langle \overline{\nabla} \alpha, \Psi_*(Dz) \rangle$ do not change of sign and the result follows from Corollary 5.5. ■

Taking

$$\alpha = \left(e^{\frac{|y|^2}{2} + \log (2\pi)^{\frac{n}{2}}} \right)^{\frac{1}{2}} \tag{5.12}$$

in Theorem 5.11, we obtain the following consequence:

Corollary 5.12 *Consider the weighted standard static spacetime $\mathbb{G}^n \times_\alpha \mathbb{R}_1$, where $\mathbb{G}^n$ is the Gaussian space and α is defined in (5.12). If the entire Killing graph associate to z is such that $\langle \nabla \alpha, \Psi_*(Dz) \rangle$ does not change sign, then the only bounded solutions of the problem*

$$\begin{cases} \mathrm{Div}_f \left(\dfrac{\alpha Dz}{(1 - \alpha^2 |Dz|^2_{\mathbb{G}})^{1/2}} \right) + \dfrac{\langle Dz, D\alpha \rangle}{(1 - \alpha^2 |Dz|^2_{\mathbb{G}})^{1/2}} = C, \qquad z \in C^\infty(\mathbb{G}) \\ \\ \sup_{\Sigma(z)} (\alpha^2 |Dz|^2_{\mathbb{G}}) < 1, \end{cases}$$

are the constant ones.

Applying the Theorem 5.6 we obtain the following result:

Theorem 5.13 *Let $M^n_{\log \alpha^{-2}} \times_\alpha \mathbb{R}_1$ be a weighted standard static spacetimes which is spatially $\log \widetilde{\alpha}^{\,-2}$-parabolic. If the entire Killing graph associate to z is such that $|Dz|^2_M$ is bounded and $\mathrm{Ric}_{\log \alpha^{-2}} \geq \kappa$, for some constant $\kappa > 0$, then the only bounded solutions of the problem*

$$\begin{cases} \mathrm{Div} \left(\dfrac{\alpha Dz}{(1 - \alpha^2 |Dz|^2_M)^{1/2}} \right) + \dfrac{\langle Dz, D\alpha \rangle}{(1 - \alpha^2 |Dz|^2_M)^{1/2}} = 0, \qquad z \in C^\infty(M) \\ \\ \sup_{\Sigma(z)} (\alpha^2 |Dz|^2_M) < 1, \end{cases} \tag{5.13}$$

are the constant ones.

Proof. We observe that if $z \in C^\infty(M)$ is solution of problem (5.13), then the entire Killing graph $\Sigma(z)$ is spacelike and maximal. Moreover using (5.10), we note that the boundness of $|\nabla h|^2$ follows from the boundness of $|Dz|_M^2$. Then, the result follows from Theorem 5.6. ■

Finally, considering

$$\alpha = \left(e^{\frac{|y|^2}{2}+\log{(2\pi)^{\frac{n}{2}}}}\right)^{\frac{-1}{2}} \tag{5.14}$$

in Theorem 5.13, we have:

Corollary 5.14 *Consider the weighted standard static spacetime $\mathbb{G}^n \times_\alpha \mathbb{R}_1$, where $\mathbb{G}^n$ is the Gaussian space and α is defined in (5.14). If the entire Killing graph associate to z is such that $|Dz|_M^2$ is bounded and $\mathrm{Ric}_{\log \alpha^{-2}} \geq \kappa$, for some constant $\kappa > 0$, then the only bounded solutions of the problem*

$$\begin{cases} \mathrm{Div}\left(\dfrac{\alpha Dz}{(1-\alpha^2|Dz|_{\mathbb{G}}^2)^{1/2}}\right) + \dfrac{\langle Dz, D\alpha\rangle}{(1-\alpha^2|Dz|_{\mathbb{G}}^2)^{1/2}} = 0, \qquad z \in C^\infty(\mathbb{G}) \\ \\ \sup_{\Sigma(z)} (\alpha^2|Dz|_{\mathbb{G}}^2) < 1, \end{cases}$$

are the constant ones.

Chapter 6

Bifurcation and local rigidity in Riemannian warped products

In this chapter, we use equivariant bifurcation theory in order to establish sufficient conditions that allow us to guarantee the existence of bifurcation instants or the local rigidity of a certain family of open subsets of the warped product $I \times_\alpha M^n$, in the Section 6.2, and of the weighted Killing warped product $M^n_f \times_\alpha \mathbb{R}$, in the Section 6.3. Unless stated otherwise, all manifolds considered on this chapter will be connected, while *closed* means compact without boundary. The results presented in this chapter are part of [27] and [31]

6.1 The Variation concept

In a Riemannian manifold $\overline{M}^{n+1}$ as described in Section 1.1, let $\mathcal{M}$ be the space of open subsets $\Omega \subset \overline{M}^{n+1}$ with compact closure $\overline{\Omega}$ and whose smooth compact boundary $\partial\Omega$ is a closed, connected and orientable hypersurface. We denote by $d\overline{M}$ and dV the volume elements of $\overline{M}^{n+1}$ and $\partial\Omega$, respectively. If $\Omega \in \mathcal{M}$, the unit normal vector field globally defined on $\partial\Omega$ will be denoted by N. Moreover,

$$\mathrm{Vol}(\Omega) = \int_\Omega d\overline{M}\,, \quad \mathrm{Vol}_f(\Omega) = \int_\Omega d\mu$$

will denote respectively the *volume* and the *f-volume* of Ω and

$$\mathrm{Area}(\partial\Omega) = \int_{\partial\Omega} dV, \quad 1\text{-}\mathrm{Area}(\partial\Omega) = n\int_{\partial\Omega} H_1 dV \quad \text{and} \quad \mathrm{Area}_f(\partial\Omega) = \int_{\partial\Omega} d\mu$$

will denote the *area*, the 1-*area* and the *f-area* of $\partial\Omega$, respectively, where H_1 is the mean curvature of $\partial\Omega$ with respect to N and $d\mu = e^{-f}d(\partial\Omega)$ is the weighted volume form associated with the density function f.

For $\Omega \in \mathcal{M}$, we define a *variation* of $\partial\Omega$ as being the smooth mapping

$$\begin{array}{rcccl} X & : & (-\epsilon, \epsilon) \times \partial\Omega & \to & \overline{M}^{n+1} \\ & & (t,p) & \mapsto & X(t,p) \end{array} \tag{6.1}$$

satisfying the following two conditions:

(1) for all $t \in (-\epsilon, \epsilon)$, the map

$$\begin{array}{rcccl} X_t & : & \partial\Omega & \looparrowright & \overline{M}^{n+1} \\ & & p & \mapsto & X_t(p) = X(t,p) \end{array} \tag{6.2}$$

is a immersion;

(2) $X(0,p) = \iota(p)$ for all $p \in \partial\Omega$, where $\iota : \partial\Omega \looparrowright \overline{\Omega}$ is the inclusion map.

In this context, given $\Omega \in \mathcal{M}$ and a variation $X : (-\epsilon, \epsilon) \times \partial\Omega \to \overline{M}^{n+1}$ of $\partial\Omega$ we adopted the notation $\partial\Omega_t = X_t(\partial\Omega)$. For values of t small enough, $\partial\Omega_t$ is also a connected and oriented smooth submanifold. Moreover, it bounds an open subset Ω_t whose closure is also compact. Thus, $X : (-\epsilon, \epsilon) \times \partial\Omega \to \overline{M}^{n+1}$ induces us naturally a variation of the open subset Ω denoted by Ω_t, which is also an element of $\mathcal{M}$.

In all that follows, we let dV_t denote the volume element of the metric induced on $\partial\Omega_t$ by (6.2) and N_t will denote the unit normal vector field of (6.2). Moreover, we also consider in $\partial\Omega_t$ the weighted volume form given by $d\mu_t = e^{-f} dV_t$. When $t = 0$, all these objects coincide with those already defined on $\partial\Omega$.

The *variational field* associated to $X : (-\epsilon, \epsilon) \times \partial\Omega \to \overline{M}^{n+1}$ is the vector field $\frac{\partial X}{\partial t}|_{t=0}$ and, letting

$$u_t = \left\langle \frac{\partial X}{\partial t}, N_t \right\rangle, \tag{6.3}$$

we get that

$$\frac{\partial X}{\partial t}\Big|_{t=0} = u_0 N + \left(\frac{\partial X}{\partial t}\Big|_{t=0} \right)^{\top}.$$

We are now in a position to describe our variational problems and proceed with our study of bifurcation and local rigidity. This is what we will do next.

6.2 Bifurcation and local rigidity of constant second mean curvature hypersurfaces in riemannian warped products

In this section, based in [27], we will establish sufficient conditions that allow us to guarantee the existence of bifurcation instants or the local rigidity of a certain family of open sets of a Riemannian warped product $I \times_{\alpha} M^n$, where M^n is a compact

Riemannian manifold without boundary. Such family is formed by the open sets whose boundaries are H_2-hypersurfaces, namely, whose boundaries are hypersurfaces with constant second mean curvature H_2. For each of our results, we have provided a considerable number of examples that verify all the assumptions under consideration.

6.2.1 Description of the variational problem associated with the 1-Area functional

The *balance of volume* of $X : (-\epsilon, \epsilon) \times \partial\Omega \to \overline{M}^{n+1}$ is the functional

$$\begin{array}{rcl} \mathcal{V} & : (-\epsilon,\epsilon) \to & \mathbb{R} \\ & t \quad \mapsto & \mathcal{V}(t) = \mathrm{Vol}(\Omega_t) \end{array} \tag{6.4}$$

and we say that the variation $X : (-\epsilon, \epsilon) \times \partial\Omega \to \overline{M}^{n+1}$ is *volume-preserving* of Ω if $\mathcal{V}(t) = \mathcal{V}(0)$, for all $t \in (-\epsilon, \epsilon)$.

The formula of the first variation $\frac{d}{dt}\mathcal{V}(t)$ of the balance of volume $\mathcal{V}(t)$ is given in the following lemma, a formula that is well known and can be found in [69].

Lemma 6.1 *If $\Omega \in \mathcal{M}$ and $X : (-\epsilon, \epsilon) \times \partial\Omega \to \overline{M}^{n+1}$ is a variation of $\partial\Omega$, then*

$$\frac{d}{dt}\mathcal{V}(t) = \int_{\partial\Omega_t} u_t \, dV_t,$$

for each $t \in (-\epsilon, \epsilon)$, where u_t is the smooth function defined in (6.3). In particular, $X : (-\epsilon, \epsilon) \times \partial\Omega \to \overline{M}^{n+1}$ is volume-preserving of Ω if, and only if, $\int_{\partial\Omega_t} u_t \, dV_t = 0$ for all $t \in (-\epsilon, \epsilon)$.

Remark 6.2 *From Lemma 2.2 of [8], we have that if $u_0 : \partial\Omega \to \mathbb{R}$ is a smooth function such that $\int_{\partial\Omega} u_0 \, dV = 0$, then there exists a volume-preserving variation $X : (-\epsilon, \epsilon) \times \partial\Omega \to \overline{M}^{n+1}$ of $\partial\Omega$ whose variational field is $\frac{\partial X}{\partial t}|_{t=0} = u_0 N$.*

Taking into account [10], we define the 1-*area functional* associated to the variation $X : (-\epsilon, \epsilon) \times \partial\Omega \to \overline{M}^{n+1}$ by

$$\begin{array}{rcl} \mathcal{A}_1 & : (-\epsilon,\epsilon) \to & \mathbb{R} \\ & t \quad \mapsto & \mathcal{A}_1(t) = 1\text{-Area}(\partial\Omega_t) = n \displaystyle\int_{\partial\Omega_t} H_1^t \, dV_t, \end{array} \tag{6.5}$$

where $H_1^t = H_1(t, \cdot)$ denotes the mean curvature of $\partial\Omega_t$ with respect to the metric induced by the immersion X_t defined in (6.2).

The following result follows from Proposition 3.2 of [44].

Lemma 6.3 *If $\Omega \in \mathcal{M}$ and $X : (-\epsilon, \epsilon) \times \partial\Omega \to \overline{M}^{n+1}$ is a variation of $\partial\Omega$, then*

$$\begin{aligned}\frac{\partial}{\partial t} H_2^t &= \frac{2}{n(n-1)} \Box_t(u_t) \\ &+ \left\{ nH_1^t H_2^t - (n-2)H_3^t + \frac{2}{n(n-1)} \operatorname{tr}(T_t \circ \overline{R}_t) \right\} u_t \\ &+ \Big\langle \left(\frac{\partial X}{\partial t} \right)^{\top}, \nabla(H_2^t) \Big\rangle,\end{aligned}$$

where $\Box_t$ is the Cheng-Yau's square operator on $\partial\Omega_t$, $H_2^t = H_2(t, \cdot)$ and $H_3^t = H_3(t, \cdot)$ are the second and third mean curvatures of $\partial\Omega_t$, respectively, u_t is the function defined in (6.3), T_t is the Newton transformation on $\partial\Omega_t$ and $\overline{R}_t$ is the linear operator on $\partial\Omega_t$ given by $\overline{R}_t(Y) = \overline{R}(N_t, Y)N_t$ for all $Y \in \mathfrak{X}(\partial\Omega_t)$.

The previous lemma allows us to compute the first variation $\frac{d}{dt}\mathcal{A}_1(t)$ of the 1-Area functional $\mathcal{A}_1(t)$ (cf. Proposition 3.4 of [44]).

Lemma 6.4 *If $\Omega \in \mathcal{M}$ and $X : (-\epsilon, \epsilon) \times \partial\Omega \to \overline{M}^{n+1}$ is a variation of $\partial\Omega$, then*

$$\frac{d}{dt}\mathcal{A}_1(t) = \int_{\partial\Omega_t} \left\{ -n(n-1)H_2^t + \operatorname{Ric}_{\overline{M}}(N_t, N_t) \right\} u_t \, dV_t,$$

for all $t \in (-\epsilon, \epsilon)$, where u_t is the function defined in (6.3) and $\operatorname{Ric}_{\overline{M}}$ is the Ricci curvature of $\overline{M}^{n+1}$.

In order to characterize open subsets Ω of $\overline{M}^{n+1}$ whose boundary $\partial\Omega$ is a closed hypersurface with constant second mean curvature, we consider the variational problem of

(VP-1): *minimizing the 1-area functional $\mathcal{A}_1(t)$ given in (6.5) for all variations of $\partial\Omega$ that preserve the volume of Ω.*

The Lagrange multiplier method leads us then to the *Jacobi functional*

$$\begin{aligned} \mathcal{F}^\lambda : (-\epsilon, \epsilon) &\to \mathbb{R} \\ t &\mapsto \mathcal{F}^\lambda(t) = \mathcal{A}_1(t) + \lambda \mathcal{V}(t), \end{aligned} \tag{6.6}$$

where $\lambda \in \mathbb{R}$ is a constant to be determined. As an immediate consequence of Lemmas 6.1 and 6.4 we get that the first variation $\frac{d}{dt}\mathcal{F}^\lambda(t)$ of $\mathcal{F}^\lambda(t)$ takes the following form

$$\frac{d}{dt}\mathcal{F}^\lambda(t) = \int_{\partial\Omega_t} \left\{ -n(n-1)H_2^t + \operatorname{Ric}_{\overline{M}}(N_t, N_t) + \lambda \right\} u_t \, dV_t, \tag{6.7}$$

for each $t \in (-\epsilon, \epsilon)$.

