

Universidade Federal da Paraíba
Universidade Federal de Campina Grande
Programa Associado de Pós-Graduação em Matemática
Doutorado em Matemática

On Milnor classes of constructible functions

por

Mauri Pereira da Silva

João Pessoa - PB

Setembro/2019

On Milnor classes of constructible functions

por

Mauri Pereira da Silva [†]

sob orientação do

Prof. Dr. Roberto Callejas Bedregal

Tese apresentada ao Corpo Docente do Programa Associado de Pós-Graduação em Matemática - UFPB/UFCG, como requisito parcial para obtenção do título de Doutor em Matemática.

João Pessoa - PB

Setembro/2019

[†]Este trabalho contou com apoio financeiro da CAPES

Catálogo na publicação
Seção de Catalogação e Classificação

S586o Silva, Mauri Pereira da.

On Milnor classes of constructible functions / Mauri Pereira da Silva. - João Pessoa, 2019.

77 f.

Orientação: Roberto Callejas-Bedregal.

Tese (Doutorado) - UFPB/CCEN.


1. Número de Milnor. 2. Classe de Milnor. 3. Classe de Segre. 4. Classes características. 5. Variedades singulares. I. Callejas-Bedregal, Roberto. II. Título.

UFPB/BC

Universidade Federal da Paraíba
Universidade Federal de Campina Grande
Programa Associado de Pós-Graduação em Matemática
Doutorado em Matemática

Área de Concentração: Geometria complexa

Aprovada em:



Prof. Dr. Aurélio Menegon Neto



Prof. Dr. José Seade Kuri



Prof.^a Dr.^a Michelle Ferreira Zanchetta Morgado



Prof. Dr. Roberto Callejas Bedregal

Orientador

Tese apresentada ao Corpo Docente do Programa Associado de Pós-Graduação em Matemática - UFPB/UFCG, como requisito parcial para obtenção do título de Doutor em Matemática.

Setembro/2019

Resumo

O principal objetivo deste trabalho é apresentar uma generalização do importante invariante da Teoria das Singularidades, chamado número de Milnor. Tal generalização é o que chamamos de número de Milnor logarítmico. Bem como explicar sobre definições um pouco mais gerais no contexto de funções construtíveis, apresentando observações, exemplos e propriedades. Dentre os conceitos que trabalhamos estão também a classe de Fulton-Johnson, a classe de Schwartz-MacPherson, a classe de Milnor e a classe de Segre.

Palavras-chave: Número de Milnor; Classe de Milnor; Classe de Segre; Classes características; variedades singulares.

Abstract

The main goal of this thesis is to present a generalization of the important invariant of the singularity theory, called the Milnor number. Such generalization is what we call the logarithmic Milnor number. As well as to discuss about more general definitions in the context of constructible functions, presenting observations, examples and properties. Among the concepts we work on are also the Fulton-Johnson class, the Schwartz-MacPherson class, the Milnor class and the Segre class.

Keywords: Milnor number; Milnor class; Segre class; Characteristic classes; singular varieties.

Agradecimentos

A Deus por tudo.

Aos meus pais, Mário e Maria, pelo carinho e apoio.

A minha família.

A Jéssica pela paciência e apoio.

Ao professor Roberto pela orientação e paciência.

Aos professores José Seade, Aurélio Menegon e Michelle Morgado por aceitarem compor a banca.

Ao Departamento de Matemática.

Aos meus amigos do curso.

A CAPES pelo apoio financeiro.

Dedicatória

Aos meus pais Mário e Maria

Sumário

Introdução	1
1 Preliminary	7
1.1 Index of vector fields	7
1.2 Holomorphic vector bundles	8
1.3 Logarithmic forms and free divisors	9
1.4 Constructible functions	10
1.5 Intersection theory	11
1.5.1 Algebraic Schemes	11
1.5.2 Rational Equivalence	14
1.5.3 Gysin map for divisors	16
1.5.4 Chern classes	17
1.5.5 Refined Gysin homomorphisms	18
1.5.6 Segre Classes	20
1.6 Milnor number	21
1.7 Borel-Moore homology	23
1.8 Schwartz-MacPherson class	23
2 Logarithmic Milnor number and some generalizations	26
2.1 The logarithmic Milnor number	26
2.2 On the generalization of Schürmann for Milnor classes	36
3 Intersection product formulas relative to constructible functions	39
3.1 Product formulas for the Milnor class of constructible functions	39
3.2 On the diagonal embedding	43

3.3	Intersection product formulas relative to constructible functions	48
4	Segre classes relative to a constructible function	52
5	Apêndice A	57
5.1	Algebraic sets	57
5.2	Sheaves and schemes	58
5.3	Whitney stratification	60
5.4	Chern-Weil Theory	60
5.5	Derived categories	62
	Referências	63

Introdução

The Milnor's fibration theorem is one of the main tools for the topological study of the fibers of holomorphic functions near their critical points. Among the information obtained is a well-known invariant called the Milnor number. The classic definition of this number is purely topological. However, there is an algebraic version of Milnor number which is given as follows. Consider $f : (\mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}, 0)$ a germ of holomorphic function, with isolated critical point at 0. The Milnor number, denoted by $\mu(f)$, coincides with the number $\dim_{\mathbb{C}} \mathcal{O}_{n+1} / (\partial f / \partial z_0, \dots, \partial f / \partial z_n)$, where \mathcal{O}_{n+1} is the ring of germs of analytic functions at the origin. Among the generalizations of Milnor number we highlight Parusiński-Milnor number which is defined in a context not necessarily with isolated singularity. Let M be a n -dimensional compact complex manifold and let L be a holomorphic line bundle over M . Consider $X := v^{-1}(0)$ a divisor in M , where v is a regular holomorphic section of L . The Parusiński-Milnor number is defined by

$$\mu(X) = (-1)^{n-1} (\chi(M|L) - \chi(X)),$$

where for a vector bundle E over M ,

$$\chi(M|E) := \int_M c(E)^{-1} c_{top}(E) c(M) \cap [M].$$

In collaboration with P. Pragacz, Parusiński presented some interesting properties of the above number.

Characteristic classes are certain kinds of cohomology classes associated to vector bundles over spaces. The classic case is when we have a smooth manifold and their tangent bundle. This theory began in the year 1935 with the works of H. Whitney and E. Stiefel, arising then Stiefel-Whitney classes. We also have the Pontrjagin and

Euler classes in the real case. In 1946, S.S. Chern defined characteristic classes for complex vector bundles, which are called Chern classes. In a topological approach, obstruction theory can be used to define Chern classes. However, for a differential-geometric approach, Chern-Weil theory can be used to define Chern classes.

The first notions of Chern class for singular varieties appeared in the works of W.T. Wu and J. Mather. The obstruction approach of Chern classes is due to M.H. Schwartz, and R. MacPherson presented an axiomatic theory about Chern classes. Moreover, Macpherson gave a positive answer to the Deligne and Grothendieck's Conjecture on the existence of Chern classes seen as a natural transformation from the covariant functor of the constructible functions to the homology functor with a good behavior regarding pushforward. The MacPherson's original work was in context of complex algebraic varieties and on homology groups. For a context completely algebraic, by extending to varieties over an arbitrary field of characteristic zero, see [Ken]; for a context on Chow groups, see [Ful, 19.1.7]. Using the Alexander duality, one has that the Schwartz class and the MacPherson class coincide, being so-called Schwartz-MacPherson classes. The Fulton and Fulton-Johnson classes are other known generalizations. In the definition of these classes another class appears, called the Segre class. It is known that for complete intersections the Fulton classes and Fulton-Johnson classes coincide with the Chern class of the virtual bundle.

We can find relationship between the Schwartz-MacPherson class and the Fulton-Johnson class in works of A. Parusiński, P. Pragacz, J.P. Brasselet, D. Lehmann, J. Seade, T. Suwa, S. Yokura and P. Aluffi, see [PP3], [BLSS], [Y], [A4]. Motivating to define the notion of a Milnor class as being the difference, up to sign, between the Schwartz-MacPherson class and the Fulton-Johnson class. Explicitly, let X be a n -dimensional irreducible analytic (or algebraic) variety embedded in a smooth manifold M . The Milnor class of X is defined by

$$\mathcal{M}(X) = (-1)^{n-1}(c^{FJ}(X) - c^{SM}(X)),$$

where $c^{FJ}(X)$ denotes the Fulton-Johnson class of X and $c^{SM}(X)$ denotes the Schwartz-MacPherson class of X . An interesting fact is that the degree of Milnor class is equals the Parusiński-Milnor number, that is,

$$\mu(X) = \int_X \mathcal{M}(X).$$

We will present some definitions that are generalizations of some elements that we have explained above, as well as some properties and examples. To this end, this thesis was divided into five chapters, which we will describe briefly.

The **Chapter 1** is devoted to some preliminary facts and definitions. Among these facts and definitions are the notions of vector fields with isolated singularities, Poincaré-Hopf theorem, holomorphic vector bundles, hermitian metric, logarithmic forms, free divisors, constructible functions and Milnor number. Moreover, we organized a summary about intersection theory, with definitions and results used throughout subsequent chapters.

In the **Chapter 2**, we begin with a brief explanation of the Schwartz-MacPherson class and generalize a formula due to Parusiński and Pragacz involving the Euler characteristic. Then we present the definition of a logarithmic connection due to P. Deligne (Definition 2.2.2). Such a connection is used to extend the definition of μ -number due to Parusiński (Definition 2.2.4). The Theorem 2.2.3 due to Parusiński motivated us to Proposition 2.2.6. Let M be a n -dimensional compact connected complex manifold and let $X := v^{-1}(0)$ be a divisor in M , where v is a regular holomorphic section of L . Given D another divisor in M , the logarithmic Milnor number of X with poles along D or, simply, logarithmic Milnor number of X is defined by

$$\mu_D(X) = (-1)^n \left(\chi(X; \mathbf{1}_{X \setminus D}) - \int_M c(L)^{-1} c_1(L) c(\text{Der}_M(-\log D)) \cap [M] \right).$$

In the Proposition 2.2.13 we present the following relationship between logarithmic Milnor number and Parusiński-Milnor number:

$$\mu_D(X) = \mu(X) - (-1)^n \chi(X; \mathbf{1}_{X \cap D}) + (-1)^n \int_M c(L)^{-1} c_1(L) c_*(D).$$

Now we generalize our definition of logarithmic Milnor number to an arbitrary constructible function. We define the Milnor number relative to α as

$$\mu(Y; \alpha) = (-1)^{\dim Y} \left(\int_M c(E)^{-1} c_{\text{top}}(E) c_*(\alpha) - \chi(Y; \alpha|_Y) \right)$$

where Y is a closed subvariety of M of pure dimension given as zero set of a regular holomorphic section of a holomorphic vector bundle E over M and α is a constructible function on M .

Lastly we present a generalization of Milnor class due to J. Schürmann (Definition 2.3.4). For a regular embedding $\iota : X \hookrightarrow Z$ and α constructible function on Z . Schürmann defined the Milnor class of the pair $X \subset Z$ relative to α as

$$\mathcal{M}(X \subset Z; \alpha) = (-1)^{\dim X} (c(N_X Z)^{-1} \cap \iota^*(c_*(\alpha)) - c_*(\iota^*(\alpha))) \in H_*(X).$$

Assuming that Z is a smooth variety and X is the zero-scheme of a regular section of a vector bundle E on Z . We show that

$$\mu(X; \alpha) = \int_X \mathcal{M}(X; \alpha).$$

In a natural way, from the Schürmann's definition, one defines the Fulton-Johnson class of the pair $X \subset Z$ relative to α and the Schwartz-MacPherson class of the pair $X \subset Z$ relative to α .

The **Chapter 3** starts with the result due to M. Kwieciński (Theorem 3.1.1), which states the following: Let X and Y be manifolds and let α and β be constructible functions on X and Y , respectively. Then

$$c_*(\alpha \otimes \beta) = c_*(\alpha) \times c_*(\beta).$$

The Theorem 3.1.1 will be of great use throughout this text. We use the same to show Proposition 3.1.2 and Proposition 3.1.3. As a consequence of these propositions we get the Theorem 3.1.4, which generalizes a product formula due to T. Ohmoto and S. Yokura.

Let M be an n -dimensional compact complex analytic manifold. Define $M^{(r)} := M \times \cdots \times M$. And let $Z(t)$ be the zero set of a regular holomorphic section t of a holomorphic d -vector bundle E over $M^{(r)}$. Hence, $Z(t)$ is a closed subvariety of $M^{(r)}$ of dimension $nr - d$. Consider $\Delta : M \rightarrow M^{(r)}$ the diagonal morphism, which is a regular embedding of codimension $nr - n$. The morphism Δ induces the refined Gysin homomorphism

$$\Delta^! : H_{2k}(Z(t)) \rightarrow H_{2(k-nr+n)}(Z(\Delta^*(t))).$$

The Lemma 3.2.1, due to R. Callejas-Bedregal, M. F. Z. Morgado and J. Seade, is used in Propositions 3.2.2 and 3.2.3, which are generalizations for constructible functions of results involving the refined Gysin homomorphism above and the Fulton-Johnson

and Schwartz-MacPherson classes, found in [B-M-S]. And thus, the Proposition 3.2.4, which states that

$$\Delta^!(\mathcal{M}(Z(t); \alpha)) = (-1)^{nr-n} c \left((TM|_{Z(\Delta^*t)})^{\oplus r-1} \right) \cap \mathcal{M}(Z(\Delta^*t); \Delta^* \alpha),$$

follows as a consequence of these propositions, where α is a constructible function on $M^{(r)}$.

Now, for each $i = 1, \dots, r$, let E_i be a holomorphic vector bundle of rank d_i over M and let $X_i := s_i^{-1}(0)$ be a $(n - d_i)$ -dimensional local complete intersection, where s_i is a regular holomorphic section on E_i . Set $X := X_1 \cap \dots \cap X_r$. Then we generalize, for constructible functions, some intersection formulas due to R. Callejas-Bedregal, M. F. Z. Morgado and J. Seade, involving the Fulton-Johnson class, the Schwartz-MacPherson class and the Milnor class. Explicitly, for $\alpha_1, \dots, \alpha_r$ constructible functions on M , we have that

$$\begin{aligned} c^{FJ}(X; \alpha) &= c \left((TM|_X)^{\oplus r-1} \right)^{-1} \cap c^{FJ}(X_1; \alpha_1) \cdot \dots \cdot c^{FJ}(X_r; \alpha_r), \\ c^{SM}(X; \alpha) &= c \left((TM|_X)^{\oplus r-1} \right)^{-1} \cap c^{SM}(X_1; \alpha_1) \cdot \dots \cdot c^{SM}(X_r; \alpha_r) \end{aligned}$$

and

$$\begin{aligned} \mathcal{M}(X; \alpha) &= (-1)^{\dim X} c \left((TM|_X)^{\oplus r-1} \right)^{-1} \cap (c^{FJ}(X_1; \alpha_1) \cdot \dots \cdot c^{FJ}(X_r; \alpha_r) - \\ &\quad - c_*(X_1; \alpha_1) \cdot \dots \cdot c_*(X_r; \alpha_r)) \end{aligned}$$

where α denotes the constructible function $\alpha_1 \otimes \dots \otimes \alpha_r$.

In the **Chapter 4**, we look for a notion for Segre class relative to constructible functions that had "a good behavior" with the definition of Fulton-Johnson class relative to a constructible function due to Schürmann. This reason is for the following fact. Let X be a proper closed subscheme of a variety Y . Consider \tilde{Y} the blow-up of Y along X , $\tilde{X} = P(N_X Y)$ the exceptional divisor and $\eta: \tilde{X} \rightarrow X$ the projection, where $N_X Y$ is the normal bundle. The Segre class of X in Y is characterized by

$$s(X, Y) = \sum_{i \geq 0} \eta_* \left(c_1(\mathcal{O}(1))^i \cap [P(N_X Y)] \right)$$

where $\mathcal{O}(1)$ is the canonical line bundle on $P(N_X Y)$. Suppose that $Y = M$ is non-singular. The Fulton class of X is defined by

$$c^F(X) = c(TM|_X) \cap s(X, M).$$

In this way, motivated by Schürmann's definition, we define the Segre class of X relative to a constructible function α on M as being

$$s(X \subset M; \alpha) = \eta_* \left(\sum_{i \geq 0} c_1(\mathcal{O}_{\mathbb{P}(N_{XM})}(1))^i \cap \eta^*(c(TM|_X)^{-1} \cap \iota^* c_*(\alpha)) \right),$$

where $\iota : X \hookrightarrow M$ is the regular embedding. And we define the Fulton class of X relative to α as

$$c^F(X \subset M; \alpha) = c(TM|_X) \cap s(X \subset M; \alpha).$$

Note that, this definition coincides with the definition of Fulton-Johnson class relative to a constructible function due to Schürmann. At long last, we show two results. Proposition 4.0.3 is about the Segre class of a product of schemes and Proposition 4.0.5 is about a pullback of a Segre class.

Finally, the **Chapter 5** is dedicated to some auxiliary definitions and properties.

Capítulo 1

Preliminary

1.1 Index of vector fields

As references we cite [B-S-S] and [M-T].

Let $v = \sum_{i=1}^m f_i \partial/\partial x_i$ be a vector field on an open $U \subset \mathbb{R}^m$. One says that a point $p \in U$ is a *singularity* of v if $f_i(p) = 0$ for all $i = 1, \dots, m$. The singularity is *isolated* if at every point x near p there is at least one component of v which is not zero. Consider v a continuous vector field on U with an isolated singularity at p , and consider \mathbb{S}_ϵ a small sphere in U around p . The Poincaré-Hopf index of v at p , denoted by $\text{Ind}(v, p)$, is defined to be the degree of the Gauss map $v/\|v\|$ from \mathbb{S}_ϵ to the unit sphere in \mathbb{R}^m .

Now let M be an m -dimensional smooth manifold. A vector field v on M is locally expressed as above and one defines of the local index at an isolated singularity extending in the natural way. Note that, this definition not depend on the local chart. The *total index* of v , denoted by $\text{Ind}_M v$, is the sum of all its local indices at the singular points.

Theorem 1.1.1 (Poincaré-Hopf) *Let M be a closed oriented manifold and let v be a continuous vector field on M with finitely many isolated singularities. Then,*

$$\text{Ind}_M v = \chi(M)$$

independently of v , where $\chi(M)$ denotes the Euler characteristic of M .

These notions can be extended to sections of a oriented vector bundle as follows. Let E be an oriented vector bundle over a compact oriented smooth manifold M , and let $\pi : E \rightarrow M$ be its projection. Suppose that rank of E is equals to $\dim M = m$. Consider s_0 the zero section of E and s another arbitrary smooth section of E .

Definition 1.1.2 Let $p \in M$ be a zero for s , with $s(p) = s_0(p)$. One says that s is transversal to s_0 at p if

$$D_p s(T_p M) \cap D_p s_0(T_p M) = 0 \tag{1.1}$$

and s is called transversal to s_0 if this holds for all zeros of s .

Say that s is transversal to s_0 at p is equivalent to statement that $D_p s(T_p M)$ is the graph of a linear isomorphism A from $T_p M$ to the fiber E_p . By assumption, both vector spaces are oriented. In this way, one defines the local index $\iota(s; p)$ to be $+1$ if A preserves the orientations, and -1 if not. In particular, (1.1) forces p to be an isolated zero of s . Moreover, if s is transversal to the zero section then the number of zeros of s is finite, since M was assumed compact.

Theorem 1.1.3 If s is transverse to the zero section, then

$$I(e(E)) = \sum_p \iota(s; p)$$

where the sum runs over the zeros of s , $e(E)$ is the Euler class of E and $I : H^m(M) \rightarrow \mathbb{R}$.

Proof. See [M-T, Theorem 21.9 and Theorem 21.11]. ■

As a consequence, for any oriented compact smooth manifold M , one has $I(e(TM)) = \chi(M)$.

1.2 Holomorphic vector bundles

Let M be a differentiable manifold. A C^∞ complex vector bundle on M consist of a family $\{E_x\}_{x \in M}$ of complex vector spaces, together with a C^∞ manifold structure on $E = \cup_{x \in M} E_x$ such that the projection map $\pi : E \rightarrow M$ taking E_x to x is C^∞ ; and for every $x_0 \in M$ there is as open set U in M containing x_0 and a diffeomorphism $\varphi_U : \pi^{-1}(U) \rightarrow U \times \mathbb{C}^r$, called trivialization, taking the vector space E_x isomorphically onto $\{x\} \times \mathbb{C}^r$ for each $x \in U$. The dimension of the fibers E_x of E is called the rank

of E . A vector bundle of rank 1 is called a *line bundle*. A section s of the vector bundle E over $U \subset M$ is a C^∞ map $s : U \rightarrow E$ such that $s(x) \in E_x$ for all $x \in U$. A frame for E over $U \subset M$ is a collection s_1, \dots, s_r of sections of E over U such that $\{s_1(x), \dots, s_r(x)\}$ is a basis for E_x for all $x \in U$.

Now assume that M is a complex manifold. A holomorphic vector bundle $\pi : E \rightarrow M$ is a complex vector bundle together with the structure of a complex manifold on E , such that for any $x \in M$ there is an open set U in M with $x \in U$ and a trivialization $\varphi_U : \pi^{-1}(U) \rightarrow U \times \mathbb{C}^r$ which is a biholomorphism map of complex manifolds. A section s of the holomorphic bundle E over $U \subset M$ is said to be holomorphic if $s : U \rightarrow E$ is a holomorphic map. A frame $\underline{s} = s_1, \dots, s_r$ is called holomorphic if each s_i is a holomorphic map. A *hermitian metric* on E is defined as a positive definite hermitian inner product

$$(\cdot, \cdot)_z : T'_z M \otimes \overline{T'_z M} \rightarrow \mathbb{C}$$

where $T'_z M$ denotes the holomorphic tangent space at z for each $z \in M$, depending smoothly on z , that is, for local coordinates z on M the functions $h_{ij}(z) = (\partial/\partial z_i, \partial/\partial z_j)$ are C^∞ .

1.3 Logarithmic forms and free divisors

Let U be a domain of \mathbb{C}^n , and let $D \subseteq U$ be a divisor of U defined by an equation $h(z) = 0$, where h is holomorphic on U . A meromorphic q -form ω on U is called a q -form with logarithmic pole along D or logarithmic q -form (along D) if it satisfies the following equivalent conditions:

- (a) $h\omega$ and $hd\omega$ are holomorphic on U .
- (b) $h\omega$ and $dh \wedge \omega$ are holomorphic on U .
- (c) There exists a holomorphic function $g(z)$, a holomorphic $(q-1)$ -form ξ and a holomorphic q -form η on U , such that

- $\dim_{\mathbb{C}} D \cap \{z \in U : g(z) = 0\} \leq n - 2$,
- $g\omega = \frac{dh}{h} \wedge \xi + \eta$.

