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Programa Associado de Pós-Graduação em Matemática
Doutorado em Matemática

Some generalizations of minimax theorems for lower semicontinuous functionals and a new approach for logarithmic Schrödinger equations

by

Ismael Sandro da Silva

Campina Grande - PB

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Tese apresentada ao Corpo Docente do Programa Associado de Pós-Graduação em Matemática - UFPB/UFCG, como requisito parcial para obtenção do título de Doutor em Matemática.

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Resumo

O presente trabalho é soerguido em duas direções principais: primeiro, desenvolvem-se novos teoremas abstratos para uma classe de funcionais semi-contínuos inferiormente da seguinte forma: dado X um espaço de Banach, $I = \Phi + \Psi : X \longrightarrow (-\infty, \infty]$ é uma soma de um funcional Φ de classe C^1 com um funcional convexo e semicontínuo inferiormente $\Psi : X \longrightarrow (-\infty, \infty]$ $(\Psi \not\equiv \infty)$. Nossos resultados são referentes à Teoria dos Pontos Críticos para funcionais não-diferenciáveis construída por Szulkin em [81]; é-se provada uma generalização do teorema da fonte de Bartsch [23] e também de um teorema devido a Heinz em [61] relacionado com a noção do gênero de conjuntos fechados e simétricos com respeito à origem. Uma versão do teorema do passo da montanha simétrico é também provada. Como aplicação dos resultados abstratos mencionados, mostra-se a existência de uma infinidade de soluções para uma ampla classe de problemas elípticos. Os problemas envolvem não-linearidades logarítmicas, não-lineradades descontínuas e o operador 1-Laplaciano.

Posteriormente, como uma consequência natural de nossos estudos, introduzimos uma nova abordagem para o estudo das equações logarítmicas que nos possibilita aplicar métodos variacionais clássicos para funcionais de classe C^1 no intuito de obter soluções para diferentes classes de equações logarítmicas de Schrödinger. Essa nova ideia é introduzida utilizando-se técnicas exploradas no estudo dos espaços de Orlicz. Os resultados obtidos garantem desde resultados de multiplicidade de soluções para equações logarítmicas de Schrödinger envolvendo a categoria de Lusternik-Schnirelmann, à existência de soluções positiva para uma classe de equações logarítmicas sobre um domínio exterior, considerando diferentes condições de contorno.

Palavras-chave: funcionais semicontínuos inferiormente, teoria dos tontos críticos para funcionais não-diferenciáveis, teorema da fonte, equações logarítmicas de Schrödinger.

Abstract

The current text has been constructed in two main directions: first one, we have established new abstracts theorems for a class of semicontinuous functionals of the following form: let X be a Banch space, $I = \Phi + \Psi : X \longrightarrow (-\infty, \infty]$ is a sum of a C^1-functional Φ with a convex lower semicontinuous functional $\Psi : X \longrightarrow (-\infty, \infty]$ $(\Psi \not\equiv \infty)$. Our results are referring to the nonsmooth critical point theory developed by Szulkin in [81]; it is proved a generalization of the Bartsch's fountain theorem [23] and also a theorem due to Heinz in [61] related with the genus of $\mathbb{Z}_2$-symmetric closed sets. A version of the symmetric mountain pass theorem it is also proved. As application of the mentioned abstract result, we have showed the existence of many infinitely solutions for large classes of elliptical problems. The problems involve logarithmic nonlinearities, discontinuous nonlinearities and the 1-Laplacian operator.

After that, as a byproduct of our study, we have introduced a new approach in order to study logarithmic equations which allow us to apply C^1-variational methods to get solutions for several classes of logarithmic Schrödinger equations. We have established this new approach through the Orlicz space's techniques. The produced results include the multiplicity of solutions for logarithmic Schrödinger equations involving the Lusternik-Schnirelmann category, and also they include the existence of positive solutions for a class of logarithmic equations on a exterior domain, by considering different boundary conditions.

Keywords: lower semicontinuous functionals, nonsmooth critical point theory, fountain theorem, logarithmic Schrödinger equations.

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"In the beginning was the Word, and the Word was with God, and the Word was God".

Holy Bible, ***Jhon 1:1*** *(King James Version)*

Dedication

To the most enchanting Flower (Yngrid M. A. S. da Silva), to my Parents and Brother, and, of course, to Mathematics.