To make an appropriate choice of λ, we assume from now on that there is $\overline{\varrho} \in \mathbb{R}$ such that the Ricci curvature $\mathrm{Ric}_{\overline{M}}$ of $\overline{M}^{n+1}$ satisfies the condition

$$\mathrm{Ric}_{\overline{M}}(N_t, N_t) \;=\; \overline{\varrho} \;=\; \text{const.} \qquad \text{on} \quad \partial\Omega_t, \qquad \text{for all } t \in (-\epsilon, \epsilon). \tag{6.8}$$

At the moment, when $\overline{M}^{n+1}$ is Einstein, (6.8) is naturally valid, but there is a larger class of manifolds that verify this condition, which will be described in Section 6.2.3. In addition, let

$$\Lambda \;=\; \frac{1}{\mathrm{Area}(\partial\Omega)} \int_{\partial\Omega} H_2 \, dV \tag{6.9}$$

be the integral mean of the second mean curvature H_2 on $\partial\Omega$. We call the attention to the fact that, in case H_2 is constant, we have

$$\Lambda \;=\; H_2, \tag{6.10}$$

and this notation will be used in what follows without further comments.

Hence, if we choose

$$\lambda \;=\; n(n-1)\Lambda - \overline{\varrho}, \tag{6.11}$$

from (6.7) we arrive at

$$\frac{d}{dt}\mathcal{F}^{\lambda}(t) \;=\; -n(n-1)\int_{\partial\Omega_t} \left\{ H_2^t - \Lambda \right\} u_t \, dV_t, \tag{6.12}$$

for all $t \in (-\epsilon, \epsilon)$. In particular,

$$\frac{d}{dt}\mathcal{F}^{\lambda}(0) \;=\; -n(n-1)\int_{\partial\Omega} \left\{ H_2 - \Lambda \right\} u_0 \, dV. \tag{6.13}$$

Now, following the same ideas of Proposition 2.7 of [9] we can establish the following result.

Proposition 6.5 *Let $\Omega \in \mathcal{M}$. Assume that the Ricci curvature $\mathrm{Ric}_{\overline{M}}$ of $\overline{M}^{n+1}$ satisfies (6.8). The following statements are equivalent:*

- (a) *$\partial\Omega$ is a H_2-hypersurface with constant second mean curvature H_2 equal to*

$$H_2 \;=\; \frac{\lambda + \overline{\varrho}}{n(n-1)};$$

- (b) *For all variations $X : (-\epsilon, \epsilon) \times \partial\Omega \to \overline{M}^{n+1}$ of $\partial\Omega$ which preserve the balance of volume of Ω, we have that $\frac{d}{dt}\mathcal{A}_1(0) = 0$;*

- (c) *For all variations $X : (-\epsilon, \epsilon) \times \partial\Omega \to \overline{M}^{n+1}$ of $\partial\Omega$, we have that $\frac{d}{dt}\mathcal{F}^{\lambda}(0) = 0$.*

Proof. We will show this result through the sequence $(a) \Rightarrow (c)$, $(c) \Rightarrow (b)$, $(b) \Rightarrow (a)$.

$(a) \Rightarrow (c)$: The result follows directly from (6.10), (6.11) and (6.13).

$(c) \Rightarrow (b)$: Form (6.6), $0 = \frac{d}{dt}\mathcal{F}^{\lambda}(0) = \frac{d}{dt}\mathcal{A}_1(0) + \lambda\frac{d}{dt}\mathcal{V}(0)$ or all variation $X : (-\epsilon, \epsilon) \times \partial\Omega \to \overline{M}^{n+1}$ of $\partial\Omega$. But if the variation preserves the volume of Ω, then $\frac{d}{dt}\mathcal{V}(0) = 0$. Hence, $\frac{d}{dt}\mathcal{A}_1(0) = 0$ for all volume-preserving variation $X : (-\epsilon, \epsilon) \times \partial\Omega \to \overline{M}^{n+1}$ of $\partial\Omega$.

$(b) \Rightarrow (a)$: By contradiction, suppose that there exists p_0 in $\partial\Omega$ such that

$$\left(H_2 - \frac{\lambda + \overline{\varrho}}{n(n-1)}\right)(p_0) \neq 0.$$

We can assume that

$$\left(H_2 - \frac{\lambda + \overline{\varrho}}{n(n-1)}\right)(p_0) > 0. \tag{6.14}$$

From the definition of Λ in (6.9) we can obtain another point $q_0 \in \Sigma^n$ such that

$$\left(H_2 - \frac{\lambda + \overline{\varrho}}{n(n-1)}\right)(q_0) < 0.$$

Indeed, from (6.9) and (6.11) we have

$$\begin{aligned} \int_{\partial\Omega}\left(H_2 - \frac{\lambda + \overline{\varrho}}{n(n-1)}\right) dV &= \int_{\partial\Omega} H_2\, dV - \Lambda\, \mathrm{Area}(\partial\Omega) \\ &= \int_{\partial\Omega} H_2\, dV \\ &\qquad - \frac{1}{\mathrm{Area}(\partial\Omega)}\left(\int_{\partial\Omega} H_2\, dV\right)\mathrm{Area}(\partial\Omega) \\ &= 0. \end{aligned} \tag{6.15}$$

Thus, if $\left(H_2 - \dfrac{\lambda + \overline{\varrho}}{n(n-1)}\right)(q) > 0$ for every $q \in \partial\Omega$, since there is $p_0 \in \partial\Omega$ such that (6.14) is valid, then

$$\int_{\partial\Omega}\left(H_2 - \frac{\lambda + \overline{\varrho}}{n(n-1)}\right) dV > 0,$$

which contradicts (6.15).

So, we have the sets

$$\partial\Omega^+ = \left\{ q \in \partial\Omega : \left(H_2 - \frac{\lambda + \overline{\varrho}}{n(n-1)}\right)(q) > 0 \right\}$$

and

$$\partial\Omega^- = \left\{ q \in \partial\Omega : \left(H_2 - \frac{\lambda + \overline{\varrho}}{n(n-1)}\right)(q) < 0 \right\}$$

are well defined.

Now, let us consider nonnegative smooth functions φ and ψ on $\partial\Omega$ such that

$$p_0 \in \operatorname{supp}\varphi \subset \partial\Omega^+, \qquad \operatorname{supp}\psi \subset \partial\Omega^-$$

and

$$\int_{\partial\Omega} (\varphi+\psi)\left(-n(n-1)H_2 + \overline{\varrho} + \frac{\lambda+\overline{\varrho}}{n(n-1)}\right) dV \;=\; 0,$$

where $\operatorname{supp}\varphi$ and $\operatorname{supp}\psi$ denote the support of φ and the support of ψ, respectively. If we consider the smooth function

$$u_0 = (\varphi+\psi)\left(-n(n-1)H_2 + \overline{\varrho} + \frac{\lambda+\overline{\varrho}}{n(n-1)}\right)$$

then, according to Remark 6.2, there is a volume-preserving variation $X : (-\epsilon,\epsilon) \times \partial\Omega \to \overline{M}^{n+1}$ of $\partial\Omega$ whose variational field is $\frac{\partial X}{\partial t}|_{t=0} = u_0 N$. Next, from our hypothesis, Lemma 6.4 and (6.8) we get

$$0 \;=\; \frac{d}{dt}\mathcal{A}_1(0) \;=\; \int_{\partial\Omega} \left(-n(n-1)H_2 + \overline{\varrho}\right) u_0 \, dV.$$

Furthermore, since $\int_{\partial\Omega} u_0\, dV = 0$, then

$$\begin{aligned} 0 \;&=\; \int_{\partial\Omega} \left(-n(n-1)H_2 + \overline{\varrho}\right) u_0\, dV \\ &=\; \int_{\partial\Omega} \left(-n(n-1)H_2 + \overline{\varrho}\right) u_0\, dV + \frac{\lambda+\overline{\varrho}}{n(n-1)} \int_{\partial\Omega} u_0\, dV \\ &=\; \int_{\partial\Omega} \left(-n(n-1)H_2 + \overline{\varrho} + \frac{\lambda+\overline{\varrho}}{n(n-1)}\right) u_0\, dV \\ &=\; \int_{\partial\Omega} (\varphi+\psi)\left(-n(n-1)H_2 + \overline{\varrho} + \frac{\lambda+\overline{\varrho}}{n(n-1)}\right)^2 dV \;>\; 0, \end{aligned}$$

which constitutes an absurd.

Therefore, we must have $H_2 = \dfrac{\lambda+\overline{\varrho}}{n(n-1)}$ on $\partial\Omega$. ∎

Hence, when the Riemannian manifold $\overline{M}^{n+1}$ verifies (6.8), from Proposition 6.5 we have that the critical points of (VP-1) are open subsets Ω of $\overline{M}^{n+1}$ whose boundary $\partial\Omega$ is a closed H_2-hypersurface with constant second mean curvature H_2 equal to

$$H_2 \;=\; \frac{\lambda+\overline{\varrho}}{n(n-1)}, \tag{6.16}$$

with $\lambda, \overline{\varrho} \in \mathbb{R}$. On the other hand, if we change our variational problem to

(VP-2) : *minimizing the* 1-*area functional* $\mathcal{A}_1(t)$ *given in* (6.5) *for all variations of* $\partial\Omega$, *not necessarily volume-preserving variations of* Ω, from Proposition 6.5 we obtain that the respective critical points of (VP-2) coincide with the same critical points of the initial variational problem (VP-1).

Remark 6.6 *As observed in [48], our approach is valid for the following more general configuration. Assume that $\mathcal{M}$ is the space of open subsets $\Omega \subset \overline{M}^{n+1}$ whose boundary $\partial\Omega$ is the union of two disjoint sets $\partial\Omega = \Sigma_1^n \cup \Sigma_2^n$. We will assume that one of them, Σ_1^n, is a fixed set and so that the variations considered of $\partial\Omega$ only affects Σ_2^n. Under this assumption, the critical points of* (VP-1) *or* (VP-2) *will be open subsets Ω such that their boundaries are union of a (fixed) set Σ_1^n and a closed hypersurface Σ_2^n with constant second mean curvature.*

For such a critical point (for either of the two variational problems described above), the formula for the second variation $\frac{d^2}{dt^2}\mathcal{F}^\lambda(0)$ of $\mathcal{F}^\lambda$ is given in the following result.

Proposition 6.7 *Let Ω be open subset of an $(n+1)$-dimensional Riemannian manifold $\overline{M}^{n+1}$ $(n \geq 2)$ whose boundary $\partial\Omega$ is a closed H_2-hypersurface, with constant second mean curvature H_2 given in (6.16), and let $X : (-\epsilon, \epsilon) \times \partial\Omega \to \overline{M}^{n+1}$ be a variation of $\partial\Omega$. Assume that the Ricci curvature $\mathrm{Ric}_{\overline{M}}$ of $\overline{M}^{n+1}$ satisfies (6.8). Then*

$$\frac{d^2}{dt^2}\mathcal{F}^\lambda(0)(u) = -2\int_{\partial\Omega} u\,\mathcal{J}(u)\,dV, \tag{6.17}$$

for any $u \in C^\infty(\partial\Omega)$, where $\mathcal{J} : C^\infty(\partial\Omega) \to C^\infty(\partial\Omega)$ is the Jacobi operator associated with the variational problems (VP-1) *and* (VP-2) *defined by*

$$\mathcal{J} = \Box + \left\{\frac{n(n-1)}{2}(nH_1H_2 - (n-2)H_3) + \mathrm{tr}\left(T \circ \overline{R}_0\right)\right\}. \tag{6.18}$$

In the last equation (6.18), $\Box$ is the Cheng-Yau's square operator on $\partial\Omega$, H_1, and H_3 are the first and third mean curvatures of $\partial\Omega$, respectively, T is the Newton transformation on $\partial\Omega$ and $\overline{R}_0$ is the linear operator on $\partial\Omega$ given by $\overline{R}_0(Y) = \overline{R}(N, Y)N$ for all $Y \in \mathfrak{X}(\partial\Omega)$.

With respect to the functions on $\partial\Omega$ to be evaluated in the second variation $\frac{d^2}{dt^2}\mathcal{F}^\lambda(0)$ of a critical point of (VP-1)*, they have to be considered according to Remark 6.2, that is, smooth functions on $\partial\Omega$ whose integral mean is zero. On the other hand, any smooth function on $\partial\Omega$ can be evaluated on the second variation $\frac{d^2}{dt^2}\mathcal{F}^\lambda(0)$ of a critical point of* (VP-2).

Proof. Initially, for any variation $X : (-\epsilon, \epsilon) \times \partial\Omega \to \overline{M}^{n+1}$ of $\partial\Omega$ we consider the function $u_0 \in C^\infty(\partial\Omega)$ defined in (6.3). Since H_2 is constant on $\partial\Omega$, from (6.10), (6.12) and Lemma 6.3 we get

$$\begin{aligned}\frac{d^2}{dt^2}\mathcal{F}^\lambda(0)(u_0) &= -n(n-1)\left\{\int_{\partial\Omega}\left(\frac{\partial}{\partial t}H_2^t\Big|_{t=0}\right)u_0\,dV\right.\\ &\quad\left.+\int_{\partial\Omega}\left(\underbrace{H_2-\Lambda}_{0}\right)\frac{\partial}{\partial t}(u_t\,dV_t)\Big|_{t=0}\right\}\\ &= -\int_{\partial\Omega}\{2\,\Box(u_0)+\{n(n-1)\,(nH_1H_2\\ &\qquad -(n-2)H_3)+2\,\mathrm{tr}\,(T\circ\overline{R}_0)\}\}u_0\}\,u_0\,dV.\end{aligned} \tag{6.19}$$

Now, for any $u\in C^\infty(\partial\Omega)$, considering variations $X:(-\epsilon,\epsilon)\times\partial\Omega\to\overline{M}^{n+1}$ of $\partial\Omega$ whose variational field is $\frac{\partial X}{\partial t}|_{t=0}=uN$, we obtain that the last expression (6.19) is also valid for every $u\in C^\infty(\partial\Omega)$. This shows the formula of the second variation of a critical point of (VP-2).