- (d) There exists an $(n - 2)$ -dimensional analytic set $A \subset D$ such that the germ of ω at any point $p \in D \setminus A$ belongs to $\frac{dh}{h} \wedge \Omega_{U,p}^{q-1} + \Omega_{U,p}^q$, where $\Omega_{U,p}^q$ denotes the module of germs of holomorphic q -forms on U at p .

Let S an n -dimensional complex manifold and D be a divisor of D . Consider $h_p = 0$ a reduced equation for D , locally at $p \in D$. A meromorphic q -form ω is logarithmic along D at p if $h_p\omega$ and $h_p d\omega$ are holomorphic. We denote $\Omega_{S,p}^q(\log D) := \{\text{germ of logarithmic } q\text{-form at } p\}$ and $\Omega_S^q(\log D) := \bigcup_{p \in S} \Omega_{S,p}^q(\log D)$.

Definition 1.3.1 *Let S an n -dimensional complex manifold and D be a divisor of D . Consider $h_p = 0$ a reduced equation for D , locally at $p \in D$. A holomorphic vector field δ on S is logarithmic if it satisfies the following equivalent conditions:*

- (a) *For any smooth point $p \in D$, the tangent vector $\delta(p)$ of p is tangent to D ,*
- (b) *For any point $p \in D$, the derivation δh_p of the local equation for D belongs to the ideal $(h_p)\mathcal{O}_{S,p}$.*

We denote $\text{Der}_{S,p}(\log D) := \{\delta : \text{germ of a holomorphic vector field on } S \text{ at } p \text{ such that } \delta(h_p) \in (h_p)\}$ and $\text{Der}_S(\log D) := \bigcup_{p \in S} \text{Der}_{S,p}(\log D)$. Note that, $\text{Der}_S(\log D)$ is a coherent \mathcal{O}_S -submodule of Der_S , where Der_S is a sheaf of holomorphic vector fields on S .

Definition 1.3.2 *Let D be a divisor in S and let $p \in D$, we say that D is a free divisor in p if $\Omega_{S,p}^1(\log D)$ (or its dual $\text{Der}_{S,p}(\log D)$) is a $\mathcal{O}_{S,p}$ -free module. Moreover, we say that D is a free divisor if $\Omega_{S,p'}^1(\log D)$ (or its dual $\text{Der}_{S,p'}(\log D)$) is a $\mathcal{O}_{S,p'}$ -free module for all $p' \in D$.*

1.4 Constructible functions

The references are [Scha] and [K-S].

Let X be a real analytic manifold. A function $\alpha : X \rightarrow \mathbb{Z}$ is called *constructible* if for each $m \in \mathbb{Z}$, the set $\alpha^{-1}(m)$ is subanalytic and the family $\{\alpha^{-1}(m)\}_{m \in \mathbb{Z}}$ is locally finite, or equivalent, by triangulation theorem, if there exists a locally finite covering $X = \bigcup_{i \in I} X_i$ and

$$\alpha = \sum m_i \mathbf{1}_{X_i}$$

where m_i are integers, X_i are (closed) analytic subset of X and $\mathbf{1}_{X_i}$ is the characteristic functions of X_i . The set of all constructible functions on X , denoted by $CF(X)$,

is endowed with a structure of algebra. If α has support compact, then all X_i are compacts. Thus one defines the *Weighted Euler characteristic* of α as been

$$\chi(X; \alpha) = \sum_i m_i \chi(X_i),$$

where χ is the topological Euler characteristic. Now, let $f : X \rightarrow Y$ be a morphism of analytic manifolds. Given β a constructible function on Y , one defines the *inverse image*, or *pullback*, of β by f as been the constructible function $f^*\beta$ on X defined by $f^*\beta(x) = \beta(f(x))$, for all $x \in X$. Assume that $f : X \rightarrow Y$ is proper morphism on the support of a constructible function α on X , one defines the *direct image*, or *pushforward*, as been the constructible function $f_*\alpha$ on Y defined by

$$f_*(\alpha)(y) = \chi(f^{-1}(y); \alpha|_{f^{-1}(y)}),$$

for all $y \in Y$. Note that, given a analytic subset W of X , the characteristic function $\mathbf{1}_W$ is constructible.

1.5 Intersection theory

The references are [Ful] and [H].

1.5.1 Algebraic Schemes

We say that a scheme X is *algebraic* over a field K if there is a morphism of finite type from X to $\text{Spec}(K)$. This means that, X has a finite covering by affine sets whose coordinate rings are finitely generated K -algebras. We denote the coordinate ring of an affine open U by $A(U)$. However, the word *scheme* means an algebraic scheme over some field. A closed subscheme Y of a scheme X is defined by an ideal sheaf $\mathcal{I}(Y)$ in the structure sheaf \mathcal{O}_X of X .

A *variety* is a reduced and irreducible (integral) algebraic scheme. A *subvariety* V of a scheme X is a reduced and irreducible closed subscheme of X . One has that a subvariety V corresponds to a prime ideal in the coordinate ring of any affine open set meeting V . The *local ring* of X along V , denoted by $\mathcal{O}_{V,X}$, is the localization of such a coordinate ring at the corresponding prime ideal; its maximal ideal is denoted by $\mathcal{M}_{V,X}$. The *function field* of V , denoted by $R(V)$, is the residue field $\mathcal{O}_{V,X}/\mathcal{M}_{V,X}$.

The *dimension* of a scheme X , denoted by $\dim X$, is the maximum length n of a chain

$$\emptyset \subsetneq V_0 \subsetneq V_1 \subsetneq \dots \subsetneq V_n \subset X$$

of subvarieties of X . A scheme X is *pure dimensional* if all irreducible components of X have the same dimension. A *point* on a scheme X is a 0-dimensional subvariety of X . We say that a point p of X is *regular* if $\mathcal{O}_{p,X}$ is a regular local ring. The open set of regular points in X is denoted by X_{reg} .

The affine n -space, denoted by \mathbb{A}^n , is the affine variety whose coordinate ring is the polynomial ring $K[x_1, \dots, x_n]$. The subscheme of \mathbb{A}^n defined by an ideal $I = (f_1, \dots, f_n)$ in $K[x_1, \dots, x_n]$ is denoted by $V(I)$.

A *morphism* $f : X \rightarrow Y$ of algebraic schemes is assumed to be compatible with the structure morphism to $\text{Spec}(K)$, where K is the ground field. If f maps an affine open subset U' of X into an affine open subset U of Y , then f corresponds to a homomorphism $f^* : A(U) \rightarrow A(U')$ of K -algebras.

Let $f : X \rightarrow S$ and $g : Y \rightarrow S$ be morphisms. The fibre product of X and Y over S , denoted by $X \times_S Y$, comes equipped with projections $p : X \times_S Y \rightarrow X$ and $q : X \times_S Y \rightarrow Y$ which satisfy the following universal property: for any scheme Z with morphisms $u : Z \rightarrow X$ and $v : Z \rightarrow Y$ such that $f \circ u = g \circ v$, there is a unique morphism $(u, v) : Z \rightarrow X \times_S Y$ such that $p \circ (u, v) = u$ and $q \circ (u, v) = v$. A commutative square of morphism

$$\begin{array}{ccc} X \times_S Y & \xrightarrow{q} & Y \\ p \downarrow & & \downarrow g \\ X & \xrightarrow{f} & S \end{array}$$

is called a *fibre square*. In a *fibre diagram*, all squares appearing in the diagram are required to be fibre squares. When $S = \text{Spec}(K)$, where K is the ground field, it is called *Cartesian product* of X and Y , denotes $X \times Y$ in place of $X \times_S Y$.

A morphism $f : X \rightarrow Y$ is *separated* if the diagonal morphism from X to $X \times_Y X$ is a closed imbedding. We say that a morphism $f : X \rightarrow Y$ is *proper* if it is separated, and universally closed, i.e., for all $Y' \rightarrow Y$, the induced morphism from $X \times_Y Y'$ to Y' takes closed sets to closed sets. A scheme is *complete* if the structural morphism to $\text{Spec}(K)$ is proper.

A morphism $f : X \rightarrow Y$ is *flat* if for $U \subset Y$, $U' \subset X$ affine open sets with $f(U') \subset U$, the induced map $f^* : A(U) \rightarrow A(U')$ makes $A(U')$ a flat $A(U)$ -module. A morphism $f : X \rightarrow Y$ has *relative dimension* n if for all subvarieties V of Y , and all irreducible components V' of $f^{-1}(V)$, $\dim V' = \dim V + n$. A morphism $f : X \rightarrow Y$ is called *smooth* if f is flat of some relative dimension n , and the sheaf of relative differentials $\Omega_{X/Y}^1$ is a locally free sheaf of rank n . We say that a scheme X is *nonsingular*, or *smooth*, if it is smooth over $\text{Spec}(\mathbb{K})$. If $f : X \rightarrow Y$ is smooth of relative dimension n , the *relative tangent bundle*, denoted by $T_{X/Y}$, is the vector whose sheaf of sections is the dual bundle to $\Omega_{X/Y}^1$. When $Y = \text{Spec}(\mathbb{K})$, denotes T_X in place of $T_{X/Y}$.

Let X be a scheme. A scheme E equipped with a morphism $\pi : E \rightarrow X$ is called a *vector bundle* of rank r on X if there are an open covering $\{U_i\}$ of X and isomorphisms Φ_i of $\pi^{-1}(U_i)$ with $U_i \times \mathbb{A}^r$ over U_i , such that over $U_i \cap U_j$ the composites $\varphi_i \circ \varphi_j^{-1}$ are linear. A *section* of E is a morphism $s : X \rightarrow E$ such that $\pi \circ s = \text{id}_X$. Several basic operations are defined for vector bundles, compatibly with the corresponding notions for sheaves: direct sum $E \oplus F$, tensor product $E \otimes F$, exterior product $\bigwedge^i E$, symmetric product $\text{Sym}^i E$, dual bundle E^\vee , pull-back f^*E for a morphism $f : X' \rightarrow X$. If E is a vector bundle of rank n on X , then its sheaf of sections of E is a locally free sheaf \mathcal{E} of \mathcal{O}_X -modules of rank r . Conversely, for any locally free coherent sheaf \mathcal{E} of rank n on X , one can produce a vector bundle $E = \text{Spec}(\text{Sym}(\mathcal{E}^\vee))$. The trivial bundle of rank one on X is often denoted by 1 . Lastly, a *line bundle* is a vector bundle L of rank one.

Let X be a closed subscheme of a scheme Y , defined by an ideal sheaf \mathcal{I} . The blow-up of Y along X , denoted by $Bl_X Y$, is the projective cone over Y of the sheaf of \mathcal{O}_Y -algebras $\bigoplus_{n \geq 0} \mathcal{I}^n$, that is, $Bl_X Y = \text{Proj}(\bigoplus_{n \geq 0} \mathcal{I}^n)$. Let us denote $\tilde{Y} = Bl_X Y$ and π the projection from \tilde{Y} to Y . The canonical invertible sheaf $\mathcal{O}(1)$ on the projective cone \tilde{Y} is the ideal sheaf of $\pi^{-1}(X)$, which is a Cartier divisor on \tilde{Y} , called the *exceptional divisor*. Let $\tilde{X} = \pi^{-1}(X)$. Note that, one has $\tilde{X} = \text{Proj}((\bigoplus_{n \geq 0} \mathcal{I}^n) \otimes_{\mathcal{O}_Y} \mathcal{O}_X) = \text{Proj}(\bigoplus_{n \geq 0} \mathcal{I}^n / \mathcal{I}^{n+1})$, which is the projective normal cone to X in Y , denoted by $P(C_X Y)$.

1.5.2 Rational Equivalence

Let X be an algebraic scheme. A k -cycle on X is a finite sum $\sum n_i[V_i]$, where the V_i are k -dimensional subvarieties of X and the n_i are integers. The free abelian group on the k -dimensional subvarieties of X is called group of k -cycles on X , which we denote by $Z_k X$. Consider a $(k+1)$ -dimensional subvariety W of X and $r \in R(W)^*$. Choose f and g in $\mathcal{O}_{V,W}$ such that $r = f/g$. The order of r along V is defined by

$$\text{ord}_V(r) = l(\mathcal{O}_{V,W}/(f)) - l(\mathcal{O}_{V,W}/(g)).$$

Define a k -cycle $[\text{div}(r)]$ on X by $[\text{div}(r)] = \sum \text{ord}_V(r)[V]$, the sum runs over all codimension one subvarieties V of W . Moreover, a k -cycle α is rationally equivalent to zero, denoted by $\alpha \sim 0$, if there are a finite number of $(k+1)$ -dimensional subvarieties W_i of X and $r_i \in R(W_i)^*$ such that $\alpha = \sum [\text{div}(r_i)]$. One denotes by $\text{Rat}_k(X)$ the group of The cycles rationally equivalent to zero form a subgroup $\text{Rat}_k(X)$ of $Z_k(X)$. The group of k -cycles modulo rational equivalence on X is given by

$$A_k X = Z_k X / \text{Rat}_k X.$$

Let $f : X \rightarrow Y$ be a proper morphism. Given any subvariety V of X , the image $W = f(V)$ is a (closed) subvariety of Y . It is known that if W has the same dimension as V then the induced imbedding of $R(W)$ in $R(V)$ is a finite field extension. Set

$$\text{deg}(V/W) = \begin{cases} [R(V) : R(W)] & \text{if } \dim W = \dim V \\ 0 & \text{if } \dim W < \dim V, \end{cases}$$

where $[R(V) : R(W)]$ denotes the degree of the field extension. We define $f_*([V]) = \text{deg}(V/W)[W]$. We can extend linearly to a homomorphism $f_* : Z_k X \rightarrow Z_k Y$.

Theorem 1.5.1 *Let $f : X \rightarrow Y$ be a proper morphism and let α be a k -cycle on X , such that α is rationally equivalent to zero. Then $f_*\alpha$ is rationally equivalent to zero on Y . Therefore, there exists an induced homomorphism*

$$f_* : A_k X \rightarrow A_k Y.$$

So the A_ is a covariant functor for proper morphisms.*

Proof. See [Ful, Theorem 1.4]. ■

Suppose that X is complete, that is, X is proper over $S = \text{Spec}(K)$, with K being the ground field. Consider $\alpha = \sum_P n_P[P]$ a zero-cycle on X .

Definition 1.5.2 *The degree of α , denoted $\deg(\alpha)$, or $\int_X \alpha$, is defined by*

$$\deg(\alpha) = \int_X \alpha = \sum_P n_P [R(P) : K].$$

Equivalently, $\deg(\alpha) = p_*(\alpha)$, where p is the structure morphism from X to S , and $A_0 S = \mathbb{Z}[S]$ is identified with \mathbb{Z} . One can extend the degree homomorphism to all of $A_* X = \bigoplus_{k=0}^{\dim X} A_k X$,

$$\int_X : A_* X \longrightarrow \mathbb{Z}$$

with $\int_X \alpha = 0$ if $\alpha \in A_* X$, $k > 0$. Now, for any morphism $f : X \longrightarrow Y$ of complete schemes, and any $\alpha \in A_* X$,

$$\int_X \alpha = \int_Y f_*(\alpha).$$

Now, let X be any scheme and let X_1, \dots, X_r be the irreducible components of X . Note that, the local rings $\mathcal{O}_{X_i, X}$ are all zero-dimensional. We define the geometric multiplicity m_i of X_i in X as being the length of $\mathcal{O}_{X_i, X}$, that is, $m_i = l_{\mathcal{O}_{X_i, X}}(\mathcal{O}_{X_i, X})$. The fundamental cycle $[X]$ of X is the cycle

$$[X] = \sum_{i=1}^r m_i [X_i].$$

Let $f : X \longrightarrow Y$ be a flat morphism of relative dimension n . For any subvariety V of Y , set $f^*[V] = [f^{-1}(V)]$, where $f^{-1}(V)$ is the inverse image scheme, which is a subscheme of X of pure dimension $\dim(V) + n$. By linearity, we can extend to *pull-back homomorphisms*

$$f^* : Z_k Y \longrightarrow Z_{k+n} X.$$

Theorem 1.5.3 *Let $f : X \longrightarrow Y$ be a flat morphism of relative dimension n and let α be a k -cycle on Y which is rationally equivalent to zero. Then $f^*\alpha$ is rationally equivalent to zero in $Z_{k+n} X$.*

Proof. See [Ful, Theorem 1.7]. ■

Consider X and Y two algebraic schemes over a field. Denote by $X \times Y$ the Cartesian (fiber) product of X and Y over the ground field. The *exterior product*

$$Z_k X \otimes Z_l Y \xrightarrow{\times} Z_{k+l}(X \times Y)$$

is defined by the formula $[V] \times [W] = [V \times W]$, for V, W subvarieties of X, Y , respectively; and extending bilinearly to general cycles.

Proposition 1.5.4 (a) If $\alpha \sim 0$ or $\beta \sim 0$, then $\alpha \times \beta \sim 0$.

(b) Let $f : X' \rightarrow X$, $g : Y' \rightarrow Y$ be morphisms between algebraic schemes over a field. Denote by $f \times g$ the induced morphism from $X' \times Y'$ to $X \times Y$.

(i) If f and g are proper, then $f \times g$ is proper, and

$$(f \times g)_*(\alpha \times \beta) = f_*\alpha \times g_*\beta$$

for all cycles $\alpha \in X'$ and $\beta \in Y'$.

(ii) If f and g are flat of relative dimensions m and n , respectively, then $f \times g$ is flat of relative dimension $m + n$, and

$$(f \times g)^*(\alpha \times \beta) = f^*\alpha \times g^*\beta$$

for all cycles $\alpha \in X'$ and $\beta \in Y'$.

Proof. See [Ful, Proposition 1.10]. ■

Consequently, there are *exterior products*

$$A_k X \otimes A_l Y \xrightarrow{\times} A_{k+l}(X \times Y).$$

1.5.3 Gysin map for divisors

Let D be an effective Cartier divisor on a scheme X and $i : D \rightarrow X$ be the inclusion. There are *Gysin homomorphisms*

$$i^* : Z_k X \rightarrow A_{k-1} D$$

given by $i^*(\alpha) = D \cdot \alpha$, where $D \cdot \alpha$ is the intersection class in $A_{k-1} D$.

Proposition 1.5.5 (a) If α is rationally equivalent to zero on X , then $i^*\alpha = 0$. Thus, there are induced homomorphisms

$$i^* : A_k X \rightarrow A_{k-1} D.$$

(b) If α is a k -cycle on X , then

$$i_* i^*(\alpha) = c_1(\mathcal{O}_X(D)) \cap \alpha.$$

(c) If α is a k -cycle on D , then

$$i_* i^*(\alpha) = c_1(i^* \mathcal{O}_X(D)) \cap \alpha.$$

(d) If X is purely n -dimensional, then

$$i^*[X] = [D] \in A_{n-1}D.$$

(e) If L is a line bundle on X and α is a k -cycle on X , then

$$i^*(c_1(L) \cap \alpha) = c_1(i^*L) \cap i^*(\alpha) \in A_{k-2}D.$$

Proof. See [Ful, Proposition 2.6]. ■

1.5.4 Chern classes

Let E be a vector bundle on X of rank r . We define *Segre class* operators $s_i(E)$,

$$s_i(E) \cap _ : A_k(X) \longrightarrow A_{k-i}(X)$$

as follow. Consider $p : P(E) \longrightarrow X$ the projective bundle of E , $\mathcal{O}_E(1)$ the canonical line bundle on $P(E)$, and α in $A_k X$. Set

$$s_i(E) \cap \alpha = p_*(c_1(\mathcal{O}_E(1))^{r-1+i} \cap p^*\alpha).$$

where p_* is flat

Note that $s_i(E) = 0$ for $i < 0$ and $s_0(E) = 1$. We define *Chern class* operators

$$c_i(E) \cap _ : A_k(X) \longrightarrow A_{k-i}(X)$$

formally by $1 + c_1(E) + c_2(E) + \cdots = (1 + s_1(E) + s_2(E) + \cdots)^{-1}$. Explicitly,

$$\begin{aligned} c_0(E) &= 1, & c_1(E) &= -s_1(E) \\ c_2(E) &= s_2(E)^2 - s_1(E), \cdots \\ c_n(E) &= -s_1(E)c_{n-1}(E) - s_2(E)c_{n-2}(E) - \cdots - s_n(E). \end{aligned}$$

The *total Chern class* of E is the sum

$$c(E) := 1 + c_1(E) + \cdots + c_r(E).$$

Theorem 1.5.6 *We have that:*

(a) (Vanishing) For all vector bundles E on X , all $i > \text{rank}(E)$,

$$c_i(E) = 0.$$

(b) (Commutativity) For all vector bundles E, F on X , integers i, j , and cycles α on X ,

$$c_i(E) \cap (c_j(F) \cap \alpha) = c_j(F) \cap (c_i(E) \cap \alpha).$$

(c) (Projection formula) Let E be a vector bundle on X and let $f : X' \rightarrow X$ be a proper morphism. For all cycles α on X' , and integers i , we have

$$f_*(c_i(f^*E) \cap \alpha) = c_i(E) \cap f_*(\alpha).$$

(d) (Pull-back) Let E be a vector bundle on X and let $f : X' \rightarrow X$ be a flat morphism. For all cycles α on X' , and integers i , we have

$$c_i(f^*E) \cap f^*\alpha = f^*(c_i(E) \cap \alpha).$$

(e) (Whitney sum) For any exact sequence

$$0 \rightarrow E' \rightarrow E \rightarrow E'' \rightarrow 0$$

of vector bundles on X , we have

$$c(E) = c(E') \cdot c(E''),$$

that is

$$c_k(E) = \sum_{i+j=k} c_i(E')c_j(E'').$$

(f) (Normalization) Let L be a line bundle on X and let D be a Cartier divisor on X with $\mathcal{O}(D) \cong L$. Then

$$c_1(L) \cap [X] = [D].$$

Proof. See [Ful, Theorem 3.2]. ■

1.5.5 Refined Gysin homomorphisms

Let $i : X \rightarrow Y$ be a regular imbedding of codimension d , and let $f : Y' \rightarrow Y$ be a morphism. Form the fibre square

$$\begin{array}{ccc} X' & \xrightarrow{j} & Y' \\ g \downarrow & & \downarrow f \\ X & \xrightarrow{i} & Y \end{array}$$

Now, we define homomorphisms $i^! : Z_k Y' \rightarrow A_{k-d} X'$ given by $i^! (\sum n_i [V_i]) = \sum n_i X \cdot V_i$, where $X \cdot V_i$ is the intersection product. The induced homomorphisms

$$i^! : A_k Y' \rightarrow A_{k-d} X'$$

is called *refined Gysin homomorphism*.