Contents

Introdução

No estudo das Equações Diferenciais Parciais, os denominados métodos variacionais e cálculo das variações figuram como um tópico de notável relevância, em virtude de sua ampla aplicabilidade. Em linhas gerais, tal método consiste em associar a um problema, digamos por exemplo da forma

$$(E_1) \qquad \begin{cases} -\Delta u + V(x)u = f(u), & \text{em } \Omega \\ u \equiv 0, & \text{em } \partial\Omega, \end{cases}$$

com $\Omega \subset \mathbb{R}^N$ um conjunto aberto, um funcional do tipo $J : X \longrightarrow \mathbb{R}$, com X um espaço de Banach adequado que nos permita assegurar que $J \in C^1(X, \mathbb{R})$. É esperado que os pontos críticos de tal funcional coincidam com soluções do problema. Um funcional J nestes termos é dito o *funcional energia* ou *funcional de Eüler-Lagrange* associado ao problema. Esse método é amplamente difundido e bem consolidado no estudos das Equações Diferenciais, em especial no estudo de problemas elípticos. Aqui, apenas a título de exemplo, citamos os clássicos trabalhos de Rabinowitz [75, 76] e del Pino e Felmer [51].

Esse método tem intrínseco um dificuldade natural: as condicões sobre a função $f : \mathbb{R} \longrightarrow \mathbb{R}$ devem ser convenientes de modo a permitir a regularidade do funcional J. Isso inviabiliza, em um primeiro momento, o tratameto, via métodos variacionais clássicos, de equações do tipo (E_1) nas quais a função f não contenha as propriedades desejadas, a exemplo dos casos nos quais a função apresente descontinuidades.

No intento de abranger um maior número de casos, propostas de generalizações da cognominada *Teoria dos Pontos Críticos* tem sido idealizadas. Utilizando as técnicas da Análise convexa, os pioneiros trabalhos devidos a Clarke [41] e o de Chang [36] em 1981, permitiram a extensão da noção de ponto crítico para funcionais localmente Lipschitz. Isso possibilitou o estudo de equações com a estrutura dada em (E_1) nas quais a função f apresenta uma descontinuidade; veja, e.g., [16, 36, 45].

Posteriormente, em 1986, Szulkin [81] generalizou a Teoria dos Pontos Críticos para uma classe de funcionais semicontínuos inferiormente (s.c.i.) que é objeto de estudo do presente texto. A saber, Szulkin considerou funcionais $I : X \longrightarrow (-\infty, \infty]$, X um espaço de Banach, satisfazendo a seguinte condição:

(H) : $I = \Phi + \Psi : X \longrightarrow (-\infty, \infty]$, com $\Phi \in C^1(X, \mathbb{R})$ e $\Psi : X \longrightarrow (-\infty, \infty]$ um funcional s.c.i. convexo e próprio (i.e., não ocorre $\Psi \equiv \infty$).

Dado um ponto $u \in X$, diz-se que u é um ponto crítico para para um funcional $I = \Phi + \Psi$ satisfazendo a condição (H) descrita acima se $I(u) < \infty$ e

$$\langle \Phi'(u), v - u\rangle + \Psi(v) - \Psi(u) \geq 0, \ \ \forall v \in X.$$

Nota-se que, caso $\Psi \equiv 0$, temos $I = \Phi \in C^1(X, \mathbb{R})$ e a condição de ponto crítico acima fornece, pela arbitrariedade de v, que $\Phi'(u) \equiv 0$. Assim, o estudo de Szulkin é, de fato, uma generalização do caso clássico. Os trabalhos [10, 12, 13, 62, 69, 79] ilustram como a teoria desenvolvida por Szulkin fornece uma ferramento útil e abrangente no estudo das Equações Diferenciais.

A Teoria de Pontos Críticos para funcionais que satisfazem (H) proposta em [81] ainda nos fornece uma ferramenta para o estudo de desigualdades variacionais, isto possibilita sua utilização para o estudo de algumas aplicações físicas que recaem em desigualdades variacionais. Em [52, p. XVIII] podemos encontrar o seguinte exemplo.

Problema 1: *Suponha que $u(x, t)$ represente a pressão no ponto x no, instante t, em um fluido contido numa região $\Omega \subset \mathbb{R}^3$ delimitado por uma membrana, representada por $\partial\Omega$ que é semipermeável, i.e., permite que o fluido penetre em Ω mas evita que ele vaze completamente. Então, u satisfaz*

$$\int_\Omega \left(\frac{\partial u}{\partial t}(v - u) + \nabla_x u \nabla v + g(v - u) \right) dx \geq 0, \forall v \in H^1(\Omega),$$

onde g é uma função previamente prescita, satisfazendo uma condição de fronteira.