For those critical points of (VP-1), if $X:(-\epsilon,\epsilon)\times\partial\Omega\to\overline{M}^{n+1}$ is a variation of $\partial\Omega$ which preserve the balance of volume of Ω then for $u_0\in C^\infty(\partial\Omega)$ defined in (6.3) we have from Lemma 6.1 that $\int_{\partial\Omega}u_0\,dV=0$, and, in adittion, the expression (6.19) is valid for such u_0. Finally, for any function $u\in C^\infty(\partial\Omega)$ such that $\int_{\partial\Omega}u\,dV=0$, from Remark 6.2 we get a variation $X:(-\epsilon,\epsilon)\times\partial\Omega\to\overline{M}^{n+1}$ of $\partial\Omega$ which preserve the balance of volume of Ω such that the variational field is $\frac{\partial X}{\partial t}|_{t=0}=uN$, and immediately follows that (6.19) is retrieved for such a u. ∎

6.2.2 Bifurcation instants for H_2-hypersurfaces

In what follows, we consider the one-parameter family $\{\Omega_\tau\}_\tau\subset\mathcal{M}$ of open subsets in $\overline{M}^{n+1}$ such that the boundary of each Ω_τ, denoted by $\partial\Omega_\tau$, is a closed H_2^τ-hypersurface with constant second mean curvature H_2^τ, where τ varies on a prescribed interval $I\subset\mathbb{R}$. For every $\tau\in I$, let N_τ be the unit normal vector field globally defined on $\partial\Omega_\tau$. We assume that there is $\overline{\varrho}\in\mathbb{R}$ such that the Ricci curvature $\mathrm{Ric}_{\overline{M}}$ of $\overline{M}^{n+1}$ satisfies

$$\mathrm{Ric}_{\overline{M}}(N_\tau,N_\tau)=\overline{\varrho}=\text{const.}\quad\text{on}\quad\partial\Omega_\tau,\quad\text{for all}\quad\tau\in I. \tag{6.20}$$

In this context, as a consequence of our study of Subsection 6.2.1, we have that each Ω_τ is a critical point of a certain variational problem of type (VP-2). More specifically, each Ω_τ is a critical point for the Jacobi functional

$$I\ni\tau\longmapsto\mathcal{F}^{\lambda(\tau)}=\mathcal{A}_1+\lambda(\tau)\mathcal{V}$$

defined in (6.6), where

$$\lambda(\tau)=n(n-1)H_2^\tau-\overline{\varrho}.$$

Moreover, follows from Proposition 6.7 that, associated with each closed H_2^τ-hypersurface $\partial\Omega_\tau$ we have that the second variation $\frac{d^2}{dt^2}\mathcal{F}^{\lambda(\tau)}(0)$ of $\mathcal{F}^{\lambda(\tau)}$ is given by

$$\frac{d^2}{dt^2}\mathcal{F}^{\lambda(\tau)}(0)(u) = -2\int_{\partial\Omega_\tau} u\,\mathcal{J}_\tau(u)\,dV_\tau, \tag{6.21}$$

for any $u \in C^\infty(\partial\Omega_\tau)$, where dV_τ is the volume element on $\partial\Omega_\tau$ and

$$\mathcal{J}_\tau = \square_\tau + \left\{\frac{n(n-1)}{2}\left(nH_1^\tau H_2^\tau - (n-2)H_3^\tau\right) + \operatorname{tr}\left(T_\tau \circ \overline{R}_\tau\right)\right\}$$

is the Jacobi operator on $\partial\Omega_\tau$. Here, $\square_\tau$ is the Cheng-Yau's square operator on $\partial\Omega_\tau$ defined in (1.7), H_1^τ, H_2^τ and H_3^τ are the first three mean curvatures of $\partial\Omega_\tau$ with respect to unit normal vector field N_τ, $T_\tau : \mathfrak{X}(\partial\Omega_\tau) \to \mathfrak{X}(\partial\Omega_\tau)$ is the Newton transformation on $\partial\Omega_\tau$ defined in (1.6) and $\overline{R}_\tau : \mathfrak{X}(\partial\Omega_\tau) \to \mathfrak{X}(\partial\Omega_\tau)$ is the linear operator given by $\overline{R}_\tau(Y) = \overline{R}(N_\tau, Y)N_\tau$ for all $Y \in \mathfrak{X}(\partial\Omega_\tau)$.

With respect to our family $\{\Omega_\tau\}_{\tau\in I}$ of critical points of (VP-2), we need to adopt some notions and results that correspond to equivariant bifurcation theory for geometric variational problems. For more details on this subject, we recommend the references [3], [16], [17] and [66].

Let us first remember that two elements Ω_{τ_1} and Ω_{τ_1} of $\{\Omega_\tau\}_{\tau\in I}$ are said to be *isometrically congruent* when there is an isometry ψ of $\overline{M}^{n+1}$ that carries the image of $x_1 : \partial\Omega_{\tau_1} \looparrowright \overline{M}^{n+1}$ onto the image of $x_2 : \partial\Omega_{\tau_2} \looparrowright \overline{M}^{n+1}$ (cf. Section 1.2 of [3]), where x_1 and x_2 are the immersions of $\partial\Omega_{\tau_1}$ and $\partial\Omega_{\tau_2}$ into $\overline{M}^{n+1}$, respectively, i.e., if there exists a diffeomorphism $\phi : \partial\Omega_{\tau_1} \to \partial\Omega_{\tau_2}$ and an isometry ψ of $\overline{M}^{n+1}$ such that the following diagram commutes:

$$\begin{tikzcd}
\partial\Omega_{\tau_1} \arrow[r, "x_1"] \arrow[d, "\phi"'] & \overline{M}^{n+1} \arrow[d, "\psi"] \\
\partial\Omega_{\tau_2} \arrow[r, "x_2"'] & \overline{M}^{n+1}
\end{tikzcd}$$

Taking into account the studies reported in [16], $\widetilde{\tau} \in I$ is said to be a *bifurcation instant* of $\{\Omega_\tau\}_{\tau\in I}$ if there exists a sequence $\{\tau_n\}_{n\in\mathbb{N}} \subset I$ and a sequence $\{\Omega_{\tau_n}\}_{n\in\mathbb{N}} \subset \{\Omega_\tau\}_{\tau\in I}$ such that

(a) $\lim_{n\to\infty} \tau_n = \widetilde{\tau}$,

(b) $\lim_{n\to\infty} x_n = \widetilde{x}$, where $x_n : \Omega_{\tau_n} \looparrowright \overline{M}^{n+1}$ and $\widetilde{x} : \Omega_{\widetilde{\tau}} \looparrowright \overline{M}^{n+1}$ are the immersions of Ω_{τ_n} and $\Omega_{\widetilde{\tau}}$ into $\overline{M}^{n+1}$, respectively,

(c) for all $n \in \mathbb{N}$, x_n is not isometrically congruent to $\widetilde{x}$.

Furthermore, according to the ideas set out in [17], if $\widetilde{\tau} \in I$ is not a bifurcation instant, the family $\{\Omega_\tau\}_{\tau\in I}$ is said to be *locally rigid* at $\widetilde{\tau}$.

One of the classical criterion to determine when a instant $\widetilde{\tau} \in I$ is of bifurcation is related with the so-called *Morse index* (cf. [3] and [16]). We recall that the Morse index of Ω_τ, which will be denoted by $\operatorname{Ind}\left(\mathcal{F}^{\lambda(\tau)}, \Omega_\tau\right)$, is equal to the dimension of the maximal subspace where the second variation $\frac{d^2}{dt^2}\mathcal{F}^{\lambda(\tau)}(0)$ of the Jacobi functional $\mathcal{F}^{\lambda(\tau)}$ is negative definite. Equivalently, $\operatorname{Ind}\left(\mathcal{F}^{\lambda(\tau)}, \Omega_\tau\right)$ is the number of negative eigenvalues (counted with multiplicity) of the Jacobi operator $\mathcal{J}_\tau$. With our notations, a real number $\widehat{\xi}(\tau)$ is an *eigenvalue* of $\mathcal{J}_\tau$ if and only if $\mathcal{J}_\tau(u) + \widehat{\xi}(\tau)u = 0$ for some function $u \in C^\infty(\partial\Omega_\tau)$. From Proposition 2.7 of [3] we obtain that $\operatorname{Ind}\left(\mathcal{F}^{\lambda(\tau)}, \Omega_\tau\right)$ is finite in $I \subset \mathbb{R}$. Intuitively, $\operatorname{Ind}\left(\mathcal{F}^{\lambda(\tau)}, \Omega_\tau\right)$ measures the number of independent directions in which the hypersurface $\partial\Omega_\tau$ fails to minimize the 1-area functional $\mathcal{A}_1(t)$ defined in (6.5).

Essentially, a variation of the Morse index $\operatorname{Ind}\left(\mathcal{F}^{\lambda(\tau)}, \Omega_\tau\right)$ along the interval $I \subset \mathbb{R}$ will indicate the existence of a bifurcation instant. More precisely, under suitable Fredholmness assumptions (cf. [3] and [16]), we have that if there are $\tau_1, \tau_2 \in I$, with $\tau_1 < \tau_2$, such that the second variation $\frac{d^2}{dt^2}\mathcal{F}^{\lambda(\tau_j)}(0)$ of the Jacobi functional $\mathcal{F}^{\lambda(\tau)}$ is nonsingular (namely, the eigenvalues of the Jacobi operator $\mathcal{J}_{\tau_j}$ are nonzero) for $j \in \{1, 2\}$ and

$$\operatorname{Ind}(\mathcal{F}^{\lambda(\tau_1)}, \Omega_{\tau_1}) \neq \operatorname{Ind}(\mathcal{F}^{\lambda(\tau_2)}, \Omega_{\tau_2}),$$

then $\{\Omega_\tau\}_{\tau\in I}$ admits a bifurcation instant at some $\tau_* \in (\tau_1, \tau_2)$. On the other hand, according to [17], using the Implicit Function Theorem, we obtain that if $\frac{d^2}{dt^2}\mathcal{F}^{\lambda(\widetilde{\tau})}(0)$ is nonsingular for some $\widetilde{\tau} \in I$, then the family $\{\Omega_\tau\}_{\tau\in I}$ is locally rigid at $\widetilde{\tau}$. In particular, when $\operatorname{Ind}\left(\mathcal{F}^{\lambda(\tau)}, \Omega_\tau\right) = 0$ for all $\tau \in I$, $\{\Omega_\tau\}_{\tau\in I}$ does not have bifurcation instants.

In the Subsection 6.2.3, we will study the local rigidity and the bifurcation instants of $\{\Omega_\tau\}_{\tau\in I}$ by analyzing the spectrum of $\mathcal{J}_\tau$ for all $\tau \in I$. Essentially, we will determine the number of negative eigenvalues for each τ (counting its multiplicity) and we will study the evolution of such a number.

6.2.3 Local rigidity and bifurcation results for H_2-hypersurfaces

For an open interval $I \subset \mathbb{R}$ and a given n-dimensional Riemannian manifold M^n ($n \geq 2$) with metric tensor $\langle \cdot\,, \cdot \rangle_M$, consider the warped product $I \times_\alpha M^n$. In this context, for every $\tau \in I$ we have that the slice

$$\Sigma^n_\tau = \{\tau\} \times M^n \subset I \times_\alpha M^n$$

is a totally umbilical hypersurface in $I \times_\alpha M^n$ (cf. for instance [59]), oriented by the unit normal vector field $N_\tau = -\partial_\tau$, and whose shape operator A_τ is given by

$$\begin{array}{rcl} A_\tau \ : \ \mathfrak{X}(\Sigma^n_\tau) & \to & \mathfrak{X}(\Sigma^n_\tau) \\ Y & \mapsto & A_\tau(Y) = -\overline{\nabla}_Y(-\partial_\tau) = \dfrac{\alpha'(\tau)}{\alpha(\tau)} Y. \end{array} \tag{6.22}$$

Actually, the induced metric on Σ_τ^n is given by $\alpha(\tau)^2\langle\cdot,\cdot\rangle_M$, which means that Σ_τ^n is homotetic to M^n with scale factor $\alpha(\tau)$. Therefore, the correspondence

$$I \ni \tau \longmapsto \Sigma_\tau^n = \{\tau\} \times M^n$$

determines a foliation of $I \times_\alpha M^n$ by totally umbilical hypersurfaces, whose first three (constant) mean curvatures (see equations in (1.2)) are given respectively by

$$H_1^\tau = \frac{\alpha'(\tau)}{\alpha(\tau)}, \qquad H_2^\tau = \left(\frac{\alpha'(\tau)}{\alpha(\tau)}\right)^2, \qquad H_3^\tau = \left(\frac{\alpha'(\tau)}{\alpha(\tau)}\right)^3. \tag{6.23}$$

Moreover, the Ricci curvature $\mathrm{Ric}_{I\times_\alpha M^n}(\,,\,)$ of $I \times_\alpha M^n$ obeys the condition

$$\mathrm{Ric}_{I\times_\alpha M^n}(N_\tau, N_\tau) = -n\,\frac{\alpha''(\tau)}{\alpha(\tau)} = \text{const.} \qquad \text{on} \qquad \Sigma_\tau^n, \tag{6.24}$$

that is, the Riemanniann warped product $I \times_\alpha M^n$ satisfies (1.4).

From (6.24) we observe that the slices $\Sigma_\tau^n = \{\tau\} \times M^n$ of the Riemannian warped product $I \times_\alpha M^n$ verify the conditions (6.8) and (6.20) when the warped function $\alpha : I \to \mathbb{R}$ verifies the ordinary differential equation

$$n\alpha''(\tau) + \overline{\varrho}\alpha(\tau) = 0, \quad \tau \in I, \tag{6.25}$$

whose solutions are given by

$$\alpha(\tau) = \begin{cases} c_1 \cosh\left(\sqrt{\dfrac{-\overline{\varrho}}{n}}\,\tau\right) + c_2 \sinh\left(\sqrt{\dfrac{-\overline{\varrho}}{n}}\,\tau\right) & , \text{ if } \quad \overline{\varrho} < 0, \\ a > 0 \quad \text{or} \quad c_1\tau + c_2 & , \text{ if } \quad \overline{\varrho} = 0, \\ c_1 \cos\left(\sqrt{\dfrac{\overline{\varrho}}{n}}\,\tau\right) + c_2 \sin\left(\sqrt{\dfrac{\overline{\varrho}}{n}}\,\tau\right) & , \text{ if } \quad \overline{\varrho} > 0, \end{cases}$$

where $c_1, c_2 \in \mathbb{R}$ are constants and, in each case, the interval of definition $I \subset \mathbb{R}$ of α is the maximal one where α is positive. From these solutions, in Table 6.1 we collect the options of Riemannian warped products for our study.

For all warped functions described in Table 6.1, when the Riemannian fiber M^n is closed, we have that the Riemannian warped product $I \times_\alpha M^n$ support a family of open subsets which can be realized as critical points of the variational problem that was described in Subsection 6.2.1. To see this, let τ_1 and τ_2 be arbitrary numbers in $I \subset \mathbb{R}$ and we consider the family

$$\{\Omega_\tau\}_{\tau\in(\tau_1,\tau_2]}$$

of open subsets of $I \times_\alpha M^n$ defined by

$$\Omega_\tau = (\tau_1, \tau) \times M^n, \qquad \tau \in (\tau_1, \tau_2]. \tag{6.26}$$

Riemannian warped product	$\overline{\varrho}$	c_1	c_2
$(-\infty, +\infty) \times_{e^\tau} M^n$	$-n$	1	1
$(-\infty, +\infty) \times_{\cosh\tau} M^n$	$-n$	1	0
$(0, +\infty) \times_{\sinh\tau} M^n$	$-n$	0	1
$(-\infty, +\infty) \times M^n$	0	$\cdots$	$\cdots$
$(0, +\infty) \times_\tau M^n$	0	1	0
$(-\pi/2, \pi/2) \times_{\cos\tau} M^n$	n	1	0
$(0, \pi) \times_{\sin\tau} M^n$	n	0	1
$(0, \pi/2) \times_{\sin\tau+\cos\tau} M^n$	n	1	1

Table 6.1: Riemannian warped products satisfying the ordinary differential equation (6.25)

Thus, assuming M^n closed, we have that the boundary $\partial\Omega_\tau$ of each Ω_τ is the disjoint union

$$\partial\Omega_\tau \;=\; \Sigma^n_{\tau_1} \cup \Sigma^n_\tau$$

of two closed hypersurfaces $\Sigma^n_{\tau_1} = \{\tau_1\} \times M^n$ (fixed) and $\Sigma^n_\tau = \{\tau\} \times M^n$. Since the variations of $\partial\Omega_\tau$ only affects Σ^n_τ and taking into account that Σ^n_τ is a closed H_2^τ-hypersurface, Remark 6.6 assures us that each element of $\{\Omega_\tau\}_{\tau\in(\tau_1,\tau_2]}$ is a critical point of the variational problem (VP-2). Moreover, from (6.18), the differential operator $\mathcal{J}_\tau : C^\infty(\Sigma^n_\tau) \to C^\infty(\Sigma^n_\tau)$ given by

$$\mathcal{J}_\tau(u) \;=\; \Box_\tau(u) + \left\{\frac{n(n-1)}{2}\left(nH_1^\tau H_2^\tau - (n-2)H_3^\tau\right) + \operatorname{tr}\left(T_\tau \circ \overline{R}_\tau\right)\right\} u \tag{6.27}$$

for $u \in C^\infty(\Sigma^n_\tau)$, is the Jacobi operator associated with our variational problem, where $\Box_\tau$ is the Cheng-Yau's square operator on Σ^n_τ defined in (1.7), H_1^τ, H_2^τ and H_3^τ are the first three mean curvatures of Σ^n_τ given in (6.23), $T_\tau : \mathfrak{X}(\Sigma^n_\tau) \to \mathfrak{X}(\Sigma^n_\tau)$ is the Newton transformation on Σ^n_τ defined in (1.6) and $\overline{R}_\tau : \mathfrak{X}(\Sigma^n_\tau) \to \mathfrak{X}(\Sigma^n_\tau)$ is the linear operator given by

$$\overline{R}_\tau(Y) \;=\; \overline{R}(\partial_\tau, Y)\partial_\tau \tag{6.28}$$

for all $Y \in \mathfrak{X}(\Sigma^n_\tau)$.