Theorem 1.5.7 Consider a fibre diagram

$$\begin{array}{ccc} X'' & \xrightarrow{i''} & Y'' \\ q \downarrow & & \downarrow p \\ X' & \xrightarrow{i'} & Y' \\ g \downarrow & & \downarrow f \\ X & \xrightarrow{i} & Y \end{array}$$

where i is a regular imbedding of codimension d .

(a) (Push-forward) If p is proper, $\alpha \in A_k Y''$, then

$$i^! p_*(\alpha) = q_*(i^! \alpha)$$

in $A_{k-d} X'$.

(b) (Pull-back) If p is flat of relative dimension n , and $\alpha \in A_k Y'$, then

$$i^! p^*(\alpha) = q^*(i^! \alpha)$$

in $A_{k+n-d} X''$.

(c) (compatibility) Let α be another regular imbedding of codimension d . If $\alpha \in A_k Y''$, then

$$i^! \alpha = i'^! \alpha$$

in $A_{k-d} X''$.

Proof. See [Ful, Theorem 6.2]. ■

Proposition 1.5.8 Let $i : X \rightarrow Y$ be a regular imbedding of codimension d ,

$$\begin{array}{ccc} X' & \xrightarrow{i'} & Y' \\ \downarrow & & \downarrow \\ X & \xrightarrow{i} & Y \end{array}$$

a fibre square, and let F be a vector bundle on Y' . Then, for all $\alpha \in A_k(Y')$, and all $m \geq 0$,

$$i^!(c_m(F) \cap \alpha) = c_m(i'^* F) \cap i^!(\alpha)$$

in $A_{k-d-m}(X')$.

Proof. See [Ful, Proposition 6.3]. ■

Example 1.5.9 [Ful, Example 6.3.4] Consider E a vector bundle of rank r on a scheme Y and consider s a regular section of E . Then, the inclusion ι of the zero-scheme $X = Z(s)$ in Y is a regular imbedding of codimension r and $N_X Y$ is the restriction of E to X . If $f : Y' \rightarrow Y$ is a morphism, form the fibre square

$$\begin{array}{ccc} X' & \xrightarrow{j} & Y' \\ g \downarrow & & \downarrow f \\ X & \xrightarrow{\iota} & Y \end{array}$$

Then

$$j_* i^! (\alpha) = c_r(f^* E) \cap \alpha$$

for all $\alpha \in A_* Y'$.

Example 1.5.10 Let $i_j : X_j \rightarrow Y_j$ be regular imbedding of codimensions d_j , $j = 1, \dots, r$. Let $f_j : Y'_j \rightarrow Y_j$ be morphisms, $\alpha_j \in A_{k_i}(Y'_j)$. Then $i_1 \times \dots \times i_r$ is a regular imbedding of $X_1 \times \dots \times X_r$ in $Y_1 \times \dots \times Y_r$, of codimension $\sum d_i$, and

$$(i_1 \times \dots \times i_r)^! (\alpha_1 \times \dots \times \alpha_r) = i_1^! (\alpha_1) \times \dots \times i_r^! (\alpha_r)$$

in $A_{\sum(k_j - d_j)}(X'_1 \times \dots \times X'_r)$, with $X'_i = X_i \times_{Y_i} Y'_i$.

Example 1.5.11 Consider X, Y schemes, p and q the projections from $X \times Y$ to X and Y and E and F vector bundles on X and Y . Given $\alpha \in A_* X$ and $\beta \in A_* Y$, one has

$$(c_i(E) \cap \alpha) \times \beta = c_i(p^* E) \cap (\alpha \times \beta)$$

and

$$(c(E) \cap \alpha) \times (c(F) \cap \beta) = c(p^* E \oplus q^* F) \cap (\alpha \times \beta).$$

1.5.6 Segre Classes

Let C be a cone over a scheme X , that is, $C = \text{Spec}(S^\bullet)$, where S^\bullet is a sheaf of graded \mathcal{O}_X -algebras. Let us assume $\mathcal{O}_X \rightarrow S^0$ is surjective, S^1 is coherent and S^\bullet is generated by S^1 . For a variable z , denote by $S^\bullet[z]$ the graded algebra whose n^{th} graded piece is $S^n \oplus S^{n-1}z \oplus \dots \oplus S^1 z^{n-1} \oplus S^0 z^n$. Let $P(C \oplus 1) = \text{Proj}(S^\bullet[z])$ be the projective completion of C , with projection $q : P(C \oplus 1) \rightarrow X$, and let $\mathcal{O}(1)$ be the canonical line bundle on $P(C \oplus 1)$. The *Segre class* of C , denoted by $s(C)$, is the class in $A_* X$ defined by the formula

$$s(C) = q_* \left(\sum_{i \geq 0} c_1(\mathcal{O}(1))^i \cap [P(C \oplus 1)] \right).$$

If E is a vector bundle on X , one has $s(E) = c(E)^{-1} \cap [X]$ see [Ful, Proposition 4.1(a)].

Now suppose X can be a closed subscheme of a scheme Y . Let $C_X Y = \text{Spec}(\sum_{n=0}^{\infty} \mathcal{I}^n / \mathcal{I}^{n+1})$ be the normal cone to X in Y . The *Segre class* of X in Y , denoted by $s(X, Y)$, is defined to be the Segre class of the normal cone $C_X Y$, that is,

$$s(X, Y) = s(C_X Y) \in A_* X.$$

If X is regularly imbedded in Y , it follow that the normal cone is a vector bundle on X , and consequently, $s(X, Y)$ is the cap product of the total inverse Chern class of the normal bundle with $[X]$.

Proposition 1.5.12 *Let X be a proper closed subscheme of a variety Y . Consider \tilde{Y} the blow-up of Y along X , $\tilde{X} = P(C)$ the exceptional divisor, $\eta : \tilde{X} \rightarrow X$ the projection. Then*

$$s(X, Y) = \sum_{i \geq 0} \eta_* \left(c_1(\mathcal{O}(1))^i \cap [P(C)] \right).$$

Consider \mathcal{F} a coherent sheaf on a scheme X and consider $P(\mathcal{F}) = \text{Proj}(\text{Sym}(\mathcal{F}))$, with projection $p : P(\mathcal{F}) \rightarrow X$. Denote by $\mathcal{O}_{\mathcal{F}}(1)$ the canonical invertible sheaf which is the universal quotient of $p^*(\mathcal{F})$. If the support of \mathcal{F} is X , one defines its Segre class $s(\mathcal{F})$ in $A_*(X)$ by the formula

$$\begin{aligned} s(\mathcal{F}) &= p_* \left(\sum_{i \geq 0} c_1(\mathcal{O}_{\mathcal{F}}(1))^i \cap [P(\mathcal{F})] \right) \\ &= p_* \left(c(\mathcal{O}_{\mathcal{F}}(1))^{-1} \cap [P(\mathcal{F})] \right). \end{aligned}$$

1.6 Milnor number

Consider $f : (\mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}, 0)$ a germ of holomorphic function. Set Z to the complex hypersurface given by the zero set of f . One says that Z is singular at 0 if the differential of f vanishes at 0, that is, the vector field $(\partial f / \partial z_0, \dots, \partial f / \partial z_n)$ vanishes at 0. One says that 0 is an isolated singularity of Z if there is an open neighbourhood U of 0 such that $U \setminus \{0\}$ is non-singular, that is, 0 is the only point in U that the vector field $(\partial f / \partial z_0, \dots, \partial f / \partial z_n)$ vanishes. It is known that, 0 is an isolated singularity of Z if and only if the quociente algebra $\mathcal{O}_{n+1} / (\partial f / \partial z_0, \dots, \partial f / \partial z_n)$ is a \mathbb{C} -vector space of finite complex dimesion, see [L, Proposition 1.2]. Moreover, if 0 is an isolated singularity of Z , the index of the vector field $(\partial f / \partial z_0, \dots, \partial f / \partial z_n)$ coincides with the

number $\dim_{\mathbb{C}} \mathcal{O}_{n+1}/(\partial f/\partial z_0, \dots, \partial f/\partial z_n)$. The standard example is Pham-Brieskorn polynomial $f(z_0, \dots, z_n) = z_0^{a_0} + \dots + z_n^{a_n}$, with $a_n > 1$. Note that $Z = f^{-1}(0)$ is a complex hypersurface with an isolated singularity at 0.

Suppose that 0 is an isolated singularity of Z . For a ε small enough, the sphere $S_\varepsilon = \{z \in \mathbb{C}^{n+1} \mid |z| = \varepsilon\}$ intersects transversally Z , see [Milnor, Corollary 2.9]. The smooth manifold $K := S_\varepsilon \cap Z$ is called the *link* of the singularity of Z at 0, and its diffeomorphism type does not depend on ε .

Let us see below the classic Milnor's fibration theorem.

Theorem 1.6.1 [Milnor, Theorem 4.8] *There is $\varepsilon_1 > 0$ such that, for all ε with $\varepsilon_1 > \varepsilon > 0$, the map*

$$\varphi_\varepsilon = \frac{f}{|f|} : S_\varepsilon \setminus K \longrightarrow \mathbb{S}^1$$

is a locally trivial smooth fibration.

For $\varepsilon_1 > \varepsilon > 0$, all the fibrations φ_ε are diffeomorphic. Now let $B_\varepsilon(0)$ be the open ball of \mathbb{C}^{n+1} centered at 0 with radius ε and let ∂D_η be the boundary of the closed disc D_η of \mathbb{C} centered at 0 with radius η . Using the Ehresmann's fibration lemma on manifolds with boundary or the Thom's first isotopy lemma, one has the following alternative shape:

Theorem 1.6.2 (Milnor-Lê fibration theorem) *There is $\varepsilon_0 > 0$ such that, for all ε with $0 < \varepsilon \leq \varepsilon_0$, there is $\eta > 0$ such that, for all η with $0 < \eta \leq \eta_\varepsilon$. The map f induces a locally trivial smooth fibration $\psi_{\varepsilon, \eta} : B_\varepsilon(0) \cap f^{-1}(\partial D_\eta) \longrightarrow \partial D_\eta$.*

For small enough $\varepsilon > 0$ and $\eta > 0$, φ_ε and $\psi_{\varepsilon, \eta}$ are diffeomorphic. Thus, φ_ε is called Milnor fibration of f at 0. If f has an isolated singularity at 0, then each fiber of the Milnor fibration has the homotopy type of a bouquet $\mathbb{S}^n \vee \dots \vee \mathbb{S}^n$ of n -spheres, see [Milnor, Theorem 6.5]. The number of spheres in this bouquet is called the *Milnor number* of f and is denoted by $\mu(f)$. Moreover, the Milnor number $\mu(f)$ coincides with the number $\dim_{\mathbb{C}} \mathcal{O}_{n+1}/(\partial f/\partial z_0, \dots, \partial f/\partial z_n)$. In this way, assuming that f has an isolated singularity at 0, the Milnor number may be algebraically calculated.

There are some relevant generalizations to Milnor number. In 1971, H. Hamm extended the Milnor's fibration theorem for ICIS. The Lê number was introduced by David Massey, such numbers extend the notion of Milnor number to a setup of singularities not necessarily isolated. In 1988, Adam Parusiński presented a global definition for Milnor number, we will see in more detail in the next chapter.

1.7 Borel-Moore homology

The following is a brief explanation of the homology with locally finite supports, also known as Borel-Moore homology. We use as references [Ful] and [Ful2].

Let X be a topological space imbedded as a closed subspace of \mathbb{R}^n . One may define the *Borel-Moore homology groups* $H_i X$ by

$$H_i X \cong H^{n-i}(\mathbb{R}^n, \mathbb{R}^n \setminus X)$$

where the group on the right is the relative singular cohomology with integer coefficients. More generally, if X is a closed subspace of a topological space Y , there are cap products $H^j(Y^n, Y^n \setminus X) \otimes H_k Y \xrightarrow{\cap} H_{k-j} X$. Now assuming that Y is an oriented connected real n -manifold, one has that $H_n Y$ is freely generated by a *fundamental class* μ_Y , and capping μ_Y with determines an isomorphism $H^{n-i}(Y^n, Y^n \setminus X) \xrightarrow{\cap \mu_Y} H_i X$. In particular, when $X = Y$ one has the Poincaré duality, that is, the isomorphism $H^{n-i} Y \cong H_i Y$.

For any n -dimensional complex scheme X , we have $H_i X = 0$ for all $i > 2n$, and $H_{2n} X$ is a free abelian group with one generator for each irreducible component of X . Thus, the generator of $H_{2n} X$ corresponding to an n -dimensional irreducible component X_i , which we denote by $cl(X_i)$. More generally, if V is a k -dimensional closed subvariety of X , define the *cycle class* by $cl_X(V) = i_* cl(V) \in H_{2k} X$, where i is the inclusion of V in X . We may consider the homomorphism $cl : Z_k X \rightarrow H_{2k} X$ from the algebraic k -cycles to the Borel-Moore homology, which takes $\sum n_i [V_i]$ to $\sum n_i cl_X(V_i)$. This induces a homomorphism $cl : A_* X \rightarrow H_* X$, the so-called cycle map. The cycle map has interesting properties such as being covariant for proper morphisms, and compatible with Chern classes of vector bundles.

1.8 Schwartz-MacPherson class

Let X be a n -dimensional irreducible analytic (or algebraic) variety embedded in a smooth manifold M . Over the smooth part X_{reg} of X , the tangent bundle of X defines a section of the Grassmannian bundle $G_n(TM)$. We define the Nash blowup of X as been $\nu : \tilde{X} \rightarrow X$, where \tilde{X} is the closure of the image of this section and ν is the restriction of the projection on X . The restriction of the tautological bundle over

$G_n(TM)$ is denoted by \tilde{T} or \tilde{T}_X . One should notice that, \tilde{X} , \tilde{T} and ν are analytically independent of the embedding chosen since, near each point, X has a unique minimal local analytic embedding. We have that $\tilde{T}|_{\nu^{-1}(X_{reg})}$ is isomorphic to $\nu^*T(X_{reg})$. We define the Chern-Mather class of X as been the element of the Borel-Moore homology $H_*(X)$ given as

$$c_M = \nu_*(c(\tilde{T}) \cap [\tilde{X}]),$$

where $[\tilde{X}]$ is the fundamental class of \tilde{X} . An analytic cycle on a variety X is an element of free abelian group whose basis consists of all irreducible subvarieties of X . Given an analytic cycle $\sum n_i X_i$ on a variety X , with n_i are integers and X_i are irreducible subvarieties of X , one defines

$$c_M \left(\sum n_i X_i \right) := \sum n_i (incl_i)_* c_{Ma}(X_i),$$

where $incl_i$ is the inclusion of X_i in X . Note that, if X is smooth, then $c_{Ma} = c(T_X) \cap [X]$ is the total Chern class of X , where T_X is the tangent bundle of X .

Deligne and Grothendieck conjectured on the existence of a natural transformation between the functor CF , which assigns to each variety its group of constructible functions and the functor H_* of some homology theory such as Borel-Moore homology or rational equivalence theory, such that assigns the characteristic function of a non-singular variety to the Poincaré dual of the its total Chern class. In [Mac], Robert D. Macpherson responded affirmatively to the Deligne and Grothendieck's conjecture. Macpherson defined, transcendently, the local Euler obstruction $Eu_X(x)$ of X at $x \in X$. However, there exists an equivalent algebraic definition to the local Euler obstruction of X at x given by

$$Eu_X(x) = \int_X c(TX) \cap s(\nu^{-1}(x), \tilde{X}),$$

where $\nu : \tilde{X} \rightarrow X$ is the Nash blowup of X , TX is the Nash tangent bundle of X and s denotes the Segre class, see [Ful, Chapter 4]. For this definition was used as motivation the formula of Gonzalez-Sprinberg and Verdier, see [G-S]. Now for each V irreducible subvariety of X , the functions $Eu_V(-)$ on X , defined as $Eu_V(x)$ for all $x \in V$ and zero otherwise, are constructible, see [Mac] and [Ken]. MacPherson using the local Euler obstruction defined an isomorphism T from free abelian group of analytic cycles on X

to the additive group of constructible functions on X by

$$T\left(\sum m_i V_i\right) = \sum n_i E u_{V_i}(-).$$

We define the MacPherson class of X as been the element in $H_*(X)$ given by

$$c_*(X) := c_M(T^{-1}(\mathbf{1}_X)).$$

Thus, there exists a natural transformation $c_* : CF \rightarrow H_*$, such that if X is smooth then $c_*(X) = c(T_X) \cap [X] \in H_*(X)$, where T_X is the tangent bundle of X . In this way, if $f : X \rightarrow Y$ is a proper morphism of analytic varieties then the following diagram commutes

$$\begin{array}{ccc} CF(X) & \xrightarrow{c_*} & H_*(X) \\ f_* \downarrow & & \downarrow f_* \\ CF(Y) & \xrightarrow{c_*} & H_*(Y) \end{array}$$

this is, $f_*(c_*(\alpha)) = c_*(f_*(\alpha))$, for all $\alpha \in CF(X)$. Furthermore, for a compact analytic variety X , we have the a generalization of Gauss-Bonnet theorem to the singular setting

$$\chi(X) = \int_X c_*(X). \quad (1.2)$$

More generally, given a constructible function α on X , consider the constant function $\rho : X \rightarrow \{p_0\}$. Then, $\rho_*\alpha(p_0) = \chi(X; \alpha)$, and thus,

$$\int_X c_*(\alpha) = \int_{\{p_0\}} \rho_* c_*(\alpha) = \int_{\{p_0\}} c_*(\rho_*\alpha) = \int_{\{p_0\}} c_*(\chi(X; \alpha)\mathbf{1}_{p_0}) = \chi(X; \alpha).$$

In particular, if $\alpha = \mathbf{1}_X$ we have the equation (1.2).

In the 1960s, Marie-Hélène Schwartz defined a generalization of Chern classes to singular varieties, a work independently using obstruction theory and radial vector fields. In [B-S], J. P. Brasselet and M.-H. Schwartz showed that if X is a compact complex variety then, by Alexander isomorphism, the MacPherson class of X coincides with the Schwartz class of X . Therefore, the class $c_*(X)$ is known as the Schwartz-MacPherson class of X , also denotes by $c^{SM}(X)$.

Let \mathcal{S} be a Whitney stratification of X and let E be a holomorphic vector bundle on X . Consider a holomorphic section s of E such that s intersects, on each stratum of \mathcal{S} , the zero section of E transversely. Consider $Z = s^{-1}(0)$ and $\iota : Z \rightarrow X$ the inclusion.

Capítulo 2

Logarithmic Milnor number and some generalizations

In this chapter we will use the concept of logarithmic connection to generalize the Milnor number to a logarithmic setup. For this, let us review the approach on connections in the classical sense.

2.1 The logarithmic Milnor number

The main references in this section are [G-H] and [P].

Let $E \rightarrow M$ be a holomorphic k -vector bundle on a n -dimensional complex manifold M . A connection π on E is a \mathbb{C} -linear map

$$\pi : \Gamma(E) \rightarrow \Omega^1(M) \otimes \Gamma(E)$$

satisfying Leibnitz' rule

$$\pi(gs) = d(g) \otimes s + g\pi(s)$$

where $g : M \rightarrow \mathbb{C}$ is a holomorphic function and $s \in \Gamma(E)$. Let $\underline{s} = s_1, \dots, s_n : U \rightarrow E$ be a frame, where U is open subset in M . Given a connection π on E , we can decompose $\pi(s_i)$ into its components, writing

$$\pi(s_i) = \sum_{j=1}^n \theta_{ij} s_j.$$

The matrix $\theta = (\theta_{ij})$ of 1-forms is called the *connection matrix* of π with respect to \underline{s} .

Now, we assume that M is complex and E is hermitian. Since $\Omega^1(M) = \Omega^{1,0}(M) \oplus \Omega^{0,1}(M)$, we can write $\pi = \pi' + \pi''$, where $\pi' : \Gamma(E) \rightarrow \Omega^{1,0}(M) \otimes \Gamma(E)$ and $\pi'' : \Gamma(E) \rightarrow \Omega^{0,1}(M) \otimes \Gamma(E)$. We say that a connection π on E is compatible with the complex structure if $\pi'' = \bar{\partial}$. Moreover, if E is hermitian, the connection π is said to be compatible with the metric if

$$d(\xi, \eta) = (\pi(\xi), \eta) + (\xi, \pi(\eta))$$

for all $\xi, \eta \in \Gamma(E)$. The existence and uniqueness of a connection that satisfies the above conditions is something that is answered in the result below.

Proposition 2.1.1 *Given a hermitian vector bundle E on M , there is a unique connection on E compatible with both the metric and the complex structure.*

Proof. See [G-H, Page 73] or [Huy, Proposition 4.2.14]. ■

The unique connection compatible with the metric and the complex structures on E is called metric connection or Chern connection. Note that the metric connection depends on the hermitian structure adopted for E .

For our goals, we will need a more general type of connection, the so-called logarithmic connection with poles along some divisor. Such a concept was first introduced by Pierre Deligne, in [Deligne].

Definition 2.1.2 *Let $E \rightarrow M$ be a holomorphic vector bundle of rank k on a n -dimensional complex manifold M , and let D be a divisor (hypersurface) in M . A connection with logarithmic poles along D or, simply, logarithmic connection on E is a \mathbb{C} -linear map*

$$\nabla : \Gamma(E) \rightarrow \Omega_M^1(\log D) \otimes \Gamma(E)$$

satisfying

$$\nabla(gs) = d(g) \otimes s + g\nabla(s)$$

where $g : M \rightarrow \mathbb{C}$ is a holomorphic function and $s \in \Gamma(E)$.