Em [34, 71] o leitor interessado poder encontrar mais resultados e aplicações da teoria apresentada em [81].

Os comentários acima atestam a relevância, tanto em perspectiva teóricas quanto no contexto de aplicações, da teoria proposta por Szulkin. Diante do exposto, como um dos alvos da presente tese, nos propusemos a complementar o trabalho feito em [81]. Com maior acurácia, revisando com detalhe os resultados desenvolvidos em [81], encontra-se uma extensa lista de resultados do tipo *minimax* válidos para funcionais verificando (H). Em verdade, as versões clássicas do Teoerma do Passo da Montanha de Ambrosetti-Rabinowitz [75, Theorem 2.2], do Teorema do Ponto de Sela [75, Theorem 4.6] e também do Teorema de Clark, que envolve a *teoria do gênero*, são generalizadas para a classe dos funcionais satisfazendo (H).

Atentando à literatura da Teoria dos Pontos Críticos, pudemos perceber que alguns resultados do tipo minimax não foram ainda estendidos para os funcionais verificando (H). Um exemplo importante é o do famoso Teorema da Fonte devido a Bartsch (see [23, 83]). O Teorema da Fonte tem sido explorado em muitos trabalhos no sentido de estabelecer a existência e multiplicidade de soluções para problemas elípticos; aqui referenciamos [23, 25, 26, 60, 68, 78, 85].

Em seu formato original, o Teorema da Fonte pode ser enunciado como segue: fixe X um espaço de Banach e, para cada $k \in \mathbb{N}$, fixe as notações abaixo.

$i)$: $Y_k := \bigoplus_{j=1}^{k} X_j$ e $Z_k := \overline{\bigoplus_{j=k}^{\infty} X_j}$;

$ii)$: $B_k := \{u \in Y_k;\ \|u\| \leq \rho_k\}$ e $N_k := \{u \in Z_k;\ \|u\| = r_k\}$, com $\rho_k > r_k > 0$.

Considere agora G um grupo topológico compacto agindo isometricamente em X e suponha a seguinte condição verificada:

(G_0) : O grupo G age isometricamente em X e $X = \overline{\bigoplus_{j\in\mathbb{N}} X_j}$, com $X_j \cong Y$ subespaços de dimensão finita invariantes pela ação de G e a ação de G em Y é *admissível* no sentido da Definição 1.2 no Capítulo 1.

Teorema 0.0.0.1 (Teorema da Fonte de Bartsch) *Seja $I \in C^1(X, \mathbb{R})$ um funcional G-invariante (i.e. $I(g\,\cdot) = I(\cdot), \forall g \in G$) que satisfaz a condição $(PS)_c$ para todo $c \in \mathbb{R}$. Assuma que*

$i)$: $a_k := \sup_{u\in Y_k, \|u\|=\rho_k} I(u) \leq 0$;

ii): $b_k := \inf_{u \in Z_k, \|u\| = r_k} I(u) \to \infty$.

Então, definindo $c_k := \inf_{\gamma \in \Theta_k} \sup_{u \in B_k} I(\gamma(u))$, com

$$\Theta_k := \{\gamma \in \Gamma_G(B_k); \ \ \gamma|_{\partial B_k} \equiv Id|_{\partial B_k}\}. \tag{1}$$

O funcional I tem uma sequência de pontos críticos (u_k) tal que $I(u_k) = c_k \to \infty$.

Em [45], Dai estabeleceu uma versão do resultado acima para funcionais I que são localmente Lipschitz e utilizou o resultado para estabelecer a existência de uma infinidade de soluções para um problema elíptico do tipo (E) no qual a função f possui descontinuidades. É portanto natural indagar se uma versão do Teorema da Fonte, nos termos acima e em [45], seria válida para *funcionais I do tipo Szulkin*, i.e., funcionais s.c.i satisfazendo a condição em (H).

Afirmamos, precipuamente, que a resposta à indagação suscitada no parágrafo anterior é afirmativa. Como um dos nossos principais resultados abstratos neste texto, no Capítulo 1, generalizamos o Teorema da Fonte devido a Bartsch para funcionais do tipo Szulkin (veja o Theorem 1.4).

Em [25] e [83, Chapter 3] podemos encontrar uma versão dual do Teorema da Fonte. Tal resultado pode ser interpretado como uma complemento - ou como um corolário de fato; veja a prova de tal resultado em [83, Theorem 3.18] - do clássico Teorema da Fonte de Bartsch. A versão dual do