In what follows, in a Riemannian warped products $I \times_\alpha M^n$ with closed Riemannian fiber M^n and whose warped function satisfies the ordinary differential equation (6.25), we pay more attention to the study of the conditions that guarantee either the local rigidity or the existence of bifurcation instants of the family of open subsets $\{\Omega_\tau\}_{\tau\in(\tau_1,\tau_2]}$ defined in (6.26).

In the next result we calculate the expressions that will allow us to write the Jacobi operator $\mathcal{J}_\tau$ in a more malleable way, in terms of the warped function α, of the Laplacian Δ on M^n and the constant $\overline{\varrho}$.

Proposition 6.8 *With the considerations and notations established above,*

(a) $T_\tau = (n-1)\dfrac{\alpha'(\tau)}{\alpha(\tau)}\,\mathrm{Id}_\tau$, *where* Id_τ *denotes the identity map on* $\mathfrak{X}(\Sigma^n_\tau)$*;*

(b) $\square_\tau = (n-1)\dfrac{\alpha'(\tau)}{\alpha(\tau)}\,\Delta_\tau$, *where* Δ_τ *is the Laplacian operator on* Σ^n_τ*;*

(c) $\square_\tau = (n-1)\dfrac{\alpha'(\tau)}{\alpha(\tau)^3}\,\Delta$, *where* Δ *is the Laplacian operator on* M^n*;*

(d) *The* i*-th eigenvalue* $\xi_i(\tau)$ *of the Cheng-Yau's square operator* $\square_\tau$ *on* Σ^n_τ *is*

$$\xi_i(\tau) = (n-1)\frac{\alpha'(\tau)}{\alpha(\tau)^3}\,\xi_i,$$

where ξ_i *is the* i*-th eigenvalue of the Laplacian operator* Δ *on* M^n*;*

(e) $\mathrm{tr}\left(T_\tau \circ \overline{R}_\tau\right) = (n-1)\dfrac{\alpha'(\tau)}{\alpha(\tau)}\,\overline{\varrho}$*;*

(f) $\mathcal{J}_\tau = (n-1)\dfrac{\alpha'(\tau)}{\alpha(\tau)^3}\,(\Delta + Q_0)$, *where*

$$Q_0 = n\left(\alpha'(\tau)\right)^2 + \alpha(\tau)^2\,\overline{\varrho} \tag{6.29}$$

is a constant on (τ_1, τ_2)*;*

(g) *The* i*-th eigenvalue* $\widehat{\xi}_i(\tau)$ *of the Jacobi operator* $\mathcal{J}_\tau$ *on* Σ^n_τ *is*

$$\widehat{\xi}_i(\tau) = (n-1)\frac{\alpha'(\tau)}{\alpha(\tau)^3}\,(\,\xi_i - Q_0\,),$$

where ξ_i *is the* i*-th eigenvalue of the Laplacian operator* Δ *on* M^n*;*

Proof. Item (a) is obtained immediately from (1.6) and (6.22). To obtain item (b), from (1.7) and item (a) we obtain

$$\square_\tau(u) = \mathrm{tr}\left(T_\tau\left(\mathrm{Hess}_{\Sigma^n_\tau}(u)\right)\right) = (n-1)\frac{\alpha'(\tau)}{\alpha(\tau)}\,\mathrm{tr}\left(\mathrm{Hess}_{\Sigma^n_\tau}(u)\right) = \frac{\alpha'(\tau)}{\alpha(\tau)}\,\Delta_\tau(u)$$

for all $u \in C^\infty(\Sigma^n_\tau)$. Now, through the natural identification of $C^\infty(\Sigma^n_\tau)$ with $C^\infty(M)$, item (c) follows from item (b) noting that the induced metric on Σ^n_τ is given by $\alpha(\tau)^2\langle\cdot\,,\cdot\rangle_M$; and item (d) follows directly from (c).

For the item (e), let $\{E_1,\ldots,E_n\}$ be an orthonormal frame defined in a neighborhood of some point of Σ^n_τ and let $K_{\overline{M}}(\partial_\tau, E_j)$ be the sectional curvature of $\overline{M}^{n+1}$ along the plane generated by ∂_τ and E_j, $j\in\{1,\ldots,n\}$. Then, from (6.28), item (a) and noting that our Riemannian warped products verifies (6.8), we get

$$\begin{aligned}
\operatorname{tr}\left(T_\tau\circ\overline{R}_\tau\right) &= (n-1)\frac{\alpha'(\tau)}{\alpha(\tau)}\sum_{j=1}^{n}\langle\overline{R}_\tau(E_j),E_j\rangle\\
&= (n-1)\frac{\alpha'(\tau)}{\alpha(\tau)}\sum_{j=1}^{n}\langle\overline{R}(\partial_\tau,E_j)\partial_\tau,E_j\rangle\\
&= (n-1)\frac{\alpha'(\tau)}{\alpha(\tau)}\sum_{j=1}^{n}K_{\overline{M}}(\partial_\tau,E_j)\\
&= (n-1)\frac{\alpha'(\tau)}{\alpha(\tau)}\operatorname{Ric}_{\overline{M}}(\partial_\tau,\partial_\tau) \;=\; (n-1)\frac{\alpha'(\tau)}{\alpha(\tau)}\,\overline{\varrho}.
\end{aligned}$$

On the other hand, from (6.23), (6.27) and items (c) and (e) we have

$$\begin{aligned}
\mathcal{J}_\tau &= (n-1)\frac{\alpha'(\tau)}{\alpha(\tau)^3}\Delta+\left\{\frac{n(n-1)}{2}\left(n\frac{\alpha'(\tau)}{\alpha(\tau)}\left(\frac{\alpha'(\tau)}{\alpha(\tau)}\right)^2\right.\right.\\
&\qquad\left.\left.-(n-2)\left(\frac{\alpha'(\tau)}{\alpha(\tau)}\right)^3+(n-1)\frac{\alpha'(\tau)}{\alpha(\tau)}\,\overline{\varrho}\right)\right\}\\
&= (n-1)\frac{\alpha'(\tau)}{\alpha(\tau)^3}\Delta+\left\{n(n-1)\left(\frac{\alpha'(\tau)}{\alpha(\tau)}\right)^3+(n-1)\frac{\alpha'(\tau)}{\alpha(\tau)}\,\overline{\varrho}\right\}\\
&= (n-1)\frac{\alpha'(\tau)}{\alpha(\tau)^3}\Delta+\frac{(n-1)\alpha'(\tau)}{\alpha(\tau)^3}\left(n\left(\alpha'(\tau)\right)^2+\alpha(\tau)^2\,\overline{\varrho}\right).
\end{aligned}$$

Hence, to end the proof of item (f), it remains to show that

$$\begin{aligned}
Q\;:\;(\tau_1,\tau_2)&\;\to\;\mathbb{R}\\
\tau&\;\mapsto\;Q(\tau)\;=\;n\left(\alpha'(\tau)\right)^2+\alpha(\tau)^2\,\overline{\varrho}
\end{aligned}$$

is a constant function. For this, from (6.25) we observed that

$$Q'(\tau)\;=\;2\alpha'(\tau)(\underbrace{n\alpha''(\tau)+\alpha(\tau)\overline{\varrho}}_{0})\;=\;0$$

for all $\tau\in(\tau_1,\tau_2)$, which implies that there exists $Q_0\in\mathbb{R}$ such that $Q(\tau)=Q_0$ for all $\tau\in(\tau_1,\tau_2)$.

Finally, item (g) follows directly from (f). ■

For a better understanding of the statements in the following results, let us remember that the *spectrum* of the Laplacian operator Δ on a closed Riemannian manifold M^n (cf. Section 1.3 of [24]) are determined by a sequence of eigenvalues $\{\xi_i\}_{i=0}^{+\infty}$ satisfying

$$0 = \xi_0 < \xi_1 \leq \xi_2 \leq \cdots \leq \xi_i \leq \xi_{i+1} \leq \cdots,$$

repeated according to their multiplicity, and

$$\lim_{i \to +\infty} \xi_i = +\infty.$$

Our first result provides some simple sufficient conditions to get the local rigidity of the family $\{\Omega_\tau\}_{\tau\in(\tau_1,\tau_2]}$.

Theorem 6.9 *Let $I \subset \mathbb{R}$ be an open interval, let M^n be a closed n-dimensional Riemannian manifold ($n \geq 2$) and let $I \times_\alpha M^n$ be a Riemannian warped product, whose warped function $\alpha : I \to \mathbb{R}$ satisfies the ordinary differential equation (6.25). Let $\{\Omega_\tau\}_{\tau\in(\tau_1,\tau_2]}$ be a family of open subsets of $I \times_\alpha M^n$ of the form $\Omega_\tau = (\tau_1, \tau) \times M^n$, where τ_1 and τ_2 are fixed numbers in $I \subset \mathbb{R}$ with $\tau_1 < \tau_2$. If*

(a) $Q_0 \neq \xi_i$ for all $i \in \{0, 1, 2, \ldots\}$, where Q_0 is the constant defined in (6.29) and ξ_i is the i-th eigenvalue of the Laplacian operator Δ on M^n, and

(b) $\alpha'(\tau) \neq 0$ for all $\tau \in (\tau_1, \tau_2)$,

then $\{\Omega_\tau\}_{\tau\in(\tau_1,\tau_2]}$ is locally rigid at each $\tau \in (\tau_1, \tau_2)$.

Proof. Taking into account our assumptions, from item (g) of Proposition 6.8 we obtain that the i-th eigenvalue $\widehat{\xi}_i(\tau)$ of the Jacobi operator $\mathcal{J}_\tau$ on Σ^n_τ is such that

$$\widehat{\xi}_i(\tau) = (n-1)\frac{\alpha'(\tau)}{\alpha(\tau)^3}\,(\,\xi_i - Q_0\,) \neq 0.$$

Hence, the second variation $\frac{d^2}{dt^2}\mathcal{F}^{\lambda(\tau)}(0)$ given in (6.21) is nonsingular for all $\tau \in (\tau_1, \tau_2)$ and, therefore, the family $\{\Omega_\tau\}_{\tau\in(\tau_1,\tau_2]}$ is locally rigid at each $\tau \in (\tau_1, \tau_2)$. ■

Let $\mathbb{S}^n(r)$ be the n-dimensional Euclidean sphere of radius $r > 0$. We know that all the eigenvalues ξ_i of the Laplacian operator Δ on $\mathbb{S}^n(r)$ (cf. Section 2.4 of [24]) are given by

$$\xi_i = \frac{i(i+n-1)}{r^2}, \qquad i \in \{0, 1, 2, \ldots\}. \tag{6.30}$$

Then, from Table 6.1 we can investigate the families $\{\Omega_\tau\}_{\tau\in(\tau_1,\tau_2]}$ of open sets in the warped products $I \times_\alpha \mathbb{S}^n(r)$ of the form $\Omega_\tau = (\tau_1, \tau) \times \mathbb{S}^n(r)$, with $\tau_1, \tau_2 \in I \subset \mathbb{R}$ and $\tau_1 < \tau_2$, that verify the conditions of Theorem 6.9. In Table 6.2 we collect the results of this analysis.

From the Table 6.2 we observe that the first case can be extended to a broader warped product class, exchanging the Euclidean sphere $\mathbb{S}^n(r)$ by any closed Riemannian manifold M^n.

Corollary 6.10 *Let $(-\infty, +\infty) \times_{\cosh \tau} M^n$ be a Riemannian warped product, with closed Riemannian fiber M^n $(n \geq 2)$, and let $\{\Omega_\tau\}_{\tau \in (\tau_1, \tau_2]}$ be a family of open subsets of $(-\infty, +\infty) \times_{\cosh \tau} M^n$ of the form $\Omega_\tau = (\tau_1, \tau) \times M^n$, where τ_1 and τ_2 are fixed numbers either in $(-\infty, 0)$ or in $(0, +\infty)$, in both cases with $\tau_1 < \tau_2$. Then $\{\Omega_\tau\}_{\tau \in (\tau_1, \tau_2]}$ is locally rigid at each $\tau \in (\tau_1, \tau_2)$.*

In the next result we have established a criterion that guarantees the existence of bifurcation instants of the family $\{\Omega_\tau\}_{\tau \in (\tau_1, \tau_2]}$.

Theorem 6.11 *Let $I \subset \mathbb{R}$ be an open interval, let M^n be a closed n-dimensional Riemannian manifold $(n \geq 2)$ and let $I \times_\alpha M^n$ be a Riemannian warped product, whose warped function $\alpha : I \to \mathbb{R}$ satisfies the ordinary differential equation (6.25). Let $\{\Omega_\tau\}_{\tau \in (\tau_1, \tau_2]}$ be a family of open subsets of $I \times_\alpha M^n$ of the form $\Omega_\tau = (\tau_1, \tau) \times M^n$, where τ_1 and τ_2 are fixed numbers in $I \subset \mathbb{R}$ with $\tau_1 < \tau_2$. Suppose that*

- (a) *$Q_0 \neq \xi_i$ for all $i \in \{0, 1, 2, \ldots\}$, where Q_0 is the constant defined in (6.29) and ξ_i is the i-th eigenvalue of the Laplacian operator Δ on M^n, and*
- (b) *there exist numbers $\delta_0, \eta_0 \in (\tau_1, \tau_2)$ with $\delta_0 < \eta_0$ such that either $\alpha'(\delta_0) > 0$ and $\alpha'(\eta_0) < 0$, or $\alpha'(\delta_0) < 0$ and $\alpha'(\eta_0) > 0$.*

Then $\{\Omega_\tau\}_{\tau \in (\tau_1, \tau_2]}$ admits at least a bifurcation instant at some $\tau_* \in (\delta_0, \eta_0)$.

Proof. Since $\alpha > 0$ on $I \subset \mathbb{R}$, from item (g) of Proposition 6.8 and from our hypotheses involving Q_0 and α' we obtain that the eigenvalue $\widehat{\xi}_i(\delta_0)$ and $\widehat{\xi}_i(\eta_0)$ of the Jacobi operators $\mathcal{J}_{\delta_0}$ and $\mathcal{J}_{\eta_0}$ are such that

$$\widehat{\xi}_i(\delta_0) = (n-1)\frac{\alpha'(\delta_0)}{\alpha(\delta_0)^3}(\xi_i - Q_0) \neq 0 \tag{6.31}$$

and

$$\widehat{\xi}_i(\eta_0) = (n-1)\frac{\alpha'(\eta_0)}{\alpha(\eta_0)^3}(\xi_i - Q_0) \neq 0 \tag{6.32}$$

for all $i \in \{0, 1, 2, \ldots\}$, respectively. Furthermore, for some $i_0 \in \{0, 1, 2, \ldots\}$, from (6.31) and (6.32),

$$\widehat{\xi}_{i_0}(\delta_0)\widehat{\xi}_{i_0}(\eta_0) = (n-1)^2\frac{\alpha'(\delta_0)\alpha'(\eta_0)}{\alpha(\delta_0)^3\alpha(\eta_0)^3}(\xi_{i_0} - Q_0)^2 < 0, \tag{6.33}$$

since the hypothesis (b) guarantees that $\alpha'(\delta_0)\alpha'(\eta_0) < 0$.