Consider $\underline{s} = s_1, \dots, s_k : U \rightarrow E$ a frame, where U is an open subset of M . Given $\omega \in (\Omega_M^1(\log D) \otimes \Gamma(E))(U)$, can be written uniquely as $\sum \omega_i \otimes s_i$, with $\omega_i \in \Omega_M^1(\log D)|_U$. Then, given ∇ a logarithmic connection on E , we have

$$\nabla(s_i) = \sum_{j=1}^k \omega_{ij} \otimes s_j$$

where (ω_{ij}) is a $k \times k$ -matrix of elements in $\Omega_M^1(\log D)|_U$, which is called the *logarithmic connection matrix* with respect to \underline{s} . Now consider (ω_{ij}) the logarithmic connection matrix with respect to \underline{s} . For any $s \in \Gamma(E)$, we have $s = \sum_{i=1}^k f_i s_i$ for some $f_1, \dots, f_k \in \Omega^0(M)$, and then

$$\nabla(s) = \sum_{j=1}^k \nabla(f_j s_j) = \sum_{j=1}^k \left(df_j + \sum_{i=1}^k f_i \omega_{ij} \right) \otimes s_j. \quad (2.1)$$

In this way, given (ω_{ij}) a $k \times k$ -matrix of elements in $\Omega_M^1(\log D)$, we can define a logarithmic connection given by the expression of equation (2.1).

For each $r = 1, \dots, n$, we can construct a morphism \mathbb{C} -linear

$$\nabla^r : \Omega_M^r(\log D) \otimes \Gamma(E) \longrightarrow \Omega_M^{r+1}(\log D) \otimes \Gamma(E)$$

as follows. Given $\omega \in \Omega_M^r(\log D)$ and $s \in \Gamma(E)$, define

$$\nabla^r(\omega \otimes s) = d\omega \otimes s + (-1)^r \omega \wedge \nabla(s).$$

When $\nabla^r \circ \nabla^{r+1} = 0$ for all r , we say that the connection ∇ is integrable. In this way, se ∇ is integrable, one has the complex of \mathcal{O}_M -modules

$$\begin{aligned} 0 \longrightarrow \Gamma(E) \xrightarrow{\nabla} \Omega_M^1(\log D) \otimes \Gamma(E) \xrightarrow{\nabla^1} \dots \xrightarrow{\nabla^{r-1}} \Omega_M^r(\log D) \otimes \Gamma(E) \xrightarrow{\nabla^r} \\ \Omega_M^{r+1}(\log D) \otimes \Gamma(E) \xrightarrow{\nabla^{r+1}} \dots \xrightarrow{\nabla^{n+1}} \Omega_M^n(\log D) \otimes \Gamma(E) \longrightarrow 0. \end{aligned}$$

Assuming that (E, h) is a hermitian vector bundle, we can to define the following operation, locally given by

$$\begin{aligned} h : \Omega^1(M) \otimes \mathcal{O}(D) \otimes \Gamma(E) \times \Omega^1(M) \otimes \Gamma(E) &\longrightarrow \Omega^1(M) \otimes \mathcal{O}(D) \\ \left(\sum_i \frac{w_i}{f} \otimes s_i, \sum_j \eta_j \otimes t_j \right) &\longmapsto \sum_{i,j} \frac{w_i \wedge \bar{\eta}_j}{f} h(s_i, t_j) \end{aligned}$$

Similarly, we can to define $h : \Omega^1(M) \otimes \Gamma(E) \times \Omega^1(M) \otimes \mathcal{O}(D) \otimes \Gamma(E) \rightarrow \Omega^1(M) \otimes \mathcal{O}(D)$. We say that a logarithmic connection ∇ on a hermitian vector bundle (E, h) is compatible with the metric h with respect to divisor D if, locally, we have

$$\frac{dh(s_i, s_j)}{f} = h(\nabla(s_i), s_j) + h(s_i, \nabla(s_j)),$$

where f is a local defining function of D and s_1, \dots, s_n is a local frame for E . Motivated by the Proposition 2.1.1, it is natural to inquire the existence and uniqueness of a

logarithmic connection that satisfies such definition. Suppose there is such connection, we say ∇ . Consider $\underline{s} = s_1, \dots, s_n : U \rightarrow E$ a local frame, with U an open subset of M . Let $h_{ij} := h(s_i, s_j)$ and let $H = (h_{ij})$. Thus,

$$\begin{aligned} \frac{dh_{ij}}{f} &= h(\nabla(s_i), s_j) + h(s_i, \nabla(s_j)) \\ &= \sum_k \omega_{ik} h_{kj} + \sum_k \overline{\omega}_{kj} h_{ik}. \end{aligned}$$

and then

$$\begin{cases} \frac{\partial H}{f} = \omega H \\ \frac{\bar{\partial} H}{f} = H \overline{\omega}^t \end{cases}$$

where $\omega = (\omega_{ij})$. Therefore, the unique solution of the system above is $w = \frac{\partial H H^{-1}}{f}$. Note that, the form ω not necessarily has logarithmic poles along of D , this is, the existence of a logarithmic connection compatible with the metric is not always guaranteed. In some cases, we can do it, for exemplo, if the manifold M is a Riemann surface, see [Saito, pg 267]. However, if there is a logarithmic connection on a hermitian vector bundle (E, h) compatible with the metric h with respect to divisor D , then such connection is unique.

We will consider M a n -dimensional connected complex manifold and (L, h) a hermitian line bundle on M . Let $X := v^{-1}(0)$ be a (nowhere dense) divisor in M , where v is a holomorphic section of L . Let Y be a compact connected component of $\text{Sing}X$ and U a small neighbourhood of Y . In [P], A. Parusiński defined the μ -number of X at Y as been $\mu(X, Y) := \text{ind}_U \pi' v$, where $\pi = \pi' + \pi''$ is the decomposition of metric connection of L . The original setup of Milnor number is with an isolated points. This covers several cases and there are many works in this setup. The definition of a μ -number due Parusiński is a remarkable generalization of the Milnor number in the following sense: if x_0 is an isolated singularity of X , then $\mu(X, \{x_0\})$ is equals the Milnor number of X at x_0 , see [P, Proposition 1.4].

Theorem 2.1.3 [P, Proposition 1.6] *Suppose M be compact. Then,*

$$\mu(X) = (-1)^n \chi(X) + \int c_n(T^{*'} M \otimes L) \cap [M] - (-1)^n \chi(M).$$

With a computation involving two divisors, one has the following consequence:

Corollary 2.1.4 [P, Corollary 1.7] *With the notation above, we have*

$$\mu(X) - \mu(Z) = (-1)^n(\chi(X) - \chi(Z)).$$

Now let D be another divisor in M . Assume that X and D are disjoint. Suppose there exists a logarithmic connection $\nabla : \Gamma(L) \rightarrow \Omega_M^1(\log D) \otimes \Gamma(L)$ compatible with the metric h with respect to divisor D . Note that, the logarithmic connection forms of ∇ with respect to any holomorphic frames are holomorphic outside of D and then, by analog arguments to [P], we have $\text{Sing}X = \{x \in X : \nabla v(x) = 0\}$ and $\text{Sing}X$ is closed and open in $\{x \in M : \nabla v(x) = 0\}$.

Definition 2.1.5 *We define the μ -number of X at Y with respect to D as been the intersection index $\text{ind}_U \nabla v$. We denote by $\mu_D(X, Y)$. If X is compact, then intersection index of the zero section of $\Omega_M^1(\log D) \otimes L$ and ∇v will be called μ -number of X with respect to D , and we will denote by $\mu_D(X)$.*

Note that, the intersection index over U of ∇v is the number of the zero, in U , of a small perturbation of ∇v transversal to the zero section in U , counted with signs. Moreover, we have

$$\mu_D(X) = \sum_{i=1}^r \mu_D(X, Y_i)$$

where Y_1, \dots, Y_r are the connected components of $\text{Sing}X$.

We use as motivation the Theorem 2.1.3 due to Parusiński to obtain the following result:

Proposition 2.1.6 *Assume M compact. Then,*

$$\mu_D(X) = \int_U c_n(\Omega_M^1(\log D)|_U) \cap [U] + (-1)^{n-1} \int_M c(L)^{-1} c_1(L) c(\text{Der}_M(-\log D)) \cap [M],$$

where $U = \{x \in M : |v(x)| \leq \epsilon\}$, for a fixed $0 < \epsilon \ll 1$.

Proof. Consider $U = \{x \in M : |v(x)| \leq \epsilon\}$ for a ϵ sufficiently small such that $U \cap D = \emptyset$. Then,

$$\begin{aligned} \mu_D(X) &= \text{ind}_M \nabla v - \text{ind}_{M \setminus U} \nabla v \\ &= \text{ind}_M \nabla v - \text{ind}_{M \setminus U} h(\nabla v, v) \\ &= \text{ind}_M \nabla' v - (-1)^n \text{ind}_{M \setminus U} d|v|^2 \\ &= \int_M c_n(\Omega_M^1(\log D) \otimes L) \cap [M] - \\ &\quad - \left(\int_M c_n(\Omega_M^1(\log D)) \cap [M] - \int_U c_n(\Omega_M^1(\log D)|_U) \cap [U] \right). \end{aligned}$$

The last equality follows by Theorem 1.1.3. Lastly, note that

$$\begin{aligned}
& \int_M c(L)^{-1} c_1(L) c(\text{Der}_M(-\log D)) \cap [M] \\
&= \int_M \left(\sum_{k \geq 0} (-1)^k c_1(L)^{k+1} \right) \left(\sum_{j \geq 0} c_j(\text{Der}_M(-\log D)) \right) \cap [M] \\
&= \int_M \sum_{k, j \geq 0} (-1)^k c_1(L)^{k+1} c_j(\text{Der}_M(-\log D)) \cap [M] \\
&= \int_M \sum_{k=0}^{n-1} (-1)^k c_1(L)^{k+1} c_{n-1-k}(\text{Der}_M(-\log D)) \cap [M] \\
&= (-1)^{n-1} \int_M \sum_{k=0}^{n-1} c_1(L)^{k+1} c_{n-1-k}(\Omega_M^1(\log D)) \cap [M]
\end{aligned}$$

and

$$\begin{aligned}
& \int_M c_n(\Omega_M^1(\log D) \otimes L) \cap [M] \\
&= \int_M \sum_{i=0}^n c_1(L)^i c_{n-i}(\Omega_M^1(\log D)) \cap [M] \\
&= \int_M c_n(\Omega_M^1(\log D)) \cap [M] + \int_M \sum_{i=1}^n c_1(L)^i c_{n-i}(\Omega_M^1(\log D)) \cap [M] \\
&= \int_M c_n(\Omega_M^1(\log D)) \cap [M] + \int_M \sum_{k=0}^{n-1} c_1(L)^{k+1} c_{n-k-1}(\Omega_M^1(\log D)) \cap [M].
\end{aligned}$$

■

The case in which X and D are disjoint is not very interesting. Then, we would like to define a number involving two divisors in M not necessarily disjoint. Therefore, consider M a n -dimensional connected complex variety. Let $X := v^{-1}(0)$ be a divisor in M , where v is a holomorphic section of a hermitian line bundle L on M , and let D be another divisor in M . The number

$$\begin{aligned}
\mu_D(X) = & (-1)^n \left(\int_U c_n(\text{Der}_M(-\log D)|_U) \cap [U] - \right. \\
& \left. - \int_M c(L)^{-1} c_1(L) c(\text{Der}_M(-\log D)) \cap [M] \right)
\end{aligned}$$

is called *logarithmic Milnor number* of X with poles along D or, simply, logarithmic Milnor number of X , with respect to a $0 < \epsilon \ll 1$.

In [Alu2], Paolo Aluffi established the follow conjecture: for D a locally quasi-homogeneous free divisor in a n -dimensional nonsingular variety S , is it true that

$c_*(\mathbf{1}_{S \setminus D}) = c_n(\text{Der}_S(-\log D)) \cap [S]$? This question was answered in the following settings:

- (D₁) V is a nonsingular complex surface and the Tjurina number equals the Milnor number for all singularities of D , or then, D is a locally quasi-homogeneous divisors, see [Liao1].
- (D₂) V is a nonsingular projective complex variety and D is a locally quasi-homogeneous free divisor, see [Liao2].
- (D₃) V is a nonsingular compact complex variety and D is a certain class of divisors that Aluffi called "free hypersurface arrangement", see [Alu1].
- (D₄) V is a nonsingular (complex) variety defined over an algebraically closed field k of characteristic 0 and D is a free divisor with Jacobian ideal of linear type, see [Liao3].

This motivates the following definition:

Definition 2.1.7 *Let M be a n -dimensional connected complex manifold and let D be a divisor in M . We say that D is a chern logarithmic divisor in M if*

$$c_*(\mathbf{1}_{M \setminus D}) = c(\text{Der}_M(-\log D)) \cap [M].$$

Proposition 2.1.8 *Consider M and U as in the Proposition 2.1.6. Suppose that D is a chern logarithmic divisor in M , such that $c_*(\mathbf{1}_{U \setminus U \cap D}) = c(\text{Der}_M(-\log D)|_U) \cap [U]$. Then*

$$\mu_D(X) = (-1)^n \left(\chi(X; \mathbf{1}_{X \setminus D}) - \int_M c(L)^{-1} c_1(L) c(\text{Der}_M(-\log D)) \cap [M] \right).$$

Proof. Thus,

$$\begin{aligned} \mu_D(X) &= (-1)^n \left(\int_U c_n(\text{Der}_M(-\log D)|_U) \cap [U] - \int_M c(L)^{-1} c_1(L) c(\text{Der}_M(-\log D)) \cap [M] \right) \\ &= (-1)^n \left(\int_U c_*(\mathbf{1}_{U \setminus U \cap D}) - \int_M c(L)^{-1} c_1(L) c(\text{Der}_M(-\log D)) \cap [M] \right) \\ &= (-1)^n \left(\int_U c_*(U) - \int_U c_*(\mathbf{1}_{U \cap D}) - \int_M c(L)^{-1} c_1(L) c(\text{Der}_M(-\log D)) \cap [M] \right) \\ &= (-1)^n \left(\chi(U) - \chi(U; \mathbf{1}_{U \cap D}) - \int_M c(L)^{-1} c_1(L) c(\text{Der}_M(-\log D)) \cap [M] \right) \\ &= (-1)^n \left(\chi(X; \mathbf{1}_{X \setminus D}) - \int_M c(L)^{-1} c_1(L) c(\text{Der}_M(-\log D)) \cap [M] \right). \end{aligned}$$

■

Definition 2.1.9 We define the logarithmic Milnor number of X with poles along D or, simply, logarithmic Milnor number of X as been

$$\mu_D(X) = (-1)^n \left(\chi(X; \mathbf{1}_{X \setminus D}) - \int_M c(L)^{-1} c_1(L) c(\text{Der}_M(-\log D)) \cap [M] \right).$$

Example 2.1.10 Consider the crossing divisor $D = \{xyz = 0\} = \{x = 0\} \cup \{y = 0\} \cup \{z = 0\}$ and the hypersurface $X = \{y^2z - x^3 = 0\}$ in \mathbb{P}^2 . We have that $X \cap D = \{p = (0 : 0 : 1), q = (0 : 1 : 0)\}$ and $\text{Sing} X = \{p\}$. Then, $\chi(X; \mathbf{1}_{X \setminus D}) = \chi(X \setminus \{p, q\}) = \chi(X) - \chi(\{p\}) - \chi(\{q\}) = 0$, because $\chi(X) = 2$. It is known that

$$c(\Omega_{\mathbb{P}^2}^1(\log D)) = c(\Omega_{\mathbb{P}^2}^1) \prod_{i=1}^3 c(\mathcal{O}_{D_i})$$

where $D_1 = \{x = 0\}$, $D_2 = \{y = 0\}$ and $D_3 = \{z = 0\}$, see [Alu5, pg 12], [Sil, 3.1], [D-A, Proposition 2.3]. Set $h = c_1(\mathcal{O}_{\mathbb{P}^2}(1))$ the class of a hyperplane in \mathbb{P}^2 . The line bundle of X is $L = \mathcal{O}(3h)$. Hence,

$$c(\Omega_{\mathbb{P}^2}^1(\log D)) = c(\Omega_{\mathbb{P}^2}^1) \prod_{i=1}^3 c(\mathcal{O}_{D_i}) = (1 - 3h + 3h^2)(1 + h)^3 = 1 - 3h^2$$

and then $c(\text{Der}_M(-\log D)) = 1 - 3h^2$. Note that $c(L) = 1 + 3h$, and then, $c(L)^{-1} = 1 - 3h + 9h^2$. Therefore,

$$\begin{aligned} \mu_D(X) &= \chi(X; \mathbf{1}_{X \setminus D}) - \int_{\mathbb{P}^2} c(L)^{-1} c_1(L) c(\text{Der}_{\mathbb{P}^2}(-\log D)) \cap [\mathbb{P}^2] \\ &= - \int_{\mathbb{P}^2} ((1 - 3h + 9h^2)3h(1 - 3h^2)) \cap [\mathbb{P}^2] = 9. \end{aligned}$$

In [P], Parusiński presented a remarkable definition that generalizes the initial notion of Milnor number to a setup not necessarily involving isolated singularities as follows. Let M be a compact n -dimensional complex manifold and let L be a holomorphic line bundle over M . Consider $X := v^{-1}(0)$ a divisor in M , where v is a regular holomorphic section of L . Parusiński defined a generalization of the Milnor number given by

$$\mu(X) = (-1)^{n-1} (\chi(M|L) - \chi(X)),$$

where for a vector bundle E over M ,

$$\chi(M|E) := \int_M c(E)^{-1} c_{\text{top}}(E) c(M) \cap [M].$$

Define $\chi : X \rightarrow \mathbb{Z}$ given by $\chi(x) := \chi(F_x)$ for all $x \in X$, where F_x denotes de Milnor fibre at x and $\chi(F_x)$ denotes the Euler characteristic of F_x . Now define the function $\mu : X \rightarrow \mathbb{Z}$ by $\mu = (-1)^{n-1} (\chi - \mathbf{1}_X)$.

Let us fix a stratification $\mathcal{S} = \{S\}$ of X such that μ is constant on the strata of \mathcal{S} . It is known that the topological type of the Milnor fibres is constant along the strata of a Whitney stratification of X . Therefore, a Whitney stratification of X satisfies the condition desired above, see [P3], [B-M-M] and [PP]. Thus, for each $S \in \mathcal{S}$ we denote by μ_S the value of $x \mapsto \mu(X, x)$ on S . Set

$$\gamma(S) = \mu_S - \sum_{S' \neq S, \bar{S}' \supset S} \gamma(S')$$

the numbers defined inductively on descending dimension of S .

Theorem 2.1.11 *Let M be a nonsingular suvariety of \mathbb{P}^N and let L be a holomorphic line bundle on X . Consider X as being the zero set of a holomorphic section of L . Given \mathcal{S} a Whitney stratification of X , we have*

$$\mu(X) = \sum_{s \in \mathcal{Z}} \gamma(S) \int_{\bar{S}} (c(L|_{\bar{S}})^{-1} \cap c_*(\bar{S}))$$

where $\gamma(S) = \mu_S - \sum_{S' \neq S, \bar{S}' \supset S} \gamma(S')$.

Proof. See [PP, Theorem 4]. ■

Suppose that $\text{Sing}(X) = \{x_0\}$. Then, we can take the Whitney stratification $\{S_0 := X \setminus \{x_0\}, S_1 := \{x_0\}\}$ of X . Note that, $\gamma(S_0) = \mu_{S_0} = \mu(X, x) = 0$, for all $x \in X \setminus \{x_0\}$ and $\gamma(S_1) = \mu_{S_1} - \gamma(S_0) = \mu_{S_1} = \mu(X, x_0)$. Then, using the Theorem 2.1.11, it follow that

$$\begin{aligned} \mu(X) &= \gamma(S_1) \int_{\bar{S}_1} c(L|_{\bar{S}_1})^{-1} \cap c_*(\bar{S}_1) \\ &= \mu(X, x_0) \int_{\{x_0\}} [x_0] = \mu(X, x_0) \end{aligned}$$

because $c(L|_{\bar{S}_1}) = 1$ and $c_*(\bar{S}_1) = c_*(x_0) = [x_0]$. Moreover, one can generalize this fact as follows.

Example 2.1.12 [PP] *Assume that the $\text{Sing}(X) = \{x_1, \dots, x_r\}$. We have that*

$$\mu(X) = (-1)^n \sum_{i=1}^r \mu(X, x_i).$$

In [SS, Theorem 2.4], Seade and Suwa presented a generalization of this fact to "strong" local complete intersections.

In this way, we can show the following relation between the Milnor number due to Parusiński and the logarithmic Milnor number defined above:

Proposition 2.1.13 *Let D a divisor in M . If D is a chern logarithmic divisor in M , then*

$$\mu_D(X) = \mu(X) - (-1)^n \chi(X; \mathbf{1}_{X \cap D}) + (-1)^n \int_M c(L)^{-1} c_1(L) c_*(D).$$

Proof. Indeed, we have

$$\begin{aligned} \mu_D(X) &= (-1)^n \left(\chi(X; \mathbf{1}_{X \setminus X \cap D}) - \int_M c(L)^{-1} c_1(L) c(\mathrm{Der}_M(-\log D)) \cap [M] \right) \\ &= (-1)^n \left(\int_X c_*(\mathbf{1}_{X \setminus X \cap D}) - \int_M c(L)^{-1} c_1(L) c_*(\mathbf{1}_{M \setminus D}) \right) \\ &= (-1)^n \left(\int_X c_*(\mathbf{1}_X) - \int_X c_*(\mathbf{1}_{X \cap D}) - \int_M c(L)^{-1} c_1(L) c_*(\mathbf{1}_{M \setminus D}) \right) \\ &= (-1)^n \left(\chi(X) - \int_M c(L)^{-1} c_1(L) c_*(M) - \chi(X; \mathbf{1}_{X \cap D}) + \int_M c(L)^{-1} c_1(L) c_*(D) \right) \\ &= \mu(X) - (-1)^n \chi(X; \mathbf{1}_{X \cap D}) + (-1)^n \int_M c(L)^{-1} c_1(L) c_*(D). \end{aligned}$$

■

Now let E be a holomorphic vector bundle over M of rank d . Set $X := \{p \in M : s(p) = 0\}$, where s is a regular holomorphic section of E , that is, at any point $p \in X$, the germs of the components of s with respect to a holomorphic frame near p form a regular sequence in the $\mathcal{O}_{M,p}$ of germs holomorphic functions at p . Then, X is a closed subvariety of M of pure dimension $n - d$, see [Ful, B.3].

Definition 2.1.14 *Given α a constructible function on M , we define the Milnor number relative to α as being*

$$\mu(X; \alpha) = (-1)^{\dim X} \left(\int_M c(E)^{-1} c_{top}(E) c_*(\alpha) - \chi(X; \alpha|_X) \right).$$

First fact that we highlight below is how we recover our definition of the logarithmic Milnor number.