Now, from (6.21), (6.31) and (6.32) we get that the second variations $\frac{d^2}{dt^2}\mathcal{F}^{\lambda(\delta_0)}(0)$ and $\frac{d^2}{dt^2}\mathcal{F}^{\lambda(\eta_0)}(0)$ are nonsingular. Furthermore, from (6.33) we obtain that the eigenvalue $\widehat{\xi}_i(\tau)$ of the Jacobi operator

$$\mathcal{J}_\tau = (n-1)(\alpha'(\tau)/\alpha(\tau)^3)(\Delta + Q_0)$$

which corresponds to $i = i_0$ admits a change the signal between τ_1 and τ_2. Since the eigenvalues of the Jacobi operator $\mathcal{J}_\tau$ are ordered, we can ensure that the number of negative eigenvalues between τ_1 and τ_2 has changed. Therefore,

$$\operatorname{Ind}\left(\mathcal{F}^{\lambda(\tau_1)}, \Omega_{\tau_1}\right) \neq \operatorname{Ind}\left(\mathcal{F}^{\lambda(\tau_2)}, \Omega_{\tau_2}\right)$$

and the result follows. ∎

Taking into account once again the eigenvalues of the Laplacian operator Δ of the Euclidean spheres $\mathbb{S}^n(r)$, giving in (6.30), we can list in Table 6.3 some examples of families $\{\Omega_\tau\}_{\tau\in(\tau_1,\tau_2]}$ of open sets in warped products $I \times_\alpha \mathbb{S}^n(r)$ of the form $\Omega_\tau = (\tau_1, \tau) \times \mathbb{S}^n(r)$, with $\tau_1, \tau_2 \in I \subset \mathbb{R}$ and $\tau_1 < \tau_2$, that verify the conditions of the Theorem 6.11.

We remark that the first case of Table 6.3 can be extended to a broader warped product class, exchanging the Euclidean sphere $\mathbb{S}^n(r)$ by any closed Riemannian manifold M^n.

Corollary 6.12 *Let $(-\infty, +\infty) \times_{\cosh \tau} M^n$ be a Riemannian warped product, with closed Riemannian fiber M^n $(n \geq 2)$, and let $\{\Omega_\tau\}_{\tau\in(\tau_1,\tau_2]}$ be a family of open subsets of $(-\infty, +\infty) \times_{\cosh \tau} M^n$ of the form $\Omega_\tau = (\tau_1, \tau) \times M^n$, where τ_1 and τ_2 are fixed numbers such that $\tau_1 \in (-\infty, 0)$ and $\tau_2 \in (0, +\infty)$. If δ_0 and η_0 are two real numbers such that $\tau_1 < \delta_0 < 0 < \eta_0 < \tau_2$, then $\Omega_\tau = (\tau_1, \tau) \times M^n$ admits at least a bifurcation instant at some $\tau_* \in (\delta_0, \eta_0)$.*

Another way of establishing the existence of bifurcation instants of the family $\{\Omega_t\}_{t\in(t_1,t_2]}$ is given in the following result.

Theorem 6.13 *Let $I \subset \mathbb{R}$ be an open interval, let M^n be a closed n-dimensional Riemannian manifold $(n \geq 2)$ and let $I \times_\alpha M^n$ be a Riemannian warped product, whose warped function $\alpha : I \to \mathbb{R}$ satisfies the ordinary differential equation (6.25) for some nonzero constant real $\overline{\varrho}$. Let $\{\Omega_\tau\}_{\tau\in(\tau_1,\tau_2]}$ be a family of open subsets of $I \times_\alpha M^n$ of the form $\Omega_\tau = (\tau_1, \tau) \times M^n$, where τ_1 and τ_2 are fixed numbers in $I \subset \mathbb{R}$ with $\tau_1 < \tau_2$. Suppose that*

(a) *$Q_0 \neq \xi_i$ for all $i \in \{0, 1, 2, \ldots\}$, where Q_0 is the constant defined in (6.29) and ξ_i is the i-th eigenvalue of the Laplacian operator Δ on M^n, and*

(b) *there exists $\tau_* \in (\tau_1, \tau_2)$ such that $\alpha'(\tau_*) = 0$.*

Then $\{\Omega_\tau\}_{\tau\in(\tau_1,\tau_2]}$ admits a bifurcation instant in τ_*.

Proof. From item (g) of Proposition 6.8, for every $\delta_0, \eta_0 \in (\tau_1, \tau_2)$ with $\delta_0 < \tau_* < \eta_0$ and for $i_0 \in \{0, 1, 2, \ldots\}$ we have

$$\widehat{\xi}_{i_0}(\delta_0)\widehat{\xi}_{i_0}(\eta_0) = (n-1)^2 \frac{\alpha'(\delta_0)\alpha'(\eta_0)}{\alpha(\delta_0)^3\alpha(\eta_0)^3} \left(\xi_{i_0} - Q_0\right)^2. \tag{6.34}$$

Since $\alpha > 0$ on I, $\overline{\varrho} \neq 0$ and $-n\alpha''(\tau) = \overline{\varrho}\alpha(\tau)$ on I (see equation (6.25)) then $\alpha''(\tau) \neq 0$ on I, which asserts that α' is strictly increasing or strictly decreasing on I. So, from hypothesis (b), since $\delta_0 < t_* < \eta_0$ then $\alpha'(\delta_0) < 0 < \alpha'(\eta_0)$ or $\alpha'(\eta_0) < 0 < \alpha'(\delta_0)$. In both cases, $\alpha'(\delta_0)\alpha'(\eta_0) < 0$. Hence, returning to (6.34) and considering the hypothesis (a), we have that $\widehat{\xi}_{i_0}(\delta_0)\widehat{\xi}_{i_0}(\eta_0) < 0$.

In addition, again using item (g) of Proposition 6.8 and the hypothesis (a) we get that (6.31) and (6.32) are valid.

Now, the result is obtained by following the same steps of the end of the proof of Theorem 6.11. ■

With slight changes, it is immediate to observe that the families of open sets described in Table 6.3 can fit under the conditions of Theorem 6.13. We recorded this new configuration in Table 6.4.

Here we can also establish the following immediate consequence of Theorem 6.13.

Corollary 6.14 *Let $(-\infty, +\infty) \times_{\cosh \tau} M^n$ be a Riemannian warped product, with closed Riemannian fiber M^n $(n \geq 2)$, and let $\{\Omega_\tau\}_{\tau \in (\tau_1, \tau_2]}$ be a family of open subsets of $(-\infty, +\infty) \times_{\cosh \tau} M^n$ of the form $\Omega_\tau = (\tau_1, \tau) \times M^n$, where τ_1 and τ_2 are fixed numbers such that $\tau_1 \in (-\infty, 0)$ and $\tau_2 \in (0, +\infty)$. Then $\{\Omega_\tau\}_{\tau \in (\tau_1, \tau_2]}$ admits a bifurcation point in $\tau_* = 0$.*

The requirement on the constant Q_0 that appears in the hypotheses of Theorems 6.9, 6.11 and 6.13, can be interpreted as a geometric condition on the Riemannian fiber M^n of the warped product $I \times_\alpha M^n$. To arrive at this conclusion, let us first observe from (6.29) that the constant $\overline{\varrho}$ admits the expression

$$\overline{\varrho} = \frac{Q_0}{\alpha(\tau)^2} - nH_2^\tau,$$

that when substituted in (1.5) we obtain that the scalar curvature S^τ of Σ_τ^n is given by $\mathrm{S}^\tau = Q_0/\alpha(\tau)^2$. But as the induced metric on Σ_τ^n is $\alpha(\tau)^2\langle \cdot\,, \cdot \rangle_M$, we have that the scalar curvature S^M of M^n and Q_0 are related by $\mathrm{S}^M = (n-1)Q_0$. Therefore, what is requested in item (a) of Theorems 6.9, 6.11 and 6.13 can be interpreted as the requirement that the constant scalar curvature S^M of M^n does not belong to the spectrum of the Laplacian operator Δ of M^n.

6.3 Local rigidity, bifurcation and stability of H_f-hypersurfaces in weighted Killing warped products

This section corresponds to the contents of [31]. In what follows, in a weighted Killing warped product $M_f^n \times_\alpha \mathbb{R}$ endowed with a weighted function f that does not depend on the parameter $t \in \mathbb{R}$, we will establish sufficient conditions that allow us to guarantee the existence of bifurcation instants or the local rigidity for a family of open sets $\{\Omega_\gamma\}_{\gamma\in I}$ whose boundaries $\partial\Omega_\gamma$ are hypersurfaces with constant weighted mean curvature. For this, we analyze the number of negative eigenvalues of a certain Schrödinger operator and study its evolution. Furthermore, we obtain a characterization of a stable closed hypersurface $x : \Sigma^n \looparrowright M^n_f \times_\alpha \mathbb{R}$ with constant weighted mean curvature in terms of the first eigenvalue of the f-Laplacian of Σ^n.

6.3.1 Description of the variational problem associated with the weighted area functional

The *weighted volume functional* associated to the variation $X : (-\epsilon, \epsilon) \times \partial\Omega \to M^n_f \times_\alpha \mathbb{R}$ is

$$\begin{aligned} \mathcal{V}_f : (-\epsilon,\epsilon) \to\ & \mathbb{R} \\ s \mapsto\ & \mathcal{V}_f(s) = \mathrm{Vol}_f(\Omega_s) = \int_{\Omega_s} d\mu, \end{aligned} \tag{6.35}$$

and we say that $X : (-\epsilon, \epsilon) \times \partial\Omega \to M^n_f \times_\alpha \mathbb{R}$ is *weighted volume-preserving* of Ω if $\mathcal{V}_f(s) = \mathcal{V}_f(0)$, for all $s \in (-\epsilon, \epsilon)$.

The following result is well known and, in the context of weighted manifolds, it can be found in [23].

Lemma 6.15 *If $\Omega \in \mathcal{M}$ and $X : (-\epsilon, \epsilon) \times \partial\Omega \to M^n_f \times_\alpha \mathbb{R}$ is a variation of $\partial\Omega$ then*

$$\frac{d}{dt}\mathcal{V}_f(s) = \int_{\partial\Omega_s} u_s \, d\mu_s \,, \quad \text{for all } s \in (-\epsilon,\epsilon),$$

where u_s is the function defined in (6.3). In particular, $X : (-\epsilon, \epsilon) \times \partial\Omega \to M^n_f \times_\alpha \mathbb{R}$ is weighted volume-preserving of Ω if and only if $\int_{\partial\Omega_s} u_s \, d\mu_s = 0$ for all $s \in (-\epsilon, \epsilon)$.

Remark 6.16 *We observe that is not difficult to verify that Lemma 2.2 of [8] still remains valid for the context of weighted Riemannian manifolds, that is, if $u \in C^\infty(\partial\Omega)$ is such that $\int_{\partial\Omega} u d\mu = 0$, then there exists a weighted volume-preserving variation $X : (-\epsilon, \epsilon) \times \partial\Omega \to M^n_f \times_\alpha \mathbb{R}$ of $\partial\Omega$ whose variational field is $\frac{\partial X}{\partial s}|_{s=0} = uN$.*

The *weighted area functional* associated to the variation X is given by

$$\begin{aligned} \mathcal{A}_f : (-\epsilon,\epsilon) \to\ & \mathbb{R} \\ s \mapsto\ & \mathcal{A}_f = \mathrm{Area}_f(\partial\Omega_s) = \int_{\partial\Omega_s} d\mu_s. \end{aligned} \tag{6.36}$$

Following the same steps of the proof of Lemma 3.2 of [23], we can get the following

Lemma 6.17 *If $\Omega \in \mathcal{M}$ and $X : (-\epsilon, \epsilon) \times \partial\Omega \to M^n_f \times_\alpha \mathbb{R}$ is a variation of $\partial\Omega$, then*

$$\frac{d}{ds}\mathcal{A}_f(s) = -n \int_{\partial\Omega_s} (H_f)_s \, u_s \, d\mu_s \,, \quad \text{for all } s \in (-\epsilon, \epsilon),$$

where u_s is the function given in (6.3) and $(H_f)_s = H_f(s, \cdot)$ denotes the f-mean curvature of $\partial\Omega_s$ with respect to the metric induced by the immersion X_s defined in (6.2).

In order to characterize open subsets Ω of $M^n_f \times_\alpha \mathbb{R}$ whose boundary are closed hypersurfaces with constant f-mean curvature (possibly equal to zero), we consider the variational problem

(VP-3): *Minimizing the weighted area functional $\mathcal{A}_f$ (see (6.36)) for all variations of $\partial\Omega_\gamma$ that preserve the weighted volume of Ω_γ.*

The Lagrange multiplier method leads us then to the associated *weighted Jacobi functional*

$$\begin{array}{rcl} \mathcal{F}^\lambda_f : (-\epsilon, \epsilon) & \to & \mathbb{R} \\ s & \mapsto & \mathcal{F}^\lambda_f(s) = \mathrm{Area}_f(\partial\Omega_s) + \lambda \mathrm{Vol}_f(\Omega_s), \end{array} \tag{6.37}$$

where λ is a constant to be determined (eventually λ can be zero, and in this case, for $\Omega \in \mathcal{M}$, our variational problem reduces to minimizing the functional $\mathcal{A}_f$ for all variations of $\partial\Omega$).

As an immediate consequence of Lemmas 6.17 and 6.15 we get that the first variation of $\mathcal{F}^\lambda_f$ takes the following form:

$$\frac{d}{ds}\mathcal{F}^\lambda_f(s) = \frac{d}{ds}\mathcal{A}_f(s) + \lambda \frac{d}{ds}\mathcal{V}_f(s) = \int_{\partial\Omega_s} \{-n\,(H_f)_s + \lambda\} \, u_s \, d\mu_s. \tag{6.38}$$

Thinking about making the best possible choice of λ, let

$$\overline{\mathcal{H}} = \frac{1}{\mathrm{Area}_f(\partial\Omega)} \int_{\partial\Omega} H_f \, d\mu \tag{6.39}$$

be an integral mean of the f-mean curvature H_f on $\partial\Omega$. We call the attention to the fact that, in the case where H_f is constant, we have

$$\overline{\mathcal{H}} = H_f, \tag{6.40}$$

and this notation will be used in what follows without further comments. Therefore, if we choose $\lambda = n\overline{\mathcal{H}}$, from (6.38) we arrive at

$$\frac{d}{ds}\mathcal{F}^\lambda_f(s) = -n \int_{\partial\Omega_s} \{(H_f)_s - \overline{\mathcal{H}}\} \, u_s \, d\mu_s. \tag{6.41}$$

In particular,

$$\frac{d}{ds}\mathcal{F}^\lambda_f(0) = -n \int_{\partial\Omega} \{H_f - \overline{\mathcal{H}}\} \, u_0 \, d\mu. \tag{6.42}$$

Now, from (6.42) and following the same ideas of Proposition 2.7 of [9] we can establish the following result.

Proposition 6.18 *Let $\Omega \in \mathcal{M}$. The following statements are equivalent:*

(a) *$\partial\Omega$ is a closed H_f-hypersurface with constant f-mean curvature H_f equal to $H_f = \lambda/n$;*

(b) *for all weighted volume-preserving variations $X : (-\epsilon, \epsilon) \times \partial\Omega \to M_f^n \times_\alpha \mathbb{R}$ of $\partial\Omega$, we have $\frac{d}{ds}\mathcal{A}_f(0) = 0$;*

(c) *for all variations $X : (-\epsilon, \epsilon) \times \partial\Omega \to M_f^n \times_\alpha \mathbb{R}$ of $\partial\Omega$, we have $\frac{d}{ds}\mathcal{F}_f^\lambda(0) = 0$.*

Hence, from Proposition 6.18 we have that the critical points of (VP-5) are open subsets Ω of $M_f^n \times_\alpha \mathbb{R}$ whose boundary $\partial\Omega$ is a closed H_f-hypersurface with constant second mean curvature H_f equal to

$$H_f = \frac{\lambda}{n}, \tag{6.43}$$

with $\lambda \in \mathbb{R}$. On the other hand, if we change (VP-3) to

(VP-4) : *Minimizing the weighted area functional $\mathcal{A}_f$ (see (6.36)) for all variations of $\partial\Omega_\gamma$, not necessarily weighted volume-preserving variations of Ω_γ,*

from Proposition 6.18 we obtain that the respective critical points of (VP-4) coincide with the same critical points of the initial variational problem (VP-3).