Example 2.1.15 *Let D be a chern logarithmic divisor in M . When the rank of the vector bundle E is 1, this is $\dim(X) = n - 1$, and $\alpha = \mathbf{1}_{M \setminus D}$, we have that*

$$\begin{aligned} \mu(X; \mathbf{1}_{M \setminus D}) &= (-1)^{\dim X} \left(\int_M c(E)^{-1} c_{top}(E) c_*(\mathbf{1}_{M \setminus D}) - \chi(X; (\mathbf{1}_{M \setminus D})|_X) \right) \\ &= (-1)^{n-1} \left(\int_M c(E)^{-1} c_{top}(E) c(\mathrm{Der}_M(-\log D)) \cap [M] - \chi(X; \mathbf{1}_{X \setminus D}) \right) \\ &= \mu_D(X). \end{aligned}$$

2.2 On the generalization of Schürmann for Milnor classes

Let X be a scheme which can be imbedded as a closed subscheme of a nonsingular variety M . One define the Fulton class of X as being the class

$$c^F(X) = c(TM|_X) \cap s(X, M)$$

in $A_*(X)$. Fulton showed that $c^{FJ}(X)$ does not depend on the choice of imbedding, see [Ful, Example 4.2.6(a)]. Moreover, the Fulton-Johnson class of X is defined by

$$c^{FJ}(X) = c(TM|_X) \cap s(\mathcal{N})$$

where $s(\mathcal{N})$ is the total Chern class of the conormal sheaf of the embedding of X in M , see [FJ]. Assuming that X is a local complete intersection, the Fulton and Fulton-Johnson classes coincide and are equal to

$$c(TM|_X)c(N_X M)^{-1} \cap [X] = c(T_X) \cap [X]$$

where $T_X = TM|_X - N_X M$ denotes the virtual tangent bundle on X , which is a well-defined element of the Grothendieck group of vector bundles on X , see [Ful, Example 3.2.7].

Let M be a nonsingular compact complex analytic variety of pure dimension n and let L be a holomorphic line bundle over M . Consider $X := v^{-1}(0)$ a divisor in M , where v is a regular holomorphic section of L . In this way, we have

$$c^F(X) = c(TM|_X - L|_X) \cap [X].$$

Definition 2.2.1 [PP3], [BLSS], [Y], [A4] *The Milnor class of X is defined as being*

$$\mathcal{M}(X) = (-1)^{n-1}(c^{FJ}(X) - c_*(X)).$$

Due to Parusiński and Pragacz, there is the following formula to Milnor number $\mathcal{M}(X)$ in terms of a Whitney stratification of X .

Theorem 2.2.2 *Take $\mathcal{S} = \{S\}$ a Whitney stratification of X . Then*

$$\mathcal{M}(X) = \sum_{s \in \mathcal{S}} \gamma(S)c(L|_X)^{-1} \cap (\iota_{\bar{S}, X})_* c_*(\bar{S})$$

where $\gamma(S) = \mu_S - \sum_{S' \neq S, \bar{S}' \supset S} \gamma(S')$ and the application $\iota_{\bar{S}, X}$ denotes the inclusion from \bar{S} to X .

Proof. See [PP3, Theorem 0.2]. ■

In particular, one has

$$\int_X \mathcal{M}(X) = \sum_{s \in \mathcal{S}} \gamma(S) \int_{\bar{S}} c(L|_{\bar{S}})^{-1} \cap c_*(\bar{S}).$$

Note that Parusiński and Pragacz had already proven the above formula to M projective and X not necessarily compact, see Theorem 2.1.11.

Example 2.2.3 [PP3, Example 0.1] *Assume that $\text{Sing}(X) = \{x_1, \dots, x_r\}$. Then, one has*

$$\mathcal{M}(X) = \sum_{i=1}^r \mu(X, x_i)[x_i] \in H_0(X).$$

There is a generalization of this result for "strong" local complete intersections, for more details see [Suwa].

Let $\iota : X \hookrightarrow Z$ be a regular embedding between algebraic (or complex) possibly singular varieties and let $N_X Z$ be a normal bundle of X in Z . The diagram

$$\begin{array}{ccc} CF(Z) & \xrightarrow{c_*} & H_*(Z) \\ \iota^* \downarrow & & \downarrow \iota^* \cap c(N_X Z)^{-1} \\ CF(X) & \xrightarrow{c_*} & H_*(X) \end{array}$$

does not commute in general, where $\iota^* : H_*(Z) \rightarrow H_*(X)$ is the Gysin homomorphism. Indeed, suppose the particular case in which $Z = M$ a smooth variety. On the one hand, $c_*(\iota^*(\mathbf{1}_M)) = c_*(\mathbf{1}_X)$ which is equals to Schwartz-MacPherson class of X . On the other hand,

$$\begin{aligned} c(N_X M)^{-1} \cap \iota^*(c_*(\mathbf{1}_M)) &= c(N_X M)^{-1} \cap \iota^*(c(TM) \cap [M]) \\ &= c(N_X M)^{-1} \cdot c(\iota^* TM) \cap \iota^*[M] \\ &= c(N_X M)^{-1} \cdot c(TM|_X) \cap [X] = c^{FJ}(X) \end{aligned}$$

and it is known that, in general, the Schwartz-MacPherson class and the Fulton-Johnson class are different, see [Schür1], [PP3], [BLSS], [Y], [A4]. This motivated J. Schürmann the following definition:

Definition 2.2.4 [Schür1] *Let $\iota : X \hookrightarrow Z$ be a regular embedding and let α be a constructible function on Z . One defines the Milnor class of the pair $X \subset Z$ relative to α as been*

$$\mathcal{M}(X \subset Z; \alpha) = (-1)^{\dim X} (c(N_X Z)^{-1} \cap \iota^*(c_*(\alpha)) - c_*(\iota^*(\alpha))) \in H_*(X),$$

where $N_X Z$ is the normal cone of X in Z .

When Z is a smooth variety, we also denote $\mathcal{M}(X \subset Z; \alpha)$ simply by $\mathcal{M}(X; \alpha)$.

Theorem 2.2.5 *Suppose that Z is a smooth variety and suppose that X is the zero-scheme of a regular section of a vector bundle E on Z . Given a constructible function α on Z , it follows that*

$$\mu(X; \alpha) = \int_X \mathcal{M}(X; \alpha).$$

Proof. We will take a particular situation of the Example 1.5.9 given by the following fiber square

$$\begin{array}{ccc} X & \xrightarrow{\iota} & Z \\ Id \downarrow & & \downarrow Id \\ X & \xrightarrow{\iota} & Z \end{array}$$

Therefore, $\iota_* \iota^*(\beta) = c_{top}(Id^*E) \cap \beta = c_{top}(E) \cap \beta$, for all $\beta \in A_*Z$. By definition, we have

$$\mu(X; \alpha) = (-1)^{\dim X} \left(\int_X c(E)^{-1} c_{top}(E) c_*(\alpha) - \chi(X; \alpha|_X) \right).$$

Thus, using the Example 1.5.9 and Proposition 1.5.8, we obtain

$$\begin{aligned} \mu(X; \alpha) &= (-1)^{\dim X} \left(\int_X c(E)^{-1} \cap \iota_* \iota^* c_*(\alpha) - \int_X c_*(\iota^* \alpha) \right) \\ &= (-1)^{\dim X} \left(\int_X \iota_* (c(\iota^* E)^{-1} \cap \iota^* c_*(\alpha)) - \int_X c_*(\iota^* \alpha) \right) \\ &= (-1)^{\dim X} \left(\int_X c(N_X Z)^{-1} \cap \iota^* c_*(\alpha) - \int_X c_*(\iota^* \alpha) \right) \\ &= \int_X \mathcal{M}(X; \alpha) \end{aligned}$$

because $\iota^* E = E|_X = N_X Z$ and $\chi(X; \alpha|_X) = \int_X c_*(\iota^* \alpha)$. ■

In this way, the class $c^{FJ}(X \subset Z; \alpha) := c(N_X Z)^{-1} \cap \iota^*(c_*(\alpha))$ is called Fulton-Johnson class of the pair $X \subset Z$ relative to α ; and the class $c^{SM}(X \subset Z; \alpha) = c_*(\iota^* \alpha)$ is called Schwartz-MacPherson class of the pair $X \subset Z$ relative to α .

Question. Let L be a holomorphic line bundle over a smooth manifold M and let $X := v^{-1}(0)$ be a divisor in M , where v is a regular holomorphic section of L . Consider D another divisor in M . Assume that $\text{Sing} X = \{x_1, \dots, x_s\} \subset D$. One defines the number

$$\mu^{BR}(X, D) := \sum_{i=1}^s \mu^{BR}(f_i, D),$$

where f_1, \dots, f_s are local defining functions of X at x_1, \dots, x_s , respectively, with f_i being finitely $\mathcal{R}(D)$ -determined, for more details see [BR].

Is there any relationship between the numbers $\mu^{BR}(X, D)$ and $\mu_D(X)$?

Capítulo 3

Intersection product formulas relative to constructible functions

This chapter is about some product formulas and some intersection formulas of varieties, generalized for arbitrary constructible functions.

3.1 Product formulas for the Milnor class of constructible functions

Let X and Y be manifolds and let α and β be constructible functions on X and Y , respectively. One defines $(\alpha \otimes \beta)(x, y) := \alpha(x)\beta(y)$, for all $(x, y) \in X \times Y$. We have that $\alpha \otimes \beta$ is a constructible function on $X \times Y$.

Theorem 3.1.1 [Kwie],[Kwei-Yoka] *Consider α a constructible function on X and β a constructible functions on Y . Then,*

$$c_*(\alpha \otimes \beta) = c_*(\alpha) \times c_*(\beta).$$

In particular, $c_*(X \times Y) = c_*(X) \times c_*(Y)$ and $c_{Ma}(X \times Y) = c_{Ma}(X) \times c_{Ma}(Y)$, because $1_X \otimes 1_Y = 1_{X \times Y}$ and $Eu_{X \times Y} = Eu_X \otimes Eu_Y$.

For each $i = 1, 2$, let M_i be an $(n_i + k_i)$ -dimensional compact complex analytic manifold and let E_i be a holomorphic vector bundle over M_i of rank k_i . Consider s_i a regular holomorphic section of E_i , and consider the n_i -dimensional local complete intersection. For each $i = 1, 2$, consider the projection $p_i : M_1 \times M_2 \rightarrow M_i$. Then,

consider the holomorphic section $s_1 \oplus s_2 : M_1 \times M_2 \longrightarrow p_1^*E_1 \oplus p_2^*E_2$ given by $(s_1 \oplus s_2)(x, y) = (s_1(x), s_2(y))$, for all $(x, y) \in M_1 \times M_2$. Note that $X_1 \times X_2 = (s_1 \oplus s_2)^{-1}(0)$. Let α and β be constructible functions on M_1 and M_2 , respectively.

Proposition 3.1.2 *We have*

$$c_*(X_1 \times X_2 \subset M_1 \times M_2; \alpha \otimes \beta) = c_*(X_1 \subset M_1; \alpha) \times c_*(X_2 \subset M_2; \beta).$$

Proof. For all $(x_1, x_2) \in X_1 \times X_2$, it follows that

$$\begin{aligned} (\iota_1 \times \iota_2)^*(\alpha \otimes \beta)(x_1, x_2) &= (\alpha \otimes \beta)(\iota_1(x_1), \iota_2(x_2)) \\ &= \alpha(\iota_1(x_1))\beta(\iota_2(x_2)) \\ &= ((\iota_1^*\alpha) \otimes (\iota_2^*\beta))(x_1, x_2). \end{aligned}$$

Thus, $(\iota_1 \times \iota_2)^*(\alpha \otimes \beta) = (\iota_1^*\alpha) \otimes (\iota_2^*\beta)$. Then, using Theorem 3.1.1

$$\begin{aligned} c_*(X_1 \times X_2 \subset M_1 \times M_2; \alpha \otimes \beta) &= c_*((\iota_1 \times \iota_2)^*(\alpha \otimes \beta)) \\ &= c_*((\iota_1^*\alpha) \otimes (\iota_2^*\beta)) \\ &= c_*(\iota_1^*\alpha) \times c_*(\iota_2^*\beta) \\ &= c_*(X_1 \subset M_1; \alpha) \times c_*(X_2 \subset M_2; \beta). \end{aligned}$$

■

One should notice that the above proposition generalizes the Theorem 3.1.1. For this, take $\alpha = \mathbf{1}_{X_1}$ and $\beta = \mathbf{1}_{X_2}$. Now we look for, in similar fashion, a result for the Fulton-Johnson classes.

Proposition 3.1.3 *In the same conditions, we have*

$$c^{FJ}(X_1 \times X_2 \subset M_1 \times M_2; \alpha \otimes \beta) = c^{FJ}(X_1 \subset M_1; \alpha) \times c^{FJ}(X_2 \subset M_2; \beta).$$

Proof. By definition,

$$c^{FJ}(X_1 \times X_2 \subset M_1 \times M_2; \alpha \otimes \beta) = c(N_{X_1 \times X_2} M_1 \times M_2)^{-1} \cap (\iota_1 \times \iota_2)^*(c_*(\alpha \otimes \beta)).$$

By Theorem 3.1.1 and Proposition 1.5.4(ii), it follows that

$$\begin{aligned} (\iota_1 \times \iota_2)^*(c_*(\alpha \otimes \beta)) &= (\iota_1 \times \iota_2)^*(c_*(\alpha) \times c_*(\beta)) \\ &= (\iota_1^*(c_*(\alpha))) \times (\iota_2^*(c_*(\beta))). \end{aligned}$$

Since $N_{X_1 \times X_2} M_1 \times M_2 = p_1^*(N_{X_1} M_1) \oplus p_2^*(N_{X_2} M_2)$ and using the Example 1.5.11, for all $\gamma \in A_* X_1$ and $\eta \in A_* X_2$, we have

$$\begin{aligned} \gamma \times \eta &= ((c(N_{X_1} M_1) c(N_{X_1} M_1)^{-1}) \cap \gamma) \times ((c(N_{X_2} M_2) c(N_{X_2} M_2)^{-1}) \cap \eta) \\ &= c(p_1^*(N_{X_1} M_1)) c(p_2^*(N_{X_2} M_2)) (c(N_{X_1} M_1)^{-1} \cap \gamma) \times (c(N_{X_2} M_2)^{-1} \cap \eta) \\ &= c(p_1^*(N_{X_1} M_1) \oplus p_2^*(N_{X_2} M_2)) (c(N_{X_1} M_1)^{-1} \cap \gamma) \times (c(N_{X_2} M_2)^{-1} \cap \eta) \end{aligned}$$

where the last equality follows from the Whitney product formula. Consequently,

$$c(p_1^*(N_{X_1} M_1) \oplus p_2^*(N_{X_2} M_2))^{-1} \cap (\gamma \times \eta) = (c(N_{X_1} M_1)^{-1} \cap \gamma) \times (c(N_{X_2} M_2)^{-1} \cap \eta).$$

Therefore,

$$\begin{aligned} c^{FJ}(X_1 \times X_2 \subset M_1 \times M_2; \alpha \otimes \beta) &= c(p_1^*(N_{X_1} M_1) \oplus p_2^*(N_{X_2} M_2))^{-1} \cap \\ &\quad \cap (\iota_1^*(c_*(\alpha)) \times \iota_2^*(c_*(\beta))) \\ &= (c(p_1^*(N_{X_1} M_1))^{-1} \cap \iota_1^*(c_*(\alpha))) \times \\ &\quad \times (c(p_2^*(N_{X_2} M_2))^{-1} \cap \iota_2^*(c_*(\beta))) \\ &= c^{FJ}(X_1 \subset M_1; \alpha) \times c^{FJ}(X_2 \subset M_2; \beta). \end{aligned}$$

■

By definition, the Milnor class is the difference, up to sign, between the Schwartz-MacPherson class and the Fulton-Johnson class. Then, we can use the above results to show the following theorem.

Theorem 3.1.4 *Let M_1, \dots, M_r be compact complex manifolds of dimension n_i , respectively. For each i , consider a holomorphic vector bundle E_i of rank d_i over M_i , a regular holomorphic section $s_i : M_i \rightarrow E_i$ and $X_i := s_i^{-1}(0)$ the local complete intersection of dimension $n_i - d_i$. For each i , let α_i be a constructible function on M_i . Then,*

$$\mathcal{M}(X_1 \times \dots \times X_r; \alpha_1 \otimes \dots \otimes \alpha_r) = \sum (-1)^{(n_1 - d_1)\varepsilon_1 + \dots + (n_r - d_r)\varepsilon_r} P_1 \dots P_r$$

where the sum runs over all choices of $P_i \in \{c_*(X_i; \alpha_i), \mathcal{M}(X_i; \alpha_i)\}$, $i = 1, \dots, r$, except $(P_1, \dots, P_r) = (c_*(X_1; \alpha_1), \dots, c_*(X_r; \alpha_r))$ and

$$\varepsilon_i = \begin{cases} 1, & \text{if } P_i = c_*(X_i; \alpha_i) \\ 0, & \text{if } P_i = \mathcal{M}(X_i; \alpha_i). \end{cases}$$

Proof. First of all, on the one hand, we have

$$\begin{aligned}
\mathcal{M}(X_1 \times X_2; \alpha_1 \otimes \alpha_2) &= (-1)^{n_1+n_2-d_1-d_2} (c^{FJ}(X_1 \times X_2; \alpha_1 \otimes \alpha_2) - \\
&\quad - c_*(X_1 \times X_2; \alpha_1 \otimes \alpha_2)) \\
&= (-1)^{n_1+n_2-d_1-d_2} (c^{FJ}(X_1; \alpha_1) \times c^{FJ}(X_2; \alpha_2) - \\
&\quad - c_*(X_1; \alpha_1) \times c_*(X_2; \alpha_2)).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\mathcal{M}(X_1; \alpha_1) \times \mathcal{M}(X_2; \alpha_2) + (-1)^{n_1-d_1} c_*(X_1; \alpha_1) \times \mathcal{M}(X_2; \alpha_2) + \\
&\quad + (-1)^{n_2-d_2} \mathcal{M}(X_1; \alpha_1) \times c_*(X_2; \alpha_2) \\
= &(-1)^{n_1-d_1} (c^{FJ}(X_1; \alpha_1) - c_*(X_1; \alpha_1)) \times (-1)^{n_2-d_2} (c^{FJ}(X_2; \alpha_2) - c_*(X_2; \alpha_2)) + \\
&\quad + (-1)^{n_1-d_1} c_*(X_1; \alpha_1) \times (-1)^{n_2-d_2} (c^{FJ}(X_2; \alpha_2) - c_*(X_2; \alpha_2)) + \\
&\quad + (-1)^{n_2-d_2} (-1)^{n_1-d_1} (c^{FJ}(X_1; \alpha_1) - c_*(X_1; \alpha_1)) \times c_*(X_2; \alpha_2) \\
= &(-1)^{n_1+n_2-d_1-d_2} (c^{FJ}(X_1; \alpha_1) \times c^{FJ}(X_2; \alpha_2) - c^{FJ}(X_1; \alpha_1) \times c_*(X_2; \alpha_2) - \\
&\quad - c_*(X_1; \alpha_1) \times c^{FJ}(X_2; \alpha_2) + c_*(X_1; \alpha_1) \times c_*(X_2; \alpha_2)) + \\
&\quad + (-1)^{n_1+n_2-d_1-d_2} (c_*(X_1; \alpha_1) \times c^{FJ}(X_2; \alpha_2) - c_*(X_1; \alpha_1) \times c_*(X_2; \alpha_2)) + \\
&\quad + (-1)^{n_1+n_2-d_1-d_2} (c^{FJ}(X_1; \alpha_1) \times c_*(X_2; \alpha_2) - c_*(X_1; \alpha_1) \times c_*(X_2; \alpha_2)).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathcal{M}(X_1 \times X_2; \alpha_1 \otimes \alpha_2) &= (-1)^{n_1-d_1} c_*(X_1; \alpha_1) \times \mathcal{M}(X_2; \alpha_2) + \\
&\quad + (-1)^{n_2-d_2} \mathcal{M}(X_1; \alpha_1) \times c_*(X_2; \alpha_2) + \mathcal{M}(X_1; \alpha_1) \times \mathcal{M}(X_2; \alpha_2).
\end{aligned}$$

Now take $r > 2$ and suppose that the result holds for $r - 1$. Thus,

$$\mathcal{M}(X_1 \times \cdots \times X_{r-1}; \alpha_1 \otimes \cdots \otimes \alpha_{r-1}) = \sum (-1)^{n_1 \varepsilon_1 + \cdots + n_{r-1} \varepsilon_{r-1}} P_1 \cdots P_{r-1},$$

where the sum runs over all choices of $P_i \in \{c_*(X_i; \alpha_i), \mathcal{M}(X_i; \alpha_i)\}$, $i = 1, \dots, r - 1$, except $(P_1, \dots, P_{r-1}) = (c_*(X_1; \alpha_1), \dots, c_*(X_{r-1}; \alpha_{r-1}))$ and

$$\varepsilon_i = \begin{cases} 1, & \text{if } P_i = c_*(X_i; \alpha_i) \\ 0, & \text{if } P_i = \mathcal{M}(X_i; \alpha_i). \end{cases}$$

Then, it follows that

$$\begin{aligned}
& \mathcal{M}(X_1 \times \cdots \times X_r; \alpha_1 \otimes \cdots \otimes \alpha_r) \\
= & (-1)^{n_1 + \cdots + n_{r-1}} c_*(X_1 \times \cdots \times X_{r-1}; \alpha_1 \otimes \cdots \otimes \alpha_{r-1}) \times \mathcal{M}(X_r; \alpha_r) + \\
& + (-1)^{n_r} \mathcal{M}(X_1 \times \cdots \times X_{r-1}; \alpha_1 \otimes \cdots \otimes \alpha_{r-1}) \times c_*(X_r; \alpha_r) + \\
& + \mathcal{M}(X_1 \times \cdots \times X_{r-1}; \alpha_1 \otimes \cdots \otimes \alpha_{r-1}) \times \mathcal{M}(X_r; \alpha_r) \\
= & (-1)^{n_1 + \cdots + n_{r-1}} c_*(X_1; \alpha_1) \times \cdots \times c_*(X_{r-1}; \alpha_{r-1}) \times \mathcal{M}(X_r; \alpha_r) + \\
& + (-1)^{n_r} \sum (-1)^{n_1 \varepsilon_1 + \cdots + n_{r-1} \varepsilon_{r-1}} P_1 \cdots P_{r-1} \times c_*(X_r; \alpha_r) + \\
& + \sum (-1)^{n_1 \varepsilon_1 + \cdots + n_{r-1} \varepsilon_{r-1}} P_1 \cdots P_{r-1} \times \mathcal{M}(X_r; \alpha_r) \\
= & \sum (-1)^{n_1 \varepsilon_1 + \cdots + n_r \varepsilon_r} P_1 \cdots P_r.
\end{aligned}$$

■

In particular, when $\alpha_i = \mathbf{1}_{X_i}$, for all i , we obtain the principal result of [O-Y].

Explicitly:

Corollary 3.1.5 *In the same conditions, one has*

$$\mathcal{M}(X_1 \times \cdots \times X_r) = \sum (-1)^{(n_1 - d_1) \varepsilon_1 + \cdots + (n_r - d_r) \varepsilon_r} P_1 \cdots P_r$$

where the sum runs over all choices of $P_i \in \{c_*(X_i), \mathcal{M}(X_i)\}$, $i = 1, \dots, r$, except $(P_1, \dots, P_r) = (c_*(X_1), \dots, c_*(X_r))$ and

$$\varepsilon_i = \begin{cases} 1, & \text{if } P_i = c_*(X_i) \\ 0, & \text{if } P_i = \mathcal{M}(X_i). \end{cases}$$

Proof. Note that $\mathcal{M}(X_1 \times \cdots \times X_r; \mathbf{1}_{X_1} \otimes \cdots \otimes \mathbf{1}_{X_r}) = \mathcal{M}(X_1 \times \cdots \times X_r; \mathbf{1}_{X_1 \times \cdots \times X_r}) = \mathcal{M}(X_1 \times \cdots \times X_r)$. Moreover, for each $i = 1, \dots, r$, we know that $c_*(X_i; \mathbf{1}_{X_i}) = c_*(X_i)$ and $\mathcal{M}(X_i; \mathbf{1}_{X_i}) = \mathcal{M}(X_i)$. ■

3.2 On the diagonal embedding

Consider M a complex manifold and X an analytic subvariety of M . Let \mathcal{S} be a Whitney stratification of M adapted to X . Given $x \in X \cap S$ for some $S \in \mathcal{S}$, consider $g : (M, x) \rightarrow (\mathbb{C}, 0)$ a germ of holomorphic function such that $d_x g$ is a non-degenerate covector at x with respect to \mathcal{S} , that is, $d_x g \in T_S^* M$ and $d_x g \notin T_{S'}^* M$ for all stratum $S' \neq S$. For N a germ of a closed complex submanifold of M which is transversal to \mathcal{S}

with $N \cap S = \{x\}$, one define the complex link l_S of S by

$$l_S := X \cap N \cap B_\delta(x) \cap \{g = \omega\}$$

for $0 < |\omega| \ll \delta \ll 1$, and one defines the normal Morse index

$$\eta(S; F^\bullet) := \chi(X \cap N \cap B_\delta(x), l_S; F^\bullet)$$

where the right-hand side is the Euler characteristic of the relative hypercohomology. the number $\eta(S; F^\bullet)$ does not depend on the choices of $x \in S$, g and N , see [G-M, Section 2.3]. Moreover, we have

$$\eta(S; F^\bullet) := \chi(X \cap N \cap B_\delta(x); F^\bullet) - \chi(l_S; F^\bullet).$$

The conormal variety of a subvariety X in a complex manifold M is given by

$$T_X^* M := \text{closure} \{ (x, \theta) \in T^* M \mid x \in X_{\text{reg}} \text{ and } \theta|_{T_x X_{\text{reg}}} \equiv 0 \}.$$

Let $L(M)$ be the free abelian group generated by all the conormal spaces $T_X^* M$, where X varies over all subvarieties of M . Given a constructible function α on M with respect to a Whitney stratification \mathcal{S} defines an element in $L(M)$ by

$$\text{Ch}(\alpha) := \sum_{S \in \mathcal{S}} (-1)^{\dim S} \eta(S; \alpha) \cdot T_S^* M.$$

Inducing an isomorphism $\text{Ch} : CF(M) \longrightarrow L(M)$.

Let M be an n -dimensional compact complex analytic manifold. Define $M^{(r)} := M \times \cdots \times M$. And let E be a holomorphic vector bundle over $M^{(r)}$ of rank d . Consider $\Delta : M \longrightarrow M^{(r)}$ the diagonal morphism, which is a regular embedding of codimension $nr - n$. Let t be a regular holomorphic section of E . This means that set $Z(t) := \{p \in M^{(r)} : t(p) = 0\}$ is a closed subvariety of $M^{(r)}$ of dimension $nr - d$. In addition, the morphism Δ induces the refined Gysin homomorphism

$$\Delta^! : H_{2k}(Z(t)) \longrightarrow H_{2(k-nr+n)}(Z(\Delta^*(t))).$$

The refined intersection product is defined by $\gamma_1 \cdots \gamma_r := \Delta^!(\gamma_1 \times \cdots \times \gamma_r)$, see [Ful, Example 8.1.9].

Consider the projectivized cotangent bundles $\mathbb{P}(T^*M)$ and $\mathbb{P}(T^*(M^{(r)}))$. We will denote the vector bundle $\mathbb{P}(T^*M \oplus \cdots \oplus T^*M)$ by $\mathbb{P}(T^*(M^{(r)}))$. We have the following fibre square diagram:

$$\begin{array}{ccc}
\mathbb{P}((T^*M)^{\oplus r}) & \xrightarrow{\delta} & \mathbb{P}(T^*M^{(r)}) \\
p \downarrow & & \downarrow \pi^{(r)} \\
M & \xrightarrow{\Delta} & M^{(r)} \\
\iota \uparrow & & \uparrow \iota \\
Z(\Delta^*t) & \xrightarrow{\Delta} & Z(t)
\end{array} \tag{3.1}$$

where $\pi^{(r)}$ is the natural proper map. Let $i : \mathbb{P}(T^*M) \rightarrow \mathbb{P}((T^*M)^{\oplus r})$ be the morphism induced by the diagonal embedding $T^*M \rightarrow T^*M \oplus \cdots \oplus T^*M$.

Lemma 3.2.1 *Let β be a constructible function on $M^{(r)}$ with respect to a Whitney stratification \mathcal{S} , that is, β is constant at each stratum of \mathcal{S} . We assume transversal to $\Delta(M)$ and such that the intersections $S \cap \Delta(M)$ are connected. Then*

$$\delta^![\mathbb{P}(\text{Ch}(\beta))] = (-1)^{nr-n} i_*[\mathbb{P}(\text{Ch}(\Delta^*\beta))].$$

Proof. See [B-M-S, Proposition 1.4]. ■

There is a classical description of the Schwartz-MacPherson class due to C. Sabbah, see [Sab]. In our context, such a generalization is as follows: given a constructible function α on $M^{(r)}$, we have

$$c_*(Z(t); \alpha) = (-1)^{nr-1} c(TM^{(r)}|_{Z(t)}) \cap \pi_*^{(r)}(c(\mathcal{O}_r(1)^{-1}) \cap [\mathbb{P}(\text{Ch}(\alpha))]), \tag{3.2}$$

where $\mathcal{O}_r(1)$ is the tautological line bundle on the projectivisation $\mathbb{P}(T^*M^{(r)}) \rightarrow M^{(r)}$, see [P4, pg 13], [P5, pg 352], [PP3, pg 4] and [Ken].

Proposition 3.2.2 *With the same notation, we have*

$$\Delta^!(c_*(Z(t); \alpha)) = c\left((TM|_{Z(\Delta^*t)})^{\oplus r-1}\right) \cap c_*(Z(\Delta^*t); \Delta^*\alpha).$$

Proof. Using the description (3.2) and Proposition 1.5.8, one has

$$\Delta^! c_*(Z(t); \alpha) = (-1)^{nr-1} c(\Delta^*(TM^{(r)}|_{Z(t)})) \cap \Delta^! \pi_*^{(r)}(c(\mathcal{O}_r(1))^{-1} \cap [\mathbb{P}(\text{Ch}(\alpha))]).$$

By the Theorem 1.5.7 and the Proposition 1.5.8, it follows that

$$\begin{aligned}
\Delta^! \pi_*^{(r)}(c(\mathcal{O}_r(1))^{-1} \cap [\mathbb{P}(\text{Ch}(\alpha))]) &= p_* \delta^!(c(\mathcal{O}_r(1))^{-1} \cap [\mathbb{P}(\text{Ch}(\alpha))]) \\
&= p_* (c(\delta^* \mathcal{O}_r(1))^{-1}) \cap \delta^! [\mathbb{P}(\text{Ch}(\alpha))].
\end{aligned}$$

It is known that $\delta^* \mathcal{O}_r(1) = c(\mathcal{O}_{\mathbb{P}((T^*M)^{\oplus r})}(1))$ is the tautological line bundle on the projectivisation $\mathbb{P}((T^*M)^{\oplus r}) \rightarrow M^{(r)}$ and $c(\Delta^*(TM^{(r)}|_{Z(t)})) = c((TM|_{Z(\Delta^*t)})^{\oplus r})$. Thus, by Lemma 3.2.1, we have that $\delta^![\mathbb{P}(\text{Ch}(\alpha))] = (-1)^{nr-n} i_*[\mathbb{P}(\text{Ch}(\Delta^*\alpha))]$, and then

$$\begin{aligned} \Delta^! c_*(Z(t); \alpha) &= (-1)^{n-1} c(TM^{\oplus r}|_{Z(\Delta^*t)}) \cap \\ &\quad \cap p_* \left(c(\mathcal{O}_{\mathbb{P}((T^*M)^{\oplus r})}(1))^{-1} \right) \cap i_*[\mathbb{P}(\text{Ch}(\Delta^*\alpha))] \\ &= (-1)^{n-1} c(TM^{\oplus r}|_{Z(\Delta^*t)}) \cap \\ &\quad \cap (p \circ i)_* (c(i_* \mathcal{O}_{\mathbb{P}((T^*M)^{\oplus r})}(1))^{-1} \cap [\mathbb{P}(\text{Ch}(\Delta^*\alpha))]), \end{aligned}$$

where the last equality follows from the projection formula. Since that $i_* \mathcal{O}_{\mathbb{P}((T^*M)^{\oplus r})}(1) = \mathcal{O}_{\mathbb{P}(T^*M)}(1)$ and that $q := (p \circ i) : \mathbb{P}(T^*M) \rightarrow M$ is the projectived cotangent morphism, one has

$$\begin{aligned} \Delta^! c_*(Z(t); \alpha) &= (-1)^{n-1} c(TM^{\oplus r}|_{Z(\Delta^*t)}) \cap q_*(c(\mathcal{O}_{\mathbb{P}(T^*M)}(1))^{-1} \cap [\mathbb{P}(\text{Ch}(\Delta^*\alpha))]) \\ &= c((TM|_{Z(\Delta^*t)})^{\oplus r-1}) \cap c_*(Z(\Delta^*t); \Delta^*\alpha). \end{aligned}$$

■

Through this text we will assume that $Z(t) \subset \text{Supp}(\alpha)$.

Proposition 3.2.3 *With the same notation, we have*

$$\Delta^!(c^{FJ}(Z(t); \alpha)) = c\left((TM|_{Z(\Delta^*t)})^{\oplus r-1}\right) \cap c^{FJ}(Z(\Delta^*t); \Delta^*\alpha).$$

Proof. By definition of Fulton-Johnson class of $Z(t)$ relative to α and by the commutativity of the diagram (3.1), one has

$$\begin{aligned} \Delta^! c^{FJ}(Z(t); \alpha) &= \Delta^!(c(E|_{Z(t)})^{-1} \cap \iota^* c_*(\alpha)) \\ &= c(\Delta^*(E|_{Z(t)}))^{-1} \cap \Delta^! \iota^* c_*(\alpha) \\ &= c(\Delta^* E|_{Z(\Delta^*t)})^{-1} \cap \iota^* \Delta^! c_*(\alpha). \end{aligned}$$

In [PP3], the Schwartz-MacPherson class of the constructible function α has the following description:

$$c_*(\alpha) = (-1)^{nr-1} c(TM^{(r)}|_{\text{Supp}(\alpha)}) \cap \pi_*^{(r)}(c(\mathcal{O}_r(1))^{-1} \cap [\mathbb{P}(\text{Ch}(\alpha))]),$$

where $\mathcal{O}_r(1)$ is the tautological line bundle on the projectivisation $\mathbb{P}(T^*M^{(r)}) \longrightarrow M^{(r)}$. Then,

$$\Delta^! c_*(\alpha) = (-1)^{nr-1} c(\Delta^*(TM^{(r)}|_{\text{Supp}(\alpha)})) \cap \Delta^! \pi_*^{(r)}(c(\mathcal{O}_r(1))^{-1} \cap [\mathbb{P}(\text{Ch}(\alpha))]).$$

Similarly the proof of the Proposition 3.2.2, we have that

$$\Delta^! \pi_*^{(r)}(c(\mathcal{O}_r(1))^{-1} \cap [\mathbb{P}(\text{Ch}(\alpha))]) = q_*(c(\mathcal{O}_{\mathbb{P}(T^*M)}(1))^{-1} \cap [\mathbb{P}(\text{Ch}(\Delta^*\alpha))])$$

where $q : \mathbb{P}(T^*M) \longrightarrow M$ is the projectived cotangent morphism. Since that $c(\Delta^*(TM^{(r)}|_{\text{Supp}(\alpha)})) = c((TM|_{\text{Supp}(\Delta^*\alpha)})^{\oplus r})$, it follows that

$$\Delta^! c_*(\alpha) = (-1)^{n-1} c((TM|_{\text{Supp}(\Delta^*\alpha)})^{\oplus r}) \cap q_*(c(\mathcal{O}_{\mathbb{P}(T^*M)}(1))^{-1} \cap [\mathbb{P}(\text{Ch}(\Delta^*\alpha))])$$

and then,

$$\begin{aligned} \Delta^! c^{FJ}(Z(t); \alpha) &= (-1)^{n-1} c(\Delta^* E|_{Z(\Delta^*t)})^{-1} \cap \iota^* \left(c((TM|_{\text{Supp}(\Delta^*\alpha)})^{\oplus r}) \cap \right. \\ &\quad \left. \cap q_*(c(\mathcal{O}_{\mathbb{P}(T^*M)}(1))^{-1} \cap [\mathbb{P}(\text{Ch}(\Delta^*\alpha))]) \right) \\ &= (-1)^{n-1} c(\Delta^* E|_{Z(\Delta^*t)})^{-1} c(\iota^*(TM|_{\text{Supp}(\Delta^*\alpha)})^{\oplus r}) \cap \\ &\quad \cap \iota^* q_*(c(\mathcal{O}_{\mathbb{P}(T^*M)}(1))^{-1} \cap [\mathbb{P}(\text{Ch}(\Delta^*\alpha))]) \\ &= (-1)^{n-1} c(\Delta^* E|_{Z(\Delta^*t)})^{-1} c(\iota^*(TM|_{\text{Supp}(\Delta^*\alpha)})^{\oplus r-1}) \cdot \\ &\quad \cdot c(\iota^*(TM|_{\text{Supp}(\Delta^*\alpha)})) \cap \iota^* q_*(c(\mathcal{O}_{\mathbb{P}(T^*M)}(1))^{-1} \cap [\mathbb{P}(\text{Ch}(\Delta^*\alpha))]) \\ &= (-1)^{n-1} c(\Delta^* E|_{Z(\Delta^*t)})^{-1} c((TM|_{Z(\Delta^*t)})^{\oplus r-1}) \cap \\ &\quad \cap \iota^* (c(TM|_{\text{Supp}(\Delta^*\alpha)}) \cap q_*(c(\mathcal{O}_{\mathbb{P}(T^*M)}(1))^{-1} \cap [\mathbb{P}(\text{Ch}(\Delta^*\alpha))])) \\ &= c(\Delta^* E|_{Z(\Delta^*t)})^{-1} c((TM|_{Z(\Delta^*t)})^{\oplus r-1}) \cap \iota^* c_*(\Delta^*\alpha) \\ &= c((TM|_{Z(\Delta^*t)})^{\oplus r-1}) \cap c^{FJ}(Z(\Delta^*t); \Delta^*\alpha). \end{aligned}$$

■

As a consequence of the above propositions, we have the following result on the Milnor class.

Proposition 3.2.4 *We have that,*

$$\Delta^!(\mathcal{M}(Z(t); \alpha)) = (-1)^{nr-n} c((TM|_{Z(\Delta^*t)})^{\oplus r-1}) \cap \mathcal{M}(Z(\Delta^*t); \Delta^*\alpha).$$

Proof. Indeed, using the definition of the Milnor class and the above propositions, we have that

$$\begin{aligned}
\Delta^!(\mathcal{M}(Z(t); \alpha)) &= \Delta^!((-1)^{nr-n}(c^{FJ}(Z(t); \alpha) - c_*(Z(t); \alpha))) \\
&= (-1)^{nr-n} c \left((TM|_{Z(\Delta^*t)})^{\oplus r-1} \right) \cap \\
&\quad \cap (c^{FJ}(Z(\Delta^*t); \Delta^*\alpha) - c_*(Z(\Delta^*t); \Delta^*\alpha)) \\
&= (-1)^{nr-n} c \left((TM|_{Z(\Delta^*t)})^{\oplus r-1} \right) \cap \mathcal{M}(Z(\Delta^*t); \Delta^*\alpha).
\end{aligned}$$

■

3.3 Intersection product formulas relative to constructible functions

Consider M a compact complex manifold with dimension equals to n . For each $i = 1, \dots, r$, let E_i be a holomorphic vector bundle of rank d_i over M and let $X_i := s_i^{-1}(0)$ be a $(n - d_i)$ -dimensional local complete intersection, where s_i is a regular holomorphic section on E_i . Here we are assuming that the product $X_1 \times \dots \times X_r$ is equipped with a Whitney stratification such that the diagonal embedding Δ is transversal to all strata. For each i , let $p_i : M^{(r)} \rightarrow M$ be the i^{th} -projection. Now, consider the holomorphic exterior product section

$$s = s_1 \oplus \dots \oplus s_r : M^{(r)} \rightarrow (p_1^*E_1) \oplus \dots \oplus (p_r^*E_r),$$

given by $s(x_1, \dots, x_r) = (s_1(x_1), \dots, s_r(x_r))$. Therefore, $Z(s) = X_1 \times \dots \times X_r$ and $X := Z(\Delta^*(s)) = X_1 \cap \dots \cap X_r$.

Let $\alpha_1, \dots, \alpha_r$ be constructible functions on M and let $\Delta : M \rightarrow M^{(r)}$ be the diagonal morphism. Set $\alpha := \alpha_1 \otimes \dots \otimes \alpha_r$ the constructible function on $M^{(r)}$ such that $(\alpha_1 \otimes \dots \otimes \alpha_r)(x_1, \dots, x_r) = \alpha_1(x_1) \cdots \alpha_r(x_r)$, for all $(x_1, \dots, x_r) \in M^{(r)}$. Note that, $\Delta^*\alpha(x) = (\alpha_1 \otimes \dots \otimes \alpha_r) \circ \Delta(x) = \alpha_1(x) \cdots \alpha_r(x)$ for all $x \in M$. Moreover, $\Delta^*\alpha(x) = 0$ if $x \notin X$; and $\Delta^*\alpha(x) = \alpha_1(x) \cdots \alpha_r(x)$ if $x \in X$. In this way, $\Delta^*(\alpha_1 \otimes \dots \otimes \alpha_r)$ is also denoted by $\alpha_1 \otimes \dots \otimes \alpha_r$.

Lemma 3.3.1 *With the same notation, we have*

$$\mathcal{M}(X; \alpha) = (-1)^{nr-n} c \left((TM|_X)^{\oplus r-1} \right)^{-1} \cap \sum (-1)^{(n-d_1)\epsilon_1 + \dots + (n-d_r)\epsilon_r} P_1 \cdots P_r \in H_*(X),$$

where

$$\epsilon_i = \begin{cases} 1, & \text{if } P_i = c_*(X_i; \alpha_i) \\ 0, & \text{if } P_i = \mathcal{M}(X_i; \alpha_i). \end{cases}$$

Proof. Using the Proposition 3.2.4,

$$\Delta^!(\mathcal{M}(Z(s); \alpha)) = (-1)^{nr-n} c \left((TM|_{Z(\Delta^*s)})^{\oplus r-1} \right) \cap \mathcal{M}(Z(\Delta^*s); \Delta^*\alpha).$$

Then,

$$\mathcal{M}(X; \alpha) = (-1)^{nr-n} c(TM^{\oplus r-1}|_X)^{-1} \cap \Delta^!(\mathcal{M}(X_1 \times \cdots \times X_r; \alpha))$$

By Theorem 3.1.4, it follows that

$$\mathcal{M}(X; \alpha) = (-1)^{nr-n} c(TM^{\oplus r-1}|_X)^{-1} \cap \sum (-1)^{(n-d_1)\epsilon_1 + \cdots + (n-d_r)\epsilon_r} \Delta^!(P_1 \times \cdots \times P_r)$$

where the sum runs over all choices of $P_i \in \{c_*(X_i; \alpha_i), \mathcal{M}(X_i; \alpha_i)\}$, $i = 1, \dots, r-1$, except $(P_1, \dots, P_{r-1}) = (c_*(X_1; \alpha_1), \dots, c_*(X_{r-1}; \alpha_{r-1}))$ and

$$\epsilon_i = \begin{cases} 1, & \text{if } P_i = c_*(X_i; \alpha_i) \\ 0, & \text{if } P_i = \mathcal{M}(X_i; \alpha_i). \end{cases}$$

Since $\Delta^!(P_1 \times \cdots \times P_r) = P_1 \cdots P_r$, the result follows. ■

This means that, in particular, for $r = 2$ we have

$$\begin{aligned} \mathcal{M}(X; \alpha_1 \otimes \alpha_2) &= c(TM|_X)^{-1} \cap ((-1)^n \mathcal{M}(X_1; \alpha_1) \mathcal{M}(X_2; \alpha_2) + \\ &\quad + (-1)^{d_1} c_*(X_1; \alpha_1) \mathcal{M}(X_2; \alpha_2) + (-1)^{d_2} \mathcal{M}(X_1; \alpha_1) c_*(X_2; \alpha_2)). \end{aligned}$$

Moreover, for $r = 3$, we have

$$\begin{aligned} \mathcal{M}(X; \alpha) &= c((TM|_X)^{\oplus 2})^{-1} \cap (\mathcal{M}(X_1; \alpha_1) \mathcal{M}(X_2; \alpha_2) \mathcal{M}(X_3; \alpha_3) + \\ &\quad + (-1)^{d_1+d_2} c_*(X_1; \alpha_1) c_*(X_2; \alpha_2) \mathcal{M}(X_3; \alpha_3) + (-1)^{d_1+d_3} c_*(X_1; \alpha_1) \mathcal{M}(X_2; \alpha_2) c_*(X_3; \alpha_3) + \\ &\quad + (-1)^{d_2+d_3} \mathcal{M}(X_1; \alpha_1) c_*(X_2; \alpha_2) c_*(X_3; \alpha_3) + (-1)^{n-d_1} c_*(X_1; \alpha_1) \mathcal{M}(X_2; \alpha_2) \mathcal{M}(X_3; \alpha_3) + \\ &\quad + (-1)^{n-d_2} \mathcal{M}(X_1; \alpha_1) c_*(X_2; \alpha_2) \mathcal{M}(X_3; \alpha_3) + (-1)^{n-d_3} \mathcal{M}(X_1; \alpha_1) \mathcal{M}(X_2; \alpha_2) c_*(X_3; \alpha_3)). \end{aligned}$$

Now we will present the main result in this text.