Remark 6.19 *In the case $\lambda = 0$ we observe that the two variational problems* (VP-4) *and* (VP-3) *coincide, in which case the respective critical points are open subsets Ω of $M_f^n \times_\alpha \mathbb{R}$ whose boundary $\partial\Omega$ are closed f-minimal hypersurfaces. Furthermore, from (6.37) we can observe that $\mathcal{F}_f^0$ coincides with the weighted area functional $\mathcal{A}_f$ and, for each $\Omega \in \mathcal{M}$, this whole situation comes down to the variational problem of minimizing $\mathcal{A}_f$ for all variations of $\partial\Omega$ (not necessarily for those that preserve the weighted volume of Ω).*

Remark 6.20 *Taking into account the Remark 6.6, we will assume that $\mathcal{M}$ is the space of open subsets $\Omega \subset M_f^n \times_\alpha \mathbb{R}$ whose boundary $\partial\Omega = \Sigma_1^n \cup \Sigma_2^n$ is the union of two disjoint sets where Σ_1^n is a fixed set and, hence, the considered variations of $\partial\Omega$ only affects Σ_2^n. Under this assumption, the critical points of* (VP-3) *or* (VP-4) *will be open subsets Ω such that their boundaries are union of a (fixed) set Σ_1^n and a closed H_f-hypersurface Σ_2^n with constant f-mean curvature H_f given by (6.43).*

For the critical points of (VP-3) and (VP-4), the formula for the second variation of $\mathcal{F}_f^\lambda$ is given in the following result.

Proposition 6.21 *Let $\Omega \in \mathcal{M}$ be open subset of $M^n_f \times_\alpha \mathbb{R}$ whose boundary $\partial\Omega$ is a compact H_f-hypersurface, with constant f-mean curvature H_f given by (6.43). Then the second variation $\dfrac{d^2}{ds^2}\mathcal{F}^\lambda_f(0)$ of the weighted Jacobi functional $\mathcal{F}^\lambda_f$ is given by*

$$\frac{d^2}{ds^2}\mathcal{F}^\lambda_f(0)(u) = -\int_{\partial\Omega} u\,\mathcal{J}_f(u)\,d\mu, \tag{6.44}$$

for any $u \in C^\infty(\partial\Omega)$, where $\mathcal{J}_f : C^\infty(\partial\Omega) \to C^\infty(\partial\Omega)$ is the weighted Jacobi operator given by

$$\mathcal{J}_f = \Delta_f + \widetilde{\mathrm{Ric}}_f(N^*,N^*) - \frac{1}{\alpha}\widetilde{\mathrm{Hess}}\,\alpha(N^*,N^*) - \langle N,Y\rangle^2\frac{\widetilde{\Delta}_f(\alpha)}{\alpha^3} + |A|^2. \tag{6.45}$$

Here, Y is the Killing vector field on $M^n_f \times_\alpha \mathbb{R}$, $\alpha = |Y| > 0$, N is the unit normal vector field on $\partial\Omega$, Δ_f and $\widetilde{\Delta}_f$ represent the f-Laplacians on $\partial\Omega$ and M^n_f, respectively, $\widetilde{\mathrm{Ric}}_f$ and $\widetilde{\mathrm{Hess}}$ are the Bakry-Émery-Ricci tensor and the Hessian operator on M^n_f, $|A|^2$ represents the square of the norm of the shape operator A of $\partial\Omega$ with respect to the orientation given by N and N^* is the orthogonal projection of N on the tangent bundle of M^n. With respect to the functions on $\partial\Omega$ to be evaluated in $\frac{d^2}{ds^2}\mathcal{F}^\lambda_f(0)$ for a critical point of (VP-3)*, they have to be considered according to Remark 6.16, that is, smooth functions on $\partial\Omega$ whose integral mean is zero; and, on the other hand, any smooth function on $\partial\Omega$ can be evaluated in $\frac{d^2}{ds^2}\mathcal{F}^\lambda(0)$ for a critical point of* (VP-4).

Proof. Initially, for any variation $X : (-\epsilon,\epsilon)\times\partial\Omega \to M^n_f \times_\alpha \mathbb{R}$ of $\partial\Omega$ we consider the function $u_0 \in C^\infty(\partial\Omega)$ defined in (6.3). Since H_f is constant, from (6.41) and (6.40) we have that

$$\begin{aligned}\frac{d^2}{ds^2}\mathcal{F}^\lambda_f(0)(u_0) &= -n\int_{\partial\Omega}\left(\frac{\partial\,(H_f)_s}{\partial s}\Big|_{s=0}\right)u_0\,d\mu \\ &\quad -n\int_{\partial\Omega}\Big(\underbrace{H_f-\overline{\mathcal{H}}}_{0}\Big)\frac{\partial}{\partial s}\left(u_s\,d\mu_s\right)\Big|_{s=0}.\end{aligned}$$

Reasoning as in the proof of equation (3.5) of [23], we obtain

$$n\frac{\partial\,(H_f)_s}{\partial s}\Big|_{s=0} = \Delta_f\,(u_0) + \left\{\overline{\mathrm{Ric}}_f(N,N) + |A^2|\right\}u_0.$$

Hence,

$$\frac{d^2}{ds^2}\mathcal{F}^\lambda_f(0)(u_0) = -\int_{\partial\Omega}\left\{\Delta_f(u_0) + \left\{\overline{\mathrm{Ric}}_f(N,N) + |A|^2\right\}u_0\right\}u_0\,d\mu. \tag{6.46}$$

On the other hand, denoting by N^* and $N^\perp$ the orthogonal projections of N over the tangent and normal bundles of M^n, respectively, taking into account that f

is invariant along the flow determinate by Y, from [62, Proposition 7.35] we obtain

$$\begin{aligned}\overline{\mathrm{Hess}} f(N,N) &= \langle \overline{\nabla}_N \overline{\nabla} f, N\rangle \\ &= \langle \overline{\nabla}_N \widetilde{\nabla} f, N^* + N^{\perp}\rangle \\ &= \widetilde{\mathrm{Hess}} f(N^*,N^*) + \frac{1}{\alpha}\langle \widetilde{\nabla} f, \widetilde{\nabla}\alpha\rangle |N^{\perp}|^2 \\ &= \widetilde{\mathrm{Hess}} f(N^*,N^*) + \frac{1}{\alpha^3}\langle \widetilde{\nabla} f, \widetilde{\nabla}\alpha\rangle \langle N, Y\rangle^2. \end{aligned} \tag{6.47}$$

Moreover, from [62, Corollary 7.43] we get that

$$\overline{\mathrm{Ric}}(N,N) = \widetilde{\mathrm{Ric}}(N^*,N^*) - \frac{1}{\alpha}\widetilde{\mathrm{Hess}}\,\alpha(N^*,N^*) - \langle N,Y\rangle^2 \frac{\widetilde{\Delta}(\alpha)}{\alpha^3} \tag{6.48}$$

Now, from equations (6.47) and (6.48), we have that

$$\overline{\mathrm{Ric}}_f(N,N) = \widetilde{\mathrm{Ric}}_f(N^*,N^*) - \frac{1}{\alpha}\widetilde{\mathrm{Hess}}\,\alpha(N^*,N^*) - \langle N,Y\rangle^2 \frac{\widetilde{\Delta}_f(\alpha)}{\alpha^3} \tag{6.49}$$

Therefore, from equations (6.49) and (6.46) we obtain

$$\frac{d^2}{ds^2}\mathcal{F}_f^{\lambda}(0)(u_0) = -\int_{\partial\Omega} u_0\, \mathcal{J}_f(u_0)\, d\mu, \tag{6.50}$$

where $\mathcal{J}_f$ is given in (6.45).

Now, for any $u \in C^{\infty}(\partial\Omega)$, considering variations $X : (-\epsilon,\epsilon)\times\partial\Omega \to M_f^n \times_{\alpha} \mathbb{R}$ of $\partial\Omega$ whose variational field is $\frac{\partial X}{\partial t}|_{t=0} = uN$, we obtain that the last expression (6.50) is also valid for every $u \in C^{\infty}(\partial\Omega)$. All this we provide the formula of the second variation of $\mathcal{F}_f^{\lambda}$ for a critical point of (VP-4).

For the critical points of (VP-3), if $X : (-\epsilon,\epsilon)\times\partial\Omega \to M_f^n \times_{\alpha} \mathbb{R}$ is a variation of $\partial\Omega$ which preserve the weighted volume of Ω then for $u_0 \in C^{\infty}(\partial\Omega)$ defined in (6.3) we have from Lemma 6.15 that $\int_{\partial\Omega} u_0\, dV = 0$, and, in adittion, the expression (6.50) is valid for such u_0. Finally, for any function $u \in C^{\infty}(\partial\Omega)$ such that $\int_{\partial\Omega} u\, dV = 0$, from Remark 6.16 we get a variation $X : (-\epsilon,\epsilon)\times\partial\Omega \to M_f^n \times_{\alpha} \mathbb{R}$ of $\partial\Omega$ which preserve the weighted volume of Ω such that the variational field is $\frac{\partial X}{\partial t}|_{t=0} = uN$, and immediately follows that (6.50) is retrieved for such a u. ■

We conclude this subsection by noting that the weighted Jacobi operator $\mathcal{J}_f$ given in (6.45) belongs to a class of differential operators which are usually referred to as Schrödinger operators, that is, operators of the form $\Delta + q$, where Δ is the standard Laplacian on $\partial\Omega$ and q is any continuous function on $\partial\Omega$ (see, for instance, [46]). In particular, we can highlight that the behavior of the eigenvalues of $\mathcal{J}_f$ is well known, behavior that will play an important role in obtaining the main results of this section.

6.3.2 Bifurcation instants for H_f-hypersurfaces in $M_f^n \times_{\alpha} \mathbb{R}$

In what follows, we consider the one-parameter family $\{\Omega_{\gamma}\}_{\gamma}$ of open subsets in weighted Killing warped product $M_f^n \times_{\alpha} \mathbb{R}$ such that the boundary of each Ω_{γ}, denoted by $\partial\Omega_\gamma$, is a closed $H_f(\gamma)$-hypersurface with constant f-mean curvature $H_f(\gamma)$, where γ varies on a prescribed interval $I \subset \mathbb{R}$. In this context, as a consequence of our study of Subsection 6.3.1, we have that each Ω_γ is a critical point of a certain variational problem of type (VP-4). More specifically, each Ω_τ is a critical point for the one-parameter family of weighted Jacobi functionals

$$I \ni \gamma \longmapsto \mathcal{F}_f^{\lambda(\gamma)} = \mathcal{A}_f + \lambda(\gamma)\mathcal{V}_f$$

defined in (6.37), where

$$\lambda(\gamma) = nH_f(\gamma).$$

Moreover, from Proposition 6.21, associated with each closed $H_f(\gamma)$-hypersurface $\partial\Omega_\gamma$ we have that the second variation $\frac{d^2}{ds^2}\mathcal{F}_f^{\lambda(\gamma)}(0)$ of $\mathcal{F}_f^{\lambda(\gamma)}$ is given by

$$\frac{d^2}{ds^2}\mathcal{F}_f^{\lambda(\tau)}(0)(u) = -\int_{\partial\Omega} u\,\mathcal{J}_{f;\gamma}(u)\,d\mu, \tag{6.51}$$

for any $u \in C^\infty(\partial\Omega_\gamma)$, where

$$\mathcal{J}_{f;\gamma} = \Delta_{f;\gamma} + \widetilde{\mathrm{Ric}}_f(N_\gamma^*, N_\gamma^*) - \frac{1}{\alpha}\widetilde{\mathrm{Hess}}\,\alpha(N_\gamma^*, N_\gamma^*) - \langle N_\gamma, Y\rangle^2\frac{\widetilde{\Delta}_f(\alpha)}{\alpha^3} + |A_\gamma|^2 \tag{6.52}$$

is the weighted Jacobi operator on $\partial\Omega_\gamma$. Here, $\Delta_{f;\gamma}$ and $\widetilde{\Delta}_f$ are the f-Laplacians on $\partial\Omega_\gamma$ and M_f^n, respectively, $\widetilde{\mathrm{Ric}}_f$ and $\widetilde{\mathrm{Hess}}$ are the Bakry-Émery-Ricci tensor and the Hessian operator in M_f^n, A_γ is the shape operator of $\partial\Omega_\gamma$ with respect to normal vector field N_γ and N_γ^* is the orthogonal projection of N_γ on the tangent bundle of M^n.

Taking into account that the digression in the Subsection 6.2.2 can be applied to the functional $\mathcal{F}_f^{\lambda(\gamma)}$, we have that a variation of $\mathrm{Ind}_f\left(\mathcal{F}_f^{\lambda(\gamma)}, \Omega_\gamma\right)$ along the interval $I \subset \mathbb{R}$ will indicate the existence of a bifurcation instant. More precisely, under suitable Fredholmness assumptions (cf. [3] and [16]), we have that if there are $\gamma_1, \gamma_2 \in I$, with $\gamma_1 < \gamma_2$, such that the second variation $\frac{d^2}{ds^2}\mathcal{F}_f^{\lambda(\gamma_j)}(0)$ of the weighted Jacobi functional $\mathcal{F}_f^{\lambda(\gamma_j)}$ is nonsingular (namely, the eigenvalues of the weighted Jacobi operator $\mathcal{J}_{f;\gamma_j}$ are nonzero) for $j \in \{1,2\}$ and

$$\mathrm{Ind}_f\left(\mathcal{F}_f^{\lambda(\gamma_1)}, \Omega_{\gamma_1}\right) \neq \mathrm{Ind}_f\left(\mathcal{F}_f^{\lambda(\gamma_2)}, \Omega_{\gamma_2}\right), \tag{6.53}$$

then $\{\Omega_\gamma\}_{\gamma\in I}$ admits a bifurcation instant at some $\gamma_* \in (\gamma_1, \gamma_2)$. On the other hand, if $\frac{d^2}{ds^2}\mathcal{F}_f^{\lambda(\widetilde{\gamma})}(0)$ is nonsingular for some $\widetilde{\gamma} \in I$, then the family $\{\Omega_\gamma\}_{\gamma\in I}$ is locally rigid at $\widetilde{\gamma}$. In particular, when $\mathrm{Ind}_f\left(\mathcal{F}_f^{\lambda(\gamma)}, \Omega_\gamma\right) = 0$ for all $\gamma \in I$, $\{\Omega_\gamma\}_{\gamma\in I}$ does not have bifurcation instants.

Remark 6.22 *We observe that the change in the Morse index a family of hypersurfaces given by condition (6.53) is not sufficient to guarantee the bifurcation of the family $\{\Omega_\gamma\}_{\gamma\in I}$. Indeed, considering the standard context, the family of CMC spherical caps,* *starting with a pole and terminating with the entire sphere has a change in the Morse index from* 0 *to* 1 *at the hemisphere, but there is no bifurcation (for more details, see [7]). Hence, our assumption that* $\frac{d^2}{ds^2}\mathcal{F}_f^{\lambda(\gamma_j)}(0)$ *is nonsingular for* $j \in \{1,2\}$ *is a necessary condition to reach at the bifurcation.*

In the Section 6.3.3, we will study the local rigidity and the bifurcation instants of $\{\Omega_\gamma\}_{\gamma\in I}$ by analyzing the spectrum of $\mathcal{J}_{f;\gamma}$ for all $\gamma \in I$. Essentially, we will determine the number of negative eigenvalues for each γ (counting its multiplicity) and we will study the evolution of such a number.