Theorem 3.3.2 *With the same notation, we have the following formulas:*

$$c^{FJ}(X; \alpha) = c((TM|_X)^{\oplus r-1})^{-1} \cap c^{FJ}(X_1; \alpha_1) \cdots c^{FJ}(X_r; \alpha_r), \quad (3.3)$$

$$c^{SM}(X; \alpha) = c((TM|_X)^{\oplus r-1})^{-1} \cap c^{SM}(X_1; \alpha_1) \cdots c^{SM}(X_r; \alpha_r) \quad (3.4)$$

and

$$\begin{aligned} \mathcal{M}(X; \alpha) &= (-1)^{\dim X} c((TM|_X)^{\oplus r-1})^{-1} \cap (c^{FJ}(X_1; \alpha_1) \cdots c^{FJ}(X_r; \alpha_r) - \\ &\quad - c_*(X_1; \alpha_1) \cdots c_*(X_r; \alpha_r)) \end{aligned}$$

where α denotes the constructible function $\alpha_1 \otimes \cdots \otimes \alpha_r$.

Proof. By Proposition 3.2.3, one has

$$\Delta^!(c^{FJ}(Z(s); \alpha)) = c((TM|_{Z(\Delta^*s)})^{\oplus r-1}) \cap c^{FJ}(Z(\Delta^*s); \Delta^*\alpha).$$

Then, using the Proposition 3.1.3, it follows that

$$\begin{aligned} c^{FJ}(X; \alpha) &= c((TM|_X)^{\oplus r-1})^{-1} \cap \Delta^!(c^{FJ}(X_1 \times \cdots \times X_r; \alpha_1 \otimes \cdots \otimes \alpha_r)) \\ &= c((TM|_X)^{\oplus r-1})^{-1} \cap \Delta^!(c^{FJ}(X_1; \alpha_1) \times \cdots \times c^{FJ}(X_r; \alpha_r)) \end{aligned}$$

Since $\Delta^!(c^{FJ}(X_1; \alpha_1) \times \cdots \times c^{FJ}(X_r; \alpha_r)) = c^{FJ}(X_1; \alpha_1) \cdots c^{FJ}(X_r; \alpha_r)$, one has that

$$c^{FJ}(X; \alpha) = c((TM|_X)^{\oplus r-1})^{-1} \cap c^{FJ}(X_1; \alpha_1) \cdots c^{FJ}(X_r; \alpha_r).$$

Now, the Proposition 3.2.2 states that

$$\Delta^!(c^{SM}(Z(s); \alpha)) = c((TM|_{Z(\Delta^*s)})^{\oplus r-1}) \cap c^{SM}(Z(\Delta^*s); \Delta^*\alpha).$$

In similar fashion, using the Proposition 3.1.2, we have

$$c^{SM}(X; \alpha) = c((TM|_X)^{\oplus r-1})^{-1} \cap c^{SM}(X_1; \alpha_1) \cdots c^{SM}(X_r; \alpha_r).$$

Lastly, using the definition of Milnor class of X relative to α and the formulas (3.3)

(3.4), it follows that

$$\begin{aligned} \mathcal{M}(X; \alpha) &= (-1)^{\dim X} c((TM|_X)^{\oplus r-1})^{-1} \cap (c^{FJ}(X_1; \alpha_1) \cdots c^{FJ}(X_r; \alpha_r) - \\ &\quad - c_*(X_1; \alpha_1) \cdots c_*(X_r; \alpha_r)). \end{aligned}$$

■

The above result also follows from Lemma 3.3.1. To exemplify the approach of the computation, let us look at the case $r = 2$. By Lemma 3.3.1, we have

$$\begin{aligned} \mathcal{M}(X; \alpha_1 \otimes \alpha_2) &= (-1)^{2n-n} c(TM|_X)^{-1} \cap (\mathcal{M}(X_1; \alpha_1) \mathcal{M}(X_2; \alpha_2) + \\ &\quad + (-1)^{n-d_1} c_*(X_1; \alpha_1) \mathcal{M}(X_2; \alpha_2) + (-1)^{n-d_2} \mathcal{M}(X_1; \alpha_1) c_*(X_2; \alpha_2)). \end{aligned}$$

For each $i = 1, 2$, one has $\mathcal{M}(X_i; \alpha_i) = (-1)^{n-d_i}(c^{FJ}(X_i; \alpha_i) - c_*(X_i; \alpha_i))$. Then, it follows that

$$\begin{aligned} \mathcal{M}(X; \alpha_1 \otimes \alpha_2) &= (-1)^{2n-n} c(TM|_X)^{-1} \cap ((-1)^{2n-d_1-d_2}(c^{FJ}(X_1; \alpha_1) - c_*(X_1; \alpha_1)) \cdot \\ &\quad \cdot (c^{FJ}(X_2; \alpha_2) - c_*(X_2; \alpha_2)) + (-1)^{2n-d_1-d_2} c_*(X_1; \alpha_1)(c^{FJ}(X_2; \alpha_2) - \\ &\quad - c_*(X_2; \alpha_2)) + (-1)^{2n-d_1-d_2}(c^{FJ}(X_1; \alpha_1) - c_*(X_1; \alpha_1))c_*(X_2; \alpha_2)) \\ &= (-1)^{\dim X} c(TM|_X)^{-1} \cap (c^{FJ}(X_1; \alpha_1)c^{FJ}(X_2; \alpha_2) - \\ &\quad - c_*(X_1; \alpha_1)c_*(X_2; \alpha_2)) \end{aligned}$$

because $\dim X = n - d_1 - d_2$.

Forthwith, we have the following consequence:

Corollary 3.3.3 *The number $(-1)^{\dim X} \mu(X; \alpha)$ is equals to the degree*

$$\int_X c((TM|_X)^{\oplus r-1})^{-1} \cap (c^{FJ}(X_1; \alpha_1) \cdots c^{FJ}(X_r; \alpha_r) - c_*(X_1; \alpha_1) \cdots c_*(X_r; \alpha_r)).$$

Proof. The result follows immediately from Theorem 2.2.5 and Theorem 3.3.2. ■

In particular, when we have $\alpha_i = \mathbf{1}_{X_i}$ for all $i = 1, \dots, r$, we retrieve the original formulas of [B-M-S].

Corollary 3.3.4 *We have that*

$$\begin{aligned} c^{FJ}(X) &= c((TM|_X)^{\oplus r-1})^{-1} \cap c^{FJ}(X_1) \cdots c^{FJ}(X_r), \\ c^{SM}(X) &= c((TM|_X)^{\oplus r-1})^{-1} \cap c^{SM}(X_1) \cdots c^{SM}(X_r) \end{aligned}$$

and

$$\begin{aligned} \mathcal{M}(X) &= (-1)^{\dim X} c((TM|_X)^{\oplus r-1})^{-1} \cap (c^{FJ}(X_1) \cdots c^{FJ}(X_r) - \\ &\quad - c_*(X_1) \cdots c_*(X_r)). \end{aligned}$$

Capítulo 4

Segre classes relative to a constructible function

Let X be a proper closed subscheme of a variety Y . Consider \tilde{Y} the blow-up of Y along X , $\tilde{X} = P(N_X Y)$ the exceptional divisor and $\eta : \tilde{X} \rightarrow X$ the projection, where $N_X Y$ is the normal bundle. It can be organized in the following diagram:

$$\begin{array}{ccccc}
 P(N_X Y) & \xrightarrow{=} & \tilde{X} & \xrightarrow{\tilde{\iota}} & \tilde{Y} & \xrightarrow{=} & Bl_X Y \\
 & & \downarrow \eta & & \downarrow & & \\
 & & X & \xrightarrow{\iota} & Y & &
 \end{array}$$

The Segre class of X in Y is characterized by

$$s(X, Y) = \sum_{i \geq 0} \eta_* \left(c_1(\mathcal{O}(1))^i \cap [P(N_X Y)] \right)$$

where $\mathcal{O}(1)$ is the canonical line bundle on $P(N_X Y)$. Now, assume that $Y = M$ is non-singular. We know that the Fulton class of X is defined by $c^F(X) = c(TM|_X) \cap s(X, M) \in A_*(X)$.

We have already commented on the generalization of the Milnor class to an arbitrary constructible function due to Schürmann. Such a definition produces the Fulton class of X relative to a constructible function α on M which is given by $c(N_X M)^{-1} \cap \iota^*(c_*(\alpha))$, where ι^* is the Gysin homomorphism. Thus, given a constructible function α on M , we define the *Segre class* of X in M relative to α as the

following element in A_*X

$$s(X \subset M; \alpha) = \eta_* \left(\sum_{i \geq 0} c_1(\mathcal{O}_{\mathbb{P}(N_X M)}(1))^i \cap \eta^*(c(TM|_X)^{-1} \cap \iota^* c_*(\alpha)) \right),$$

where c_* represents the MacPherson class. When there is no ambiguity in relation to the variety M , we shall denote $s(X \subset M; \alpha)$ only by $s(X; \alpha)$.

Remark 4.0.1 *For any vector bundle E of rank $e + 1$ on X and for any $\beta \in A_*(X)$, we have*

$$s_i(E) \cap \beta = b_* \left(c_1(\mathcal{O}_{P(E)}(1))^{e+i} \cap b^*(\beta) \right),$$

where $P(E)$ is the projective bundle of lines in E , b is the projection from $P(E)$ to X and $\mathcal{O}_{P(E)}(1)$ denotes the canonical line bundle on $P(E)$. Since $N_X M$ is a vector bundle on X , it follows that

$$s(X \subset M; \alpha) = c(N_X M)^{-1} \cdot c(TM|_X)^{-1} \cap \iota^* c_*(\alpha),$$

for all constructible function α on M . In this case, we are making $E = N_X M$ and $\beta = c(TM|_X)^{-1} \cap \iota^* c_*(\alpha)$.

Motivating us to define a certain type of Fulton class with respect to a arbitrary constructible function.

Definition 4.0.2 *Given a constructible function α on M , we define the Fulton class of X relative to α as*

$$c^F(X \subset M; \alpha) = c(TM|_X) \cap s(X \subset M; \alpha).$$

When there is no ambiguity in relation to the variety M , we shall denote $c^F(X \subset M; \alpha)$ only by $c^F(X; \alpha)$. In this way, we have

$$\begin{aligned} c^F(X \subset M; \alpha) &= c(TM|_X) \cap s(X \subset M; \alpha) \\ &= c(TM|_X) \cap \eta_* \left(\sum_{i \geq 0} c_1(\mathcal{O}_{\mathbb{P}(N_X M)}(1))^i \cap \eta^*(c(TM|_X)^{-1} \cap \iota^* c_*(\alpha)) \right) \\ &= c(TM|_X) c(N_X M)^{-1} c(TM|_X)^{-1} \cap \iota^* c_*(\alpha) \\ &= c(N_X M)^{-1} \cap \iota^* c_*(\alpha). \end{aligned}$$

This shows that Definition 4.0.2 coincides with the definition of Fulton class relative to a constructible function due to Schürmann.

The next result shows a relevant property of the Segre class presented above.

Proposition 4.0.3 *Let X_1 and X_2 be schemes which can be imbedded as subschemes of nonsingular varieties M_1 and M_2 , respectively. For each $i = 1, 2$, consider α_i a constructible function on M_i . Then, we get*

$$s(X_1 \times X_2; \alpha_1 \otimes \alpha_2) = s(X_1; \alpha_1) \times s(X_2; \alpha_2).$$

Proof. For each $i = 1, 2$, let \widetilde{X}_i be the exceptional divisor of the blow-up of M_i along X_i , with projection $\eta_i : \widetilde{X}_i \rightarrow X_i$. Set $\eta = \eta_1 \times \eta_2$. By definition, $s(X_1 \times X_2; \alpha_1 \otimes \alpha_2)$ is equals to

$$\eta_* \left(\sum_{i \geq 0} c_1(\mathcal{O}(1))^i \cap \eta^*(c(TM|_{X_1 \times X_2})^{-1} \cap (\iota_1 \times \iota_2)^* c_*(\alpha_1 \otimes \alpha_2)) \right)$$

where M denotes the product $M_1 \times M_2$ and $\mathcal{O}(1) = \mathcal{O}_{\mathbb{P}(N_{X_1 \times X_2} M)}(1)$. Using the Theorem 3.1.1 and Example 1.5.10, it follows that $s(X_1 \times X_2; \alpha_1 \otimes \alpha_2)$ is equals to

$$\eta_* \left(\sum_{i \geq 0} c_1(\mathcal{O}(1))^i \cap \eta^*(c(TM|_{X_1 \times X_2})^{-1} \cap (\iota_1^* c_*(\alpha_1) \times \iota_2^* c_*(\alpha_2))) \right).$$

Note that $TM|_{X_1 \times X_2} = TM_1|_{X_1} \oplus TM_2|_{X_2}$. Thus, $s(X_1 \times X_2; \alpha_1 \otimes \alpha_2)$ is equals to

$$\eta_* \left(\sum_{i \geq 0} c_1(\mathcal{O}(1))^i \cap \eta^*(c(TM_1|_{X_1} \oplus TM_2|_{X_2})^{-1} \cap (\iota_1^* c_*(\alpha_1) \times \iota_2^* c_*(\alpha_2))) \right).$$

With a similar argument that we use in the proof of Proposition 3.1.3, we get $c(TM_1|_{X_1} \oplus TM_2|_{X_2})^{-1} \cap (\iota_1^* c_*(\alpha_1) \times \iota_2^* c_*(\alpha_2)) = (c(TM_1|_{X_1})^{-1} \cap \iota_1^* c_*(\alpha_1)) \times (c(TM_2|_{X_2})^{-1} \cap \iota_2^* c_*(\alpha_2))$.

Using successively the Example 1.5.11, $s(X_1 \times X_2; \alpha_1 \otimes \alpha_2)$ is equals to

$$\begin{aligned} \eta_* \left(\sum_{i \geq 0} \left(c_1(\mathcal{O}_{\mathbb{P}(N_{X_1} M_1)}(1))^i \cap \eta_1^*(c(TM_1|_{X_1})^{-1} \cap \iota_1^* c_*(\alpha_1)) \right) \times \right. \\ \left. \times \left(c_1(\mathcal{O}_{\mathbb{P}(N_{X_2} M_2)}(1))^i \cap \eta_2^*(c(TM_2|_{X_2})^{-1} \cap \iota_2^* c_*(\alpha_2)) \right) \right). \end{aligned}$$

Lastly, by Proposition 1.5.4(b), it follows that

$$\begin{aligned} & s(X_1 \times X_2; \alpha_1 \otimes \alpha_2) \\ &= (\eta_1)_* \left(\sum_{i \geq 0} \left(c_1(\mathcal{O}_{\mathbb{P}(N_{X_1} M_1)}(1))^i \cap \eta_1^*(c(TM_1|_{X_1})^{-1} \cap \iota_1^* c_*(\alpha_1)) \right) \right) \times \\ & \quad \times (\eta_2)_* \left(\sum_{i \geq 0} \left(c_1(\mathcal{O}_{\mathbb{P}(N_{X_2} M_2)}(1))^i \cap \eta_2^*(c(TM_2|_{X_2})^{-1} \cap \iota_2^* c_*(\alpha_2)) \right) \right) \\ &= s(X_1; \alpha_1) \times s(X_2; \alpha_2). \end{aligned}$$

■

We can use the above proposition to inductively show the following result.

Corollary 4.0.4 *Let X_1, \dots, X_r be schemes which can be imbedded as subschemes of nonsingular varieties M_1, \dots, M_r , respectively. For each $i = 1, \dots, r$, consider α_i a constructible function on M_i . Then*

$$s(X_1 \times \cdots \times X_r; \alpha_1 \otimes \cdots \otimes \alpha_r) = s(X_1; \alpha_1) \times \cdots \times s(X_r; \alpha_r).$$

Proposition 4.0.5 *Consider $f : M' \rightarrow M$ a proper and flat morphism of nonsingular schemes, $X \subset M$ a closed subscheme, $X' = f^{-1}(X)$ the inverse image scheme, $g : X' \rightarrow X$ the induced morphism. Given a constructible function α' on M' , set $\alpha = f_*\alpha'$. Then*

$$g^*(s(X \subset M; \alpha)) = c(N_{X'}M')^{-1}c(f^*(TM)|_{X'})^{-1} \cap \iota'^*c_*(\alpha').$$

Proof. Let B be the blow-up of M along X and let B' be the blow-up of M' along X' . Denote by \widetilde{X} the exceptional divisor in B with projection $\eta : \widetilde{X} \rightarrow X$, and denote by \widetilde{X}' the exceptional divisor in B' with projection $\eta' : \widetilde{X}' \rightarrow X'$. Consider $F : B' \rightarrow B$ the induced morphism such that $F^*\widetilde{X} = \widetilde{X}'$ and consider G the induced morphism from \widetilde{X}' to \widetilde{X} . Moreover, let $\mathcal{O}(1)$ be the canonical line bundle on \widetilde{X} . In this way, $G^*\mathcal{O}(1)$ is the canonical line bundle on \widetilde{X}' . Let us look at the following commutative diagram:

$$\begin{array}{ccccc} G^*\mathcal{O}(1) & \longrightarrow & \widetilde{X}' & \xrightarrow{G} & \widetilde{X} & \longleftarrow & \mathcal{O}(1) \\ & & \eta' \downarrow & & \downarrow \eta & & \\ & & X' & \xrightarrow{g} & X & & \\ & & \iota' \downarrow & & \downarrow \iota & & \\ & & M' & \xrightarrow{f} & M & & \end{array}$$

that is, $g \circ \eta' = \eta \circ G$ and $f \circ \iota' = \iota \circ g$. Below we have a sequence of equalities in which we use several times the commutativity of the previous diagram and the projection

formula:

$$\begin{aligned}
g^*(s(X \subset M; \alpha)) &= g^* \eta_* \left(\sum_{i \geq 0} c_1(\mathcal{O}(1))^i \cap \eta^*(c(TM|_X)^{-1} \cap \iota^* c_*(\alpha)) \right) \\
&= g^* \eta_* \left(\sum_{i \geq 0} c_1(\mathcal{O}(1))^i \cap \eta^*(c(\iota^*(TM))^{-1} \cap \iota^* c_*(f_* \alpha')) \right) \\
&= g^* \eta_* \left(\sum_{i \geq 0} c_1(\mathcal{O}(1))^i \cap \eta^*(\iota^*(TM))^{-1} \cap \iota^* f_* c_*(\alpha') \right) \\
&= g^* \eta_* \left(\sum_{i \geq 0} c_1(\mathcal{O}(1))^i \cap \eta^*(c(\iota^*(TM))^{-1} \cap g_* \iota'^* c_*(\alpha')) \right) \\
&= g^* \eta_* \left(\sum_{i \geq 0} c_1(\mathcal{O}(1))^i \cap \eta^* g_*(c(g^* \iota^*(TM))^{-1} \cap \iota'^* c_*(\alpha')) \right) \\
&= g^* \eta_* \left(\sum_{i \geq 0} c_1(\mathcal{O}(1))^i \cap G_* \eta'^*(c(\iota'^* f^*(TM))^{-1} \cap \iota'^* c_*(\alpha')) \right) \\
&= \eta'_* \left(\sum_{i \geq 0} c_1(G^* \mathcal{O}(1))^i \cap \eta'^*(c(f^*(TM)|_{X'})^{-1} \cap \iota'^* c_*(\alpha')) \right) \\
&= c(N_{X'} M')^{-1} c(f^*(TM)|_{X'})^{-1} \cap \iota'^* c_*(\alpha').
\end{aligned}$$

■

Capítulo 5

Apêndice A

5.1 Algebraic sets

Let us denote by k the field of real numbers \mathbb{R} or the field of complex numbers \mathbb{C} . As references we cite [Milnor] and [Suwa2].

A subset $V \subset k^n$ is called an *algebraic set* if V is the locus of common zeros of some collection of polynomial functions on k^n . Let $I(V) \subset k[x_1, \dots, x_n]$ be the ideal consisting of those polynomials which vanish throughout V . By the Hilbert basis theorem, we know that $I(V)$ is finitely generated. A non-vacuous algebraic set V is called *variety* or an *irreducible algebraic set* if it cannot be expressed as the union of two proper algebraic subsets. One has that, V is irreducible if and only if $I(V)$ is a prime ideal.

Given an irreducible algebraic set V , the integral domain $k[x_1, \dots, x_n]/I(V)$ is called the *field of rational functions* on V . Consider $f_1, \dots, f_k \in k[x_1, \dots, x_n]$ which span the ideal $I(V)$ and, for each $x \in V$, consider the $k \times n$ matrix $(\partial f_i / \partial x_j)$ evaluated at x . Let r be the largest rank which this matrix attains at any point of V . In this way, a point $x \in V$ is called *non-singular* if the matrix $(\partial f_i / \partial x_j)$ attains its maximal rank r at x ; and *singular* if $\text{rank}(\partial f(x) / \partial x_i) < r$. Note that, the set of all singular points of V forms a proper algebraic subset of V .

Now let us define an equivalence relation in the set of subsets of k^n . Let $p \in k^n$. Given A and B subsets of k^n , one defines $A \sim_p B$ if there is a neighborhood U of p

such that $A \cap U = B \cap U$. The equivalence class represented by the set A is denoted by (A, p) or, simply, A . Moreover, consider two functions $f, g : k^n \rightarrow k^m$: f and g are equivalent if there exists a neighborhood U of p such that $f|_U = g|_U$. This is an equivalence relation. The equivalence class represented by the function f is denoted by $f : (k^n, p) \rightarrow k^m$. In this case, one says that $f : (k^n, p) \rightarrow k^m$ is a germ of function at p . When $f(p) = q$, it is denoted by $f : (k^n, p) \rightarrow (k^m, q)$.

Analogously, we may define analytic sets, taking analytic functions instead polynomial functions.

5.2 Sheaves and schemes

The references are [H] and [G-W].

Let X be a topological space. A *presheaf* \mathcal{F} on X consist of the: for every open set U of X a set $\mathcal{F}(U)$; and for each pair of open set $U \subseteq V$ a map $\rho_U^V : \mathcal{F}(V) \rightarrow \mathcal{F}(U)$, called *restriction map*, such that

- (1) $\rho_U^U = id_{\mathcal{F}(U)}$ for all open set $U \subseteq X$,
- (2) for $U \subseteq V \subseteq W$ open sets of X , $\rho_U^W = \rho_U^V \circ \rho_V^W$.

Let \mathcal{F}_1 and \mathcal{F}_2 be presheaves on X . A *morphism of presheaves* $\varphi : \mathcal{F}_1 \rightarrow \mathcal{F}_2$ is a family of maps $\varphi(U) : \mathcal{F}_1(U) \rightarrow \mathcal{F}_2(U)$ for all $U \subseteq V$ open, such that for all pairs of open sets $U \subseteq V$ in X the following diagram commutes

$$\begin{array}{ccc} \mathcal{F}_1(V) & \xrightarrow{\varphi(V)} & \mathcal{F}_2(V) \\ \rho_U^V \downarrow & & \downarrow \rho_U^V \\ \mathcal{F}_1(U) & \xrightarrow{\varphi(U)} & \mathcal{F}_2(U) \end{array}$$

A presheaf \mathcal{F} is called a *sheaf* if for all open sets U in X and every open covering $U = \cup_i U_i$ the following conditions hold

- (a) Given $s, s' \in \mathcal{F}(U)$ with $\rho_{U_i}^U(s) = \rho_{U_i}^U(s')$ for all i , one has $s = s'$.
- (b) Given $s_i \in \mathcal{F}(U_i)$ for all i such that $\rho_{U_i \cap U_j}^U(s_i) = \rho_{U_i \cap U_j}^U(s_j)$ for all i, j , then there is an $s \in \mathcal{F}(U)$ such that $\rho_{U_i}^U(s) = s_i$.

A *morphism of sheaves* is a morphism of presheaves. In an analogous way, one defines the notion of a sheaf of abelian groups, a sheaf of rings, a sheaf of modules, or a sheaf of algebras. If \mathcal{F} is a presheaf on X , and if x is a point of X , we define the stalk \mathcal{F}_x of \mathcal{F} at x to be the direct limit of the groups $\mathcal{F}(U)$ for all open sets U containing x , via the restriction maps ρ .

Example 5.2.1 Let X be a complex manifold. Denote by \mathcal{O}_X the sheaf of holomorphic functions on X defined as follows: Given an open set U in X one has

$$\mathcal{O}_X(U) = \{f : U \longrightarrow \mathbb{C} \mid f \text{ is holomorphic}\}.$$

A *ringed space* is a pair (X, \mathcal{O}_X) , where X is a topological space and \mathcal{O}_X is a sheaf of (commutative) rings on X . Let (X, \mathcal{O}_X) and (Y, \mathcal{O}_Y) be ringed spaces. One defines a *morphism of ringed spaces* $(X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ as a pair $(f, f^\#)$, where $f : X \longrightarrow Y$ is a continuous map and $f^\# : \mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X$ is a morphism of sheaves of rings on Y . The sheaf $f_*\mathcal{O}_X$ is given by $f_*\mathcal{O}_X(U) = \mathcal{O}_X(f^{-1}(U))$ for all open set U in Y . Moreover, a locally ringed space is a ringed space (X, \mathcal{O}_X) such that for all $x \in X$ the stalk $\mathcal{O}_{X,x}$ is a local ring. A *morphism of locally ringed space* $(X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces $(f, f^\#)$ such that for all $x \in X$ the induced homomorphism on stalks $f_x^\# : \mathcal{O}_{Y,f(x)} \longrightarrow \mathcal{O}_{X,x}$ is a local ring homomorphism. An *isomorphism of locally ringed spaces* is a morphism with a two-sided inverse.

Let (X, \mathcal{O}_X) be a ringed space. An \mathcal{O}_X -module is a sheaf \mathcal{F} on X , such that for each open set $U \subseteq X$, the group $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$ -module, and for each inclusion of open sets $U \subseteq V$, the restriction map $\mathcal{F}(V) \longrightarrow \mathcal{F}(U)$ is compatible with the module structures.

For any ring A , one can associate the topological space $\text{Spec}A$, which is the set of all prime ideals of A equipped with the so-called Zariski topology. One can also define $\mathcal{O}_{\text{Spec}A}$ on $\text{Spec}A$. Given an open set $U \subseteq \text{Spec}A$, consider $\mathcal{O}_{\text{Spec}A}(U)$ the set of functions $s : U \longrightarrow \prod_{\mathfrak{p} \in U} A_{\mathfrak{p}}$, such that $s(\mathfrak{p}) \in A_{\mathfrak{p}}$ for each \mathfrak{p} , and such that s is locally a quotient of elements of A . Note that, $\mathcal{O}_{\text{Spec}A}$ defines a sheaf of rings. Moreover, $(\text{Spec}A, \mathcal{O}_{\text{Spec}A})$ is a locally ringed space, called the *affine scheme*.

A *scheme* is a locally ringed space (X, \mathcal{O}_X) which admits an open covering $X = \bigcup_{i \in I} U_i$ such that all locally ringed spaces $(U_i, \mathcal{O}_X|_{U_i})$ which are isomorphic to affine schemes. A *morphism of schemes* is a morphism of locally ringed spaces.

5.3 Whitney stratification

Let V be a complex analytic variety V of dimension n in a complex manifold M . An *analytic stratification* of V is a locally finite family $(V_\alpha)_\alpha$ of non-singular analytic subspaces of V , which are called strata, such that:

- (1) The family is a partition of V .
- (2) For each V_α , the closures in V of both $\overline{V_\alpha}$ and $\overline{V_\alpha} \setminus V_\alpha$ are analytic in V .
- (3) For each pair (V_α, V_β) such that $V_\alpha \cap \overline{V_\beta} \neq \emptyset$ one has $V_\alpha \subset \overline{V_\beta}$.

A *Whitney stratification* is a stratification $(V_\alpha)_\alpha$ that satisfies the following conditions, known as the Whitney conditions (a) and (b), for every pair (V_α, V_β) such that $V_\alpha \subset \overline{V_\beta}$. Given $y \in V_\alpha$, consider $x_i \in V_\beta$ a sequence converging to y , and $y_i \in V_\alpha$ another sequence that also converges to y . Suppose these sequences are such that the sequence of secant lines $l_i = \overline{x_i y_i}$ also converges to some limiting line l , and the tangent planes $T_{x_i} V_\beta$ converges to some limiting plane τ . The Whitney conditions (a) and (b) are the following:

- (a) The limit space τ contains the tangent space of the stratum V_α at y , that is, $T_y V_\alpha \subset \tau$.
- (b) The limit space τ contains all the limits of secants, that is, $l \subset \tau$.

There are some interesting facts about a Whitney stratification. Among them, we have: Every closed analytic subset of an analytic manifold admits a Whitney stratification; Whitney stratified spaces can be triangulated compatibly with the stratification; and Whitney stratifications are locally topologically trivial along the strata.

5.4 Chern-Weil Theory

The main references in this section are [Chern] and [Milnor-Stasheff].

Let E be a complex r -vector bundle on a n -dimensional smooth manifold M . Denote by $T_{\mathbb{C}}^* M = T^* M \otimes \mathbb{C}$ the complexified dual tangent bundle of M , $\Omega^1(M)$ the module of smooth sections of $T_{\mathbb{C}}^* M$ and $\Gamma(E)$ the module of smooth sections of E .

Definition 5.4.1 A connection π on E is a \mathbb{C} -linear map $\nabla : \Gamma(E) \longrightarrow \Omega^1(M) \otimes \Gamma(E)$ satisfying Leibnitz' rule

$$\nabla(fs) = df \otimes s + f\nabla(s)$$

where $s \in \Gamma(E)$ and $f : M \longrightarrow \mathbb{C}$ is smooth.

Let $\underline{s} = s_1, \dots, s_r : U \longrightarrow E$ be a frame, where U is open subset in M . Given a connection ∇ on E , we can decompose $\nabla(s_i)$ into its components, writing $\nabla(s_i) = \sum_{j=1}^r \theta_{ij} s_j$. The matrix $\theta = (\theta_{ij})$ of 1-forms is called the *connection matrix* of ∇ with respect to \underline{s} .

A connection ∇ on E induces a unique \mathbb{C} -linear function $\nabla : \Omega^1(M) \otimes \Gamma(E) \longrightarrow \Omega^2(M) \otimes \Gamma(E)$ that satisfies $\nabla(\omega \otimes s) = d\omega \otimes s - \omega \wedge \nabla s$ for all $\omega \in \Omega^1(M)$ and $s \in \Gamma(E)$. The \mathbb{C} -linear function $K_\nabla := \nabla^2$ is called the *curvature tensor* of the connection ∇ .

Denote by $M_n(\mathbb{C})$ the algebra consisting of all $n \times n$ complex matrices. An *invariant polynomial* on $M_n(\mathbb{C})$ is a function $P : M_n(\mathbb{C}) \longrightarrow \mathbb{C}$, which can be expressed as a complex polynomial in the entries of the matrix, and satisfies $P(XY) = P(YX)$, or equivalently, $P(TXT^{-1}) = P(X)$ for all non-singular matrix T . Note that, the trace and determinant functions are examples of invariant polynomials. For any invariant polynomial P , we have a well-defined global differential form, denoted by $P(K_\nabla)$.

Theorem 5.4.2 Given P an invariant polynomial, we have

- (a) The form $P(K_\nabla)$ is closed, that is $dP(K_\nabla) = 0$.
- (b) The cohomology class $[P(K)] = [P(K_\nabla)]$ is independent of the connection ∇ .

Given any $A \in M_n(\mathbb{C})$, let $\sigma_k(A)$ the k -th elementary symmetric function of the eigenvalues of A . One has $\det(Id + tA) = 1 + t\sigma_1(A) + \dots + t^n \sigma_n(A)$.

Definition 5.4.3 The k -th Chern class of E is defined by

$$c_k(E) := \left[\sigma_k \left(\frac{\sqrt{-1}}{2\pi} K_\nabla \right) \right] \in H_{DR}^{2k}(M; \mathbb{C}).$$

The total Chern class of E is defined by

$$c(E) := 1 + c_1(E) + \dots + c_r(E) \in H_{DR}^{\text{even}}(M; \mathbb{C}).$$

We have expected properties, such as $c(f^*E) = f^*c(E)$ for all smooth map $f : M' \longrightarrow M$; and $c(E \oplus F) = c(E)c(F)$, known as the Whitney sum formula.

5.5 Derived categories

Let X be a complex analytic space. One denotes by $\mathcal{D}_c^b(X)$ the derived category of bounded, constructible complexes of sheaves of \mathbb{C} -vector spaces on X . Given F^\bullet , the shifted complex $F^\bullet[l]$ is defined by $(F^\bullet[l])^k = F^{k+l}$ with differential given by $d_{[l]}^k = (-1)^l d^{k+l}$. For any $F^\bullet \in \mathcal{D}_c^b(X)$ and $p \in X$, one denotes by $\mathcal{H}^k(F^\bullet)_p$ the stalk cohomology of F^\bullet at p , and thus, the Euler characteristic of F^\bullet at p is given by

$$\chi(F^\bullet)_p = \sum_k (-1)^k \dim_{\mathbb{C}} \mathcal{H}^k(F^\bullet)_p.$$

Moreover, the Euler characteristic of X with coefficients in F^\bullet , denoted by $\chi(X, F^\bullet)$, is given by

$$\chi(X, F^\bullet) = \sum_k (-1)^k \dim_{\mathbb{C}} \mathbb{H}^k(X, F^\bullet)$$

where $\mathbb{H}^\bullet(X, F^\bullet)$ denotes the hypercohomology groups of X with coefficients in F^\bullet . Now, consider \mathcal{S} a Whitney stratification of X . Given $p \in S$, set $\chi(F_S^\bullet) := \chi(F^\bullet)_p$. Then, we have

$$\chi(X, F^\bullet) = \sum_{S \in \mathcal{S}} \chi(F_S^\bullet) \chi(S).$$

Referências Bibliográficas

- [Alu1] Paolo Aluffi, *Chern classes of free hypersurface arrangements*, Journal of Singularities, 5:19-32, (2012).
- [Alu2] Paolo Aluffi, *Grothendieck classes and Chern classes of hyperplane arrangements*, International Mathematics Research Notices 8, (2013) 1873-1900.
- [Alu3] Paolo Aluffi, *Characteristic classes of singular varieties*. Topics in cohomological studies of algebraic varieties, 1-32, Trends Math., Birkhäuser, Basel, 2005.
- [A4] Paolo Aluffi. *Chern classes for singular hypersurfaces*. Trans. Amer. Math. Soc. 351 (1999), no. 10, 3989-4026.
- [Alu5] Paolo Aluffi. *Limits of Chow groups, and a new construction of Chern-Schwartz-MacPherson classes*. Pure Appl. Math. Q. 2 (2006) 915-941.
- [B-S] J.P. Brasselet and M.-H. Schwartz, *Sur les classes de Chern d'un ensemble analytique complexe*, Asterisque 82-83 (1981), 93-147.
- [B-S-S] Jean-Paul Brasselet, José Seade, Tatsuo Suwa, *Vector Fields on Singular Varieties*, Lecture Notes in Mathematics 1987, Springer-Verlag, Berlin-Heidelberg, 2009.
- [Brasselet] Jean-Paul Brasselet, *From Chern classes to Milnor classes - A history of characteristic classes for singular varieties*. Singularities-Sapporo 1998, 31-52, Adv. Stud. Pure Math., 29, Kinokuniya, Tokyo, 2000.

- [BLSS] Jean-Paul Brasselet; Daniel Lehmann; José Seade; Tatsuo Suwa. *Milnor numbers and classes of local complete intersections*. Proc. Japan Acad. Ser. A Math. Sci. 75 (1999), no. 10, 179-183.
- [B-M-M] J. Briançon, P. Maisonobe, M. Merle, *Localisation de systèmes différentiels, stratifications de Whitney et condition de Thom*, Invent. Math. 117 (1994), 531-550.
- [BR] J. W. Bruce, R. M. Roberts, *Critical points of functions on analytic varieties*, Topology 27 (1988), No. 1, 57-90.
- [B-M-S] R. Callejas-Bedregal, M. F. Z. Morgado and J. Seade, *On the Chern classes of singular complete intersections*. to appear.
- [B-M-S 2] R. Callejas-Bedregal, M. F. Z. Morgado and J. Seade, *Lê cycles and Milnor classes*. Invent. Math. 197 (2014), no. 2, 453-482.
- [Chern] Shiing-shen Chern, *Complex manifolds without potential theory (with an appendix on the geometry of characteristic classes)*. Second edition. Springer-Verlag, New York, 1995.
- [Deligne] Pierre Deligne, *Equations Différentielles à Points Singuliers Réguliers*, Springer (1970).
- [Dimca] A. Dimca, *Sheaves in topology*. Universitext. Springer-Verlag, Berlin, (2004).
- [D-A] I. Dolgachev, M. Kapranov. *Arrangements of hyperplanes and vector bundles on \mathbb{P}^n* . Duke Math. J. 71 (1993), no. 3, 633-664.
- [Ful] W. Fulton, *Intersection Theory*, Ergeb. Math. Grenzgeb. 2, Springer-Verlag, New York, 1984.
- [Ful2] W. Fulton. *Introduction to intersection theory in algebraic geometry*. CBMS Regional Conference Series in Mathematics, 54. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1984.

- [FJ] W. Fulton and K. Johnson, *Canonical classes of singular varieties*. Manuscripta Math. 32, no. 3-4 (1980), 381-389.
- [G-S] G. Gonzalez-Sprinberg, *L'obstruction locale d'Euler et le théoreme de MacPherson*, Astérisque 82-83 (1981), 7-32.
- [G-M] M. Goresky, R. MacPherson: *Stratified Morse theory*. Springer-Verlag (1988).
- [G-W] U. Görtz, T. Wedhorn. *Algebraic geometry I*. Advanced Lectures in Mathematics. Vieweg+Teubner, Wiesbaden, 2010.
- [G-H] P. Griffiths and J. Harris. *Principles of algebraic geometry*, Wiley (1978).
- [H] R. Hartshorne. *Algebraic geometry*. Springer-Verlag, New York, 1977.
- [Huy] D. Huybrechts, *Complex Geometry: An Introduction*, Springer, 2006.
- [K-S] M. Kashiwara, P. Schapira, *Sheaves on Manifolds*. Springer, Berlin Heidelberg (1990).
- [Ken] G. Kennedy, *MacPherson's Chern classes of singular algebraic varieties*, Comm. Algebra 18 (1990), 2821-2839.
- [Kwie] M. Kwieciński, *Formule du produit pour les classes caractéristiques de Chern-Schwartz-MacPherson et homologie d'intersection*, C. R. Acad. Sci. Paris Sér. I Math. 314 (1992), no. 8, 625-628.
- [Kwei-Yoka] M. Kwieciński, and S. Yokura. *Product formula for twisted MacPherson Classe*. Proc. Japan Acad. 68 (1992), 167-171.
- [Liao1] Xia Liao, *Chern classes for logarithmic vector fields*, Journal of Singularities, 5:109-114, 2012.
- [Liao2] Xia Liao. *Chern classes of logarithmic vector fields for locally-homogenous free divisors*. arXiv:1205.3843.
- [Liao3] Xia Liao, *Chern classes of logarithmic derivations for free divisors with Jacobian ideal of linear type*, Journal of the Mathematical Society of Japan, 70(3), (2018) 975-988.

- [Liao4] Xia Liao, *Chern Classes of Sheaves of Logarithmic Vector Fields for Free Divisors* (Doctoral dissertation), 2013.
- [L] E. J. N. Looijenga, *Isolated Singular Points on Complete Intersections*, Cambridge University Press, 1 edition (1984).
- [Mac] R. D. MacPherson. *Chern classes for singular algebraic varieties*. Ann. of Math. (2), Vol. 100, (1974) 423-432.
- [M-T] Ib Madsen and Jorgen Tornehave. *From Calculus to Cohomology. De Rham Cohomology and Characteristic Classes*. Cambridge University Press, first edition (1997).
- [Milnor] J. Milnor. *Singular points of complex hypersurfaces*. Annals of Mathematics Studies, No. 61 Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo 1968.
- [Milnor-Stasheff] John Milnor, James D. Stasheff, *Characteristic classes*, Princeton University Press, 1974.
- [O-Y] T. Ohmoto, S. Yokura, *Product Formulas for the Milnor Class*. Bulletin of the Polish Acad. of Sciences Math. Vol. 48 (2000), no. 4, 387-401.
- [P] A. Parusiński. *A generalization of the Milnor number*, Math. Ann. 281 (1988), 247-254.
- [PP] A. Parusiński and P. Pragacz, *A formula for the Euler characteristic of singular hypersurfaces*, J. Alg. Geom. 4 (1995), 337-351.
- [PP2] A. Parusiński and P. Pragacz, *Chern-Schwartz-MacPherson classes and the Euler characteristic of degeneracy loci and special divisors*, Jour. Amer. Math. Soc. 8 (1995), no. 4, 793-817.
- [PP3] A. Parusiński and P. Pragacz, *Characteristic classes of hypersurfaces and characteristic cycles*. J. Algebraic Geom. 10 (2001), no. 1, 63-79.
- [P3] A. Parusiński, *Limits of tangent spaces to fibres and the ω_f condition*, Duke Math. J. 72 (1993), 99-108.

- [P4] A. Parusiński, *Characteristic Classes of Singular Varieties via Conormal Geometry*, Lectures at the annual meeting of GDR Singularités (2012).
- [P5] A. Parusiński, *Characteristic classes of singular varieties*. Singularity theory and its applications, Adv. Stud. Pure Math., 43, Math. Soc. Japan, Tokyo (2006) 347-367.
- [P2] A. Parusiński, C. McCrory. *Algebraically constructible functions*. Ann. Sci. Ec. Norm. Sup. 30, 527-552 (1997).
- [Sab] C. Sabbah, *Quelques remarques sur la géométrie des espaces conormaux*, Astérisque 130 (1985), 161-192.
- [Saito] Kyoji Saito, *Theory of logarithmic differential forms and logarithmic vector fields*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 27 (1980), no. 2, 265-291.
- [Scha] P. Schapira, *Operations on constructible functions*, Journal of Pure and Applied Algebra 72 (1991) 83-93.
- [Schür1] J. Schürmann, *Lectures on characteristic classes of constructible functions*. Notes by Piotr Pragacz and Andrzej Weber, Trends Math., Topics in cohomological studies of algebraic varieties, 175-201, Birkhäuser, Basel, 2005.
- [Schür2] J. Schürmann, *Topology of singular spaces and constructible sheaves*. "Monografie Matematyczne" New Series vol. 63, Birkhäuser-Verlag (2003).
- [SS] J. Seade, T. Suwa, *An adjunction formula for local complete intersections*, Int. J. Math. 9 (1998), 759-768.
- [Seade] J. Seade, *On Milnor's fibration theorem and its offspring after 50 years*. American Mathematical Society, Volume 56, Number 2 (2019) 281-348.
- [Sil] Roberto Silvotti. *On a conjecture of Varchenko*. Invent. Math. 126 (1996), no. 2, 235-248.
- [Suwa] T. Suwa, *Classes de Chern des intersections complètes locales*, C. R. Acad. Sci Paris 324 (1996), 67-70.

- [Suwa2] T. Suwa, *Introduction to Complex Analytic Geometry*, Advanced School and Workshop on Singularities in Geometry and Topology, 2005.
- [Le-Te] Lê D.T. and B. Teissier, *Varietes polaires locales et classes de Chern des varietes singulieres*, *Ann. of Math. (2)* 114 (1981), 457-491.
- [Y] S. Yokura, *On characteristic classes of complete intersections*. Algebraic geometry: Hirzebruch 70 (Warsaw, 1998), 349-369, *Contemp. Math.*, 241, Amer. Math. Soc., Providence, RI, 1999.