6.3.3 Local rigidity and bifurcation results for H_f-hypersurfaces

The first result of this section provides some simple sufficient conditions to get the local rigidity of the family $\{\Omega_\gamma\}_{\gamma\in I}$ of critical points of the variational problem (VP-4) described in Subsection 6.3.2.

Theorem 6.23 *Let* $\{\Omega_\gamma\}_{\gamma\in I}$ *be a family of open subsets of the weighted Killing warped product* $M_f^n \times_\alpha \mathbb{R}$ *whose boundaries* $\partial\Omega_\gamma$ *are closed* $H_f(\gamma)$*-hypersurfaces. If, for all* $\gamma \in I$*, the function*

$$Q_f(\gamma) = \widetilde{\mathrm{Ric}}_f\left(N_\gamma^*, N_\gamma^*\right) - \frac{1}{\alpha}\widetilde{\mathrm{Hess}}\,\alpha\left(N_\gamma^*, N_\gamma^*\right) - \langle N_\gamma, Y\rangle^2 \frac{\widetilde{\Delta}_f(\alpha)}{\alpha^3} + |A_\gamma|^2$$

is constant on $\partial\Omega_\gamma$ *and the first nonzero eigenvalue* $\xi_f^1(\gamma)$ *of the* f*-Laplacian* $\Delta_{f;\gamma}$ *on* $\partial\Omega_\gamma$ *satisfies*

$$\xi_f^1(\gamma) - Q_f(\gamma) > 0, \tag{6.54}$$

then $\{\Omega_\gamma\}_{\gamma\in I}$ *is locally rigid at each* γ*. In particular, such a family is locally rigid if one of the following conditions holds:*

(*a*) $\widetilde{\mathrm{Ric}}_f(N_\gamma^*, N_\gamma^*) - \dfrac{1}{\alpha}\widetilde{\mathrm{Hess}}\,\alpha(N_\gamma^*, N_\gamma^*) - \langle N_\gamma, Y\rangle^2 \dfrac{\widetilde{\Delta}_f(\alpha)}{\alpha^3} \leq -|A_\gamma|^2$*;*

(*b*) *either*

$$\widetilde{\mathrm{Ric}}_f(N_\gamma^*, N_\gamma^*) - \frac{1}{\alpha}\widetilde{\mathrm{Hess}}\,\alpha(N_\gamma^*, N_\gamma^*) - \langle N_\gamma, Y\rangle^2\frac{\widetilde{\Delta}_f(\alpha)}{\alpha^3} < 0 \quad \text{and} \quad \xi_f^1(\gamma) \geq |A_\gamma|^2,$$

or

$$\widetilde{\mathrm{Ric}}_f(N_\gamma^*, N_\gamma^*) - \frac{1}{\alpha}\widetilde{\mathrm{Hess}}\,\alpha(N_\gamma^*, N_\gamma^*) - \langle N_\gamma, Y\rangle^2\frac{\widetilde{\Delta}_f(\alpha)}{\alpha^3} \leq 0 \quad \text{and} \quad \xi_f^1(\gamma) > |A_\gamma|^2.$$

Proof.

Since $Q_f(\gamma)$ is constant, from (6.52) we have that the eigenfunctions of the weighted Jacobi operator $\mathcal{J}_{f;\gamma}$ will coincide with the eigenfunctions of f-Laplacian

$\Delta_{f;\gamma}$. More specifically, if u is an eigenfunction of $\Delta_{f;\gamma}$ associated with an eigenvalue $\xi_f(\gamma)$ then u is eigenfunction of $\mathcal{J}_{f;\gamma}$ with eigenvalue

$$\widehat{\xi}_f(\gamma) = \xi_f(\gamma) - Q_f(\gamma).$$

Moreover, by the spectral theorem we know that all the eigenvalues of $\Delta_{f;\gamma}$ are given by a sequence $\{\xi_f^j(\gamma)\}_{j=0}^{+\infty}$ satisfying

$$0 \; = \; \xi_f^0(\gamma) \; < \; \xi_f^1(\gamma) \; \leq \; \cdots \; \leq \; \xi_f^j(\gamma) \; \leq \; \xi_f^{j+1}(\gamma) \; \leq \; \cdots \, ,$$

repeated according to their multiplicity, and

$$\lim_{j \to +\infty} \; \xi_f^j(\gamma) \; = \; +\infty$$

(see, for instance, Section 1 of [72]). So, all the eigenvalues $\widehat{\xi}_f^j(\gamma)$ of $\mathcal{J}_{f;\gamma}$ have the following form

$$\widehat{\xi}_f^j(\gamma) = \xi_f^j(\gamma) - Q_f(\gamma) \quad \text{for every } j \in \{0, 1, 2, \ldots\}. \tag{6.55}$$

So, from (6.54) and (6.55) we obtain

$$\widehat{\xi}_f^j(\gamma) \; = \; \xi_f^j(\gamma) - Q_f(\gamma) \; \geq \; \xi_f^1(\gamma) - Q_f(\gamma) \; > \; 0 \quad \text{for every } j \in \{0, 1, 2, \ldots\}.$$

Hence, the second variation $\frac{d^2}{ds^2}\mathcal{F}_f^{\lambda(\gamma)}(0)$ given in (6.51) is nonsingular for all $\gamma \in I$ and, therefore, the family $\{\Omega_\gamma\}_{\gamma \in I}$ is locally rigid at each $\gamma \in I$.

■

Our next result provides a criterion that guarantees the existence of bifurcation instants of the family $\{\Omega_\gamma\}_{\gamma \in I}$.

Theorem 6.24 *Let $\{\Omega_\gamma\}_\gamma$ be a family of open subsets of the weighted Killing warped product $M_f^n \times_\alpha \mathbb{R}$ whose boundaries $\partial\Omega_\gamma$ are closed $H_f(\gamma)$-hypersurfaces. Suppose that, for all $\gamma \in I$, the function*

$$Q_f(\gamma) \; = \; \widetilde{\mathrm{Ric}}_f\left(N_\gamma^*, N_\gamma^*\right) - \frac{1}{\alpha}\widetilde{\mathrm{Hess}}\,\alpha\left(N_\gamma^*, N_\gamma^*\right) - \langle N_\gamma, Y\rangle^2 \frac{\widetilde{\Delta}_f(\alpha)}{\alpha^3} + |A_\gamma|^2$$

is constant on $\partial\Omega_\gamma$. If there are two values γ_1 and γ_2, with $\gamma_1 < \gamma_2$, such that the eigenvalues $\widehat{\xi}_f^j(\gamma_1)$ and $\widehat{\xi}_f^j(\gamma_2)$ of the weighted Jacobi operators $\mathcal{J}_{f;\gamma_1}$ and $\mathcal{J}_{f;\gamma_2}$ (respectively) satisfy

(a) $\widehat{\xi}_f^j(\gamma_1) \neq 0$ and $\widehat{\xi}_f^j(\gamma_2) \neq 0$ for all $j \in \{0, 1, 2, \ldots\}$,

(b) there exists $j_0 \in \{0, 1, 2, \ldots\}$ such that $\left(\widehat{\xi}_f^{j_0}(\gamma_1)\right)\left(\widehat{\xi}_f^{j_0}(\gamma_2)\right) < 0$,

then there exists a bifurcation instant $\gamma_* \in (\gamma_1, \gamma_2)$.

Proof. Initially, from (6.52) and (6.51) we note that the condition about $Q_f(\gamma)$ and the hypothesis (a) assures us that the second variation $\frac{d^2}{ds^2}\mathcal{F}_f^{\lambda(\gamma_j)}(0)$ of the weighted Jacobi functional $\mathcal{F}_f^{\lambda(\gamma_j)}$ is nonsingular for $j \in \{1,2\}$. On the order hand, we observe that hypothesis (b) assures us that the eigenvalue of the weighted Jacobi operator which corresponds to $j = j_0$ admits a change the signal between γ_1 and γ_2. Moreover, as the eigenvalues of the one-parameter family of weighted Jacobi functionals are ordered, we can ensure that the number of negative eigenvalues between γ_1 and γ_2 has changed. Therefore,

$$\mathrm{Ind}_f\left(\mathcal{F}_f^{\lambda(\gamma_1)}, \Omega_{\gamma_1}\right) \neq \mathrm{Ind}_f\left(\mathcal{F}_f^{\lambda(\gamma_2)}, \Omega_{\gamma_2}\right)$$

and the result follows. ■

When M^n is closed, the weighted Killing warped product $M_f^n \times_\alpha \mathbb{R}$ naturally admits a family of open subsets that can be realized as critical points of the weighted area functional $\mathcal{A}_f$ defined in (6.36). To visualize this, for $t_1, t_2 \in \mathbb{R}$ with $t_1 < t_2$, we consider the family of open subsets $\{\Omega_\gamma\}_{\gamma \in (t_1, t_2]}$ given by

$$\Omega_\gamma = M^n \times (t_1, \gamma), \quad \gamma \in (t_1, t_2], \tag{6.56}$$

whose boundary $\partial\Omega_\gamma$ of each Ω_γ is formed by the disjoint union

$$\partial\Omega_\gamma = \Sigma_1^n \cup \Sigma_2^n(\gamma)$$

of a fixed set $\Sigma_1^n = M^n \times \{t_1\}$ and other set $\Sigma_2^n(\gamma) = M^n \times \{\gamma\}$. From Remark 1.2 we have that each $\Sigma_2^n(\gamma)$, $\gamma \in (t_1, t_2]$, is an f-minimal totally geodesic closed hypersurface. So, since the variations of $\partial\Omega_\tau$ only affects $\Sigma_2^n(\gamma)$, from Remarks 6.19 and 6.20, we conclude that each element of the family $\Omega_{\gamma \in (t_1, t_2]}$ is a critical point of $\mathcal{A}_f$. For these critical points, noting that ∂_t is the vector field on $M_f^n \times_\alpha \mathbb{R}$ that determines the orientation of each $\Sigma_2^n(\gamma)$, $\gamma \in (t_1, t_2]$, we have that second variation of the weighted Jacobi functional $\mathcal{F}_f^0 = \mathcal{A}_f$ and the weighted Jacobi operator on each $\partial\Omega_\gamma$, given by the expressions (6.51) and (6.52), are reduced to

$$\frac{d^2}{ds^2}\mathcal{A}_f(0)(u) \;=\; -\int_{\Sigma_2(\gamma)} u\,\mathfrak{J}_{f;\gamma}(u)\,d\mu,$$

and

$$\mathfrak{J}_{f;\gamma}(u) \;=\; \Delta_{f;\gamma}(u) - \frac{1}{\alpha}\widetilde{\Delta}_f(\alpha)\,u$$

for any $u \in C^\infty(\Sigma_2^n(\gamma))$, respectively, where $\Delta_{f;\gamma}$ represents the f-Laplacian on $\Sigma_2^n(\gamma)$, $\widetilde{\Delta}_f$ is the f-Laplacian on M_f^n, $\alpha = |Y| > 0$ and Y is the Killing vector field that determines on $M_f^n \times_\alpha \mathbb{R}$ the foliation by totally geodesic closed slices $M^n \times \{t\}$, $t \in \mathbb{R}$. In addition, if α is an eigenfunction of $\widetilde{\Delta}_f$, with associated eigenvalue c, we have that $\mathfrak{J}_{f;\gamma}$ can be written simply as

$$\mathfrak{J}_{f;\gamma} \;=\; \Delta_{f;\gamma} + c.$$

In this scenario, we observe that the arguments of the proofs of Theorems 6.23 and 6.24 are valid, and even more, the statements can be refined, in the sense that we now ask as hypotheses a certain behavior of the spectrum of the drift Laplacian $\widetilde{\Delta}_f$ of the closed manifold M^n_f.

Corollary 6.25 *Let M^n be an n-dimensional closed Riemannian manifold and, for $t_1, t_2 \in \mathbb{R}$ with $t_1 < t_2$, let $\Omega_{\gamma\in(t_1,t_2]}$ be the family of open subsets of the weighted Killing warped product $M^n_f \times_\alpha \mathbb{R}$ given by (6.56). Let $\widetilde{\Delta}_f$ be the f-Laplacian on M^n_f. If α is an eigenfunction of $\widetilde{\Delta}_f$ (with associated eigenvalue c) and the first nonzero eigenvalue $\xi^1_f(\gamma)$ of the f-Laplacian $\Delta_{f;\gamma}$ on $\Sigma_2(\gamma) = M^n \times \{\gamma\}$, $\gamma \in (t_1, t_2]$, satisfies*

$$\xi^1_f(\gamma) > c,$$

then $\{\Omega_\gamma\}_{\gamma\in(t_1,t_2]}$ is locally rigid at each $\gamma \in (t_1, t_2]$.

Proof. Initially, it is immediate to note that the function $Q_f(\gamma)$ of Theorem 6.23 reduces to the nonnegative constant c. Then, as in the steps of the proof of Theorem 6.23, we make an analysis of the eigenvalues of $\mathfrak{J}_{f;\gamma}$ that contribute to $\mathrm{Ind}_f\left(\mathcal{A}_f, \Omega_\gamma\right)$ and the result follows. ■

Remark 6.26 *Considering once more the behavior of the eigenvalues of the f-Laplacian $\Delta_{f;\gamma}$ on an arbitrary closed weighted manifold M^n_f, from Corollary 6.25 we obtain the following consequence: The family of open subsets of the weighted product $M^n_f \times \mathbb{R}$ given by (6.56) is always locally rigid at each $\gamma \in (t_1, t_2]$.*

Thinking similarly, from Theorem 6.24 we obtain the following result:

Corollary 6.27 *Let M^n be an n-dimensional closed Riemannian manifold and, for $t_1, t_2 \in \mathbb{R}$ with $t_1 < t_2$, let $\Omega_{\gamma\in(t_1,t_2]}$ be the family of open subsets of the weighted Killing warped product $M^n_f \times_\alpha \mathbb{R}$ given by (6.56). Let $\widetilde{\Delta}_f$ be the f-Laplacian on M^n_f. If α is an eigenfunction of $\widetilde{\Delta}_f$ (with associated eigenvalue c) and if there are two values $\gamma_1, \gamma_2 \in (t_1, t_2]$, with $\gamma_1 < \gamma_2$, such that the eigenvalues $\widehat{\xi}^j_f(\gamma_1)$ and $\widehat{\xi}^j_f(\gamma_2)$ of the Jacobi operators $\mathfrak{J}_{f;\gamma_1}$ and $\mathfrak{J}_{f;\gamma_2}$ (respectively) satisfy*

(a) $\widehat{\xi}^j_f(\gamma_1) \neq 0$ and $\widehat{\xi}^j_f(\gamma_2) \neq 0$ for all $j \in \{0, 1, 2, \ldots\}$,

(b) there exists $j_0 \in \{0, 1, 2, \ldots\}$ such that $\left(\widehat{\xi}^{j_0}_f(\gamma_1)\right)\left(\widehat{\xi}^{j_0}_f(\gamma_2)\right) < 0$,

then there exists a bifurcation instant $\gamma_* \in (\gamma_1, \gamma_2)$.

6.4 Stability of H_f-hypersurfaces in $M_f^n \times_\alpha \mathbb{R}$

It is important to remark that, for all calculations in the Section 6.1, there is no real dependence on the open set $\Omega \in \mathcal{M}$ but on the hypersurface $\partial\Omega$. In fact, in the literature, it is more common to work in terms of hypersurfaces (for instance, see [9, 8] for the classical context, and [23, 54] for the weighted context). In this scenario, $\mathcal{M}$ becomes the space of all closed orientable hypersurfaces of $M_f^n \times_\alpha \mathbb{R}$.

In this last section, we study the notion of stability associated with problem (VP-3) described in Subsection 6.1 for this new set $\mathcal{M}$. We begin this study by remembering that if $x : \Sigma^n \looparrowright M_f^n \times_\alpha \mathbb{R}$ is such a hypersurface, then the weighted volume and weighted area associated with a variation $X : (-\epsilon, \epsilon) \times \Sigma^n \to M_f^n \times_\alpha \mathbb{R}$ are given by

$$\begin{array}{rccl} \mathcal{V}_f : & (-\epsilon,\epsilon) & \to & \mathbb{R} \\ & s & \mapsto & \mathcal{V}_f(s) = \mathrm{Vol}_f\left(\Sigma^n \times [0,s]\right) \;=\; \displaystyle\int_{\Sigma^n\times[0,s]} X^*(d\overline{\sigma}) \end{array}$$

and

$$\begin{array}{rccl} \mathcal{A}_f : & (-\epsilon,\epsilon) & \to & \mathbb{R} \\ & s & \mapsto & \mathcal{A}_f(s) = \mathrm{Area}_f\left(X_s(\Sigma^n)\right) \;=\; \displaystyle\int_{\Sigma^n} d\mu_s, \end{array}$$

respectively. Furthermore, the variational problem of minimizing the functional $\mathcal{A}_f$ for all variations of $x : \Sigma^n \looparrowright M_f^n \times_\alpha \mathbb{R}$ that preserve the weighted volume $\mathcal{V}_f$ is addressed by the study of the weighted Jacobi functional

$$\begin{array}{rccl} \mathcal{F}_f : & (-\epsilon,\epsilon) & \to & \mathbb{R} \\ & s & \mapsto & \mathcal{F}_f(s) = \mathcal{A}_f(s) + n\overline{\mathcal{H}}\,\mathcal{V}_f(s) \end{array}$$

where $\overline{\mathcal{H}}$ is the constant defined in (6.39), and their respective critical points are the closed H_f-hypersurfaces of $M_f^n \times_\alpha \mathbb{R}$. For these critical points, the stability of the corresponding variational problem is given by the second variation

$$\frac{d^2}{ds^2}\mathcal{F}_f(0)(u) = -\int_{\Sigma^n} u\,\mathcal{J}_f(u)\,d\mu,$$

where $\mathcal{J}_f : C^\infty(\Sigma^n) \to C^\infty(\Sigma^n)$ is the weighted Jacobi operator given in (6.45). The above discussion motivates the following notion of stability.

We say that a closed H_f-hypersurface $x : \Sigma^n \looparrowright M_f^n \times_\alpha \mathbb{R}$ is *f-stable* if

$$\frac{d^2}{ds^2}\mathcal{A}_f(0) \;\geq\; 0,$$

for all weighted volume-preserving variations $X : \Sigma^n \times (-\epsilon, \epsilon) \to M_f^n \times_\alpha \mathbb{R}$ of $x : \Sigma^n \looparrowright M_f^n \times_\alpha \mathbb{R}$.

Remark 6.28 *Let $x : \Sigma^n \looparrowright M^n_f \times_\alpha \mathbb{R}$ be a closed H_f-hypersurface as described in the last definition above. We consider the set*

$$\mathcal{G} = \left\{ u \in C^\infty(\Sigma^n) \ : \ \int_{\Sigma^n} u \, d\mu = 0 \right\}. \tag{6.57}$$

Just as [9], we can establish the following criterion of f-stability: a hypersurface $x : \Sigma^n \looparrowright M^n_f \times_\alpha \mathbb{R}$ is f-stable if and only if $\frac{d^2}{ds^2}\mathcal{F}_f(0)(u) \geq 0$, for all $u \in \mathcal{G}$.

In what follows, associated with a hypersurface $x : \Sigma^n \looparrowright M^n_f \times_\alpha \mathbb{R}$, we will consider the angle function Θ defined in (1.33). In this setting, we get the following key lemma, which provides sufficient conditions to obtain a eigenfunction of the drift Laplacian Δ_f on Σ^n. Let us denote by $\overline{\nabla}$, ∇ and $\widetilde{\nabla}$ the Levi-Civita conections of $M^n_f \times_\alpha \mathbb{R}$, Σ^n and M^n, respectively.

Proposition 6.29 *Let $x : \Sigma^n \looparrowright M^n_f \times_\alpha \mathbb{R}$ be a hypersurface immersed into weighted Killing warped product $M^n_f \times_\alpha \mathbb{R}$. If $\Theta \in C^\infty(\Sigma)$ is the function defined in (1.33) then*

$$\Delta_f \Theta + \{\widetilde{\mathrm{Ric}}_f(N^*, N^*) - \frac{1}{\alpha}\widetilde{\mathrm{Hess}}\,\alpha(N^*, N^*) - \Theta^2 \frac{\widetilde{\Delta}_f(\alpha)}{\alpha^3} + |A|^2\}\Theta = -nY^\top(H_f),$$

where we are using the same notations of Proposition 6.21. In addition, if Σ^n is closed and both H_f and

$$\xi \ = \ \widetilde{\mathrm{Ric}}_f(N^*, N^*) - \frac{1}{\alpha}\widetilde{\mathrm{Hess}}\,\alpha(N^*, N^*) - \Theta^2 \frac{\widetilde{\Delta}_f(\alpha)}{\alpha^3} + |A|^2$$

are constants, then ξ is an eigenvalue of Δ_f on Σ^n, with eigenfunction Θ.

Proof. Firstly, from (1.12) we note that

$$\begin{aligned} -nY^\top(H) &= -Y^\top\left(nH_f - \langle \overline{\nabla} f, N\rangle\right) \\ &= -nY^\top(H_f) + Y^\top\langle \overline{\nabla} f, N\rangle \\ &= -nY^\top(H_f) + \overline{\mathrm{Hess}} f(Y, N) - \Theta\overline{\mathrm{Hess}} f(N, N) - \langle AY^\top, \nabla f\rangle. \end{aligned} \tag{6.58}$$

Moreover, with a straightforward computation we can show that

$$\nabla\Theta = -AY^\top - (\overline{\nabla}_N Y)^\top,$$

and, since f is invariant along the flow determined by Y, we get that

$$\begin{aligned} \langle \nabla\Theta, \overline{\nabla} f\rangle &= -\langle AY^\top + (\overline{\nabla}_N Y)^\top, \nabla f\rangle \\ &= -\langle AY^\top, \nabla f\rangle - \langle \overline{\nabla}_N Y, \overline{\nabla} f\rangle \\ &= -\langle AY^\top, \nabla f\rangle + \langle Y, \overline{\nabla}_N \overline{\nabla} f\rangle \\ &= -\langle AY^\top, \nabla f\rangle + \overline{\mathrm{Hess}} f(Y, N). \end{aligned} \tag{6.59}$$

Taking into account the equations (6.58) and (6.59) we get that

$$-nY^{\top}(H) = -nY^{\top}(H_f) - \Theta\overline{\mathrm{Hess}}f(N,N) + \langle\nabla\Theta, \overline{\nabla}f\rangle \tag{6.60}$$

On the other hand, from Proposition 2.12 of [8] we have

$$\Delta\Theta = -nY^{\top}(H) - \Theta\left(\overline{\mathrm{Ric}}(N,N) + |A|^2\right), \tag{6.61}$$

Therefore, from (1.11), (1.15), (6.49), (6.61) and (6.60) we obtain the result. ■

Our stability result stated in Theorem 6.30, that follows, gives us a characterization of f-stable H_f-hypersurfaces in $M^n_f \times_\alpha \mathbb{R}$ through the first eigenvalue of the drift Laplacian Δ_f, which extends a classic result of Barbosa, do Carmo and Eschenburg (see Proposition 2.13 of [8]).

Theorem 6.30 *Let $x : \Sigma^n \looparrowright M^n_f \times_\alpha \mathbb{R}$ be a closed H_f-hypersurface immersed into weighted Killing warped product $M^n_f \times_\alpha \mathbb{R}$. If*

$$\xi \;=\; \widetilde{\mathrm{Ric}}_f(N^*,N^*) - \frac{1}{\alpha}\,\widetilde{\mathrm{Hess}}\,\alpha(N^*,N^*) - \Theta^2\,\frac{\widetilde{\Delta}_f(\alpha)}{\alpha^3} + |A|^2$$

is constant then $x : \Sigma^n \looparrowright M^n_f \times_\alpha \mathbb{R}$ is f-stable if and only if ξ is the first eigenvalue of drift Laplacian Δ_f on Σ^n.

Proof. Since that ξ is constant, Proposition 6.29 guarantees that ξ is in the spectrum of the drift Laplacian Δ_f. So, let ξ_1 be the first eigenvalue of Δ_f on Σ^n. If $\xi = \xi_1$, then the variational characterization of λ_1 (see, for instance, Section 1 of [12]) gives

$$\xi = \min_{u \in \mathcal{G}\setminus\{0\}} \frac{-\displaystyle\int_{\Sigma^n} u\Delta_f(u)\,d\mu}{\displaystyle\int_{\Sigma^n} u^2\,d\mu},$$

where $\mathcal{G}$ is defined in (6.57). Then, from (6.44) and (6.45) we obtain that

$$\frac{d^2}{ds^2}\mathcal{F}_f(0)(u) = \int_{\Sigma^n}\{-u\Delta_f(u) - \xi u^2\}\,d\mu \;\geq\; (\xi - \xi)\int_{\Sigma^n} u^2\,d\mu = 0,$$

for any $u \in \mathcal{G}$ and, according to Remark 6.28, $x : \Sigma^n \looparrowright M^n_f \times_\alpha \mathbb{R}$ is f-stable.

Now suppose that $x : \Sigma^n \looparrowright M^n_f \times_\alpha \mathbb{R}$ is f-stable, which according to Remark 6.28 is equivalent to $\frac{d^2}{ds^2}\mathcal{F}_f(0)(u) \geq 0$ for all $u \in \mathcal{G}$. Let u be an eigenfunction associated to the first eigenvalue ξ_1 of the drift Laplacian Δ_f on Σ^n. Consequently, by (6.44) and (6.45) we get

$$0 \leq \frac{d^2}{ds^2}\mathcal{F}_f(0)(u) \;=\; (\xi_1 - \xi)\int_{\Sigma^n} u^2\,d\mu.$$

Therefore, since $\xi_1 \leq \xi$, we must have $\xi_1 = \xi$. ■

Riemannian warped product	Family of open sets	Q_0
$(-\infty,+\infty)\times_{\cosh\tau}\mathbb{S}^n(r)$ with $r>0$	$\{\Omega_\tau\}_{t\in(\tau_1,\tau_2]}$ with τ_1,τ_2 in $(-\infty,0)$ and $\tau_1<\tau_2$, or with τ_1,τ_2 in $(0,+\infty)$ and $\tau_1<\tau_2$	$-n$
$(0,+\infty)\times_{\sinh\tau}\mathbb{S}^n(r)$ with $0<r<1$	$\{\Omega_\tau\}_{\tau\in(\tau_1,\tau_2]}$ with τ_1,τ_2 in $(0,+\infty)$ and $\tau_1<\tau_2$	n
$(0,+\infty)\times_{\tau}\mathbb{S}^n(r)$ with $0<r<1$	$\{\Omega_\tau\}_{\tau\in(\tau_1,\tau_2]}$ with τ_1,τ_2 in $(0,+\infty)$ and $\tau_1<\tau_2$	n
$(-\pi/2,\pi/2)\times_{\cos\tau}\mathbb{S}^n(r)$ with $0<r<1$	$\{\Omega_\tau\}_{\tau\in(\tau_1,\tau_2]}$ with τ_1,τ_2 in $(-\pi/2,0)$ and $\tau_1<\tau_2$, or with τ_1,τ_2 in $(0,\pi/2)$ and $\tau_1<\tau_2$	n
$(0,\pi)\times_{\sin\tau}\mathbb{S}^n(r)$ with $0<r<1$	$\{\Omega_\tau\}_{\tau\in(\tau_1,\tau_2]}$ with τ_1,τ_2 in $(0,\pi/2)$ and $\tau_1<\tau_2$, or with τ_1,τ_2 in $(\pi/2,\pi)$ and $\tau_1<\tau_2$	n
$(0,\pi/2)\times_{\sin\tau+\cos\tau}\mathbb{S}^n(r)$ with $0<r<1$	$\{\Omega_\tau\}_{\tau\in(\tau_1,\tau_2]}$ with τ_1,τ_2 in $(0,\pi/4)$ and $\tau_1<\tau_2$, or with τ_1,τ_2 in $(\pi/4,\pi/2)$ and $\tau_1<\tau_2$	n

Table 6.2: Families that are locally rigid according to Theorem 6.9

Riemannian warped product	Family of open sets	Q_0
$(-\infty,+\infty)\times_{\cosh\tau}\mathbb{S}^n(r)$ with $r>0$	$\{\Omega_\tau\}_{\tau\in(\tau_1,\tau_2]}$ with τ_1, τ_2, δ_0 and η_0 such that $-\infty<t_1<\delta_0<0<\eta_0<\tau_2<+\infty$	$-n$
$(-\pi/2,\pi/2)\times_{\cos\tau}\mathbb{S}^n(r)$ with $0<r<1$	$\{\Omega_\tau\}_{\tau\in(\tau_1,\tau_2]}$ with τ_1, τ_2, δ_0 and η_0 such that $-\pi/2<t_1<\delta_0<0<\eta_0<\tau_2<\pi/2$	n
$(0,\pi)\times_{\sin\tau}\mathbb{S}^n(r)$ with $0<r<1$	$\{\Omega_\tau\}_{\tau\in(\tau_1,\tau_2]}$ with τ_1, τ_2, δ_0 and η_0 such that $0<\tau_1<\delta_0<\pi/2<\eta_0<\tau_2<\pi$	n
$(0,\pi/2)\times_{\sin\tau+\cos\tau}\mathbb{S}^n(r)$ with $0<r<1$	$\{\Omega_\tau\}_{\tau\in(\tau_1,\tau_2]}$ with τ_1, τ_2, δ_0 and η_0 such that $0<\tau_1<\delta_0<\pi/4<\eta_0<\tau_2<\pi/2$	n

Table 6.3: Families that admit a bifurcation instant according to Theorem 6.11

Riemannian warped product	Family of open sets	Q_0
$(-\infty,+\infty)\times_{\cosh\tau}\mathbb{S}^n(r)$ with $r>0$	$\{\Omega_\tau\}_{\tau\in(\tau_1,\tau_2]}$ with τ_1, τ_2, and τ_* such that $-\infty<\tau_1<\tau_*=0<\tau_2<+\infty$	$-n$
$(-\pi/2,\pi/2)\times_{\cos\tau}\mathbb{S}^n(r)$ with $0<r<1$	$\{\Omega_\tau\}_{\tau\in(\tau_1,\tau_2]}$ with τ_1, τ_2, and τ_* such that $-\pi/2<\tau_1<\tau_*=0<\tau_2<\pi/2$	n
$(0,\pi)\times_{\sin\tau}\mathbb{S}^n(r)$ with $0<r<1$	$\{\Omega_\tau\}_{\tau\in(\tau_1,\tau_2]}$ with τ_1, τ_2, and τ_* such that $0<\tau_1<\tau_*=\pi/2<\tau_2<\pi$	n
$(0,\pi/2)\times_{\sin\tau+\cos\tau}\mathbb{S}^n(r)$ with $0<r<1$	$\{\Omega_\tau\}_{\tau\in(\tau_1,\tau_2]}$ with τ_1, τ_2, and τ_* such that $0<\tau_1<\tau_*=\pi/4<\tau_2<\pi/2$	n

Table 6.4: Families that admit a bifurcation instant at τ_* in (τ_1,τ_2) according to Theorem 6.13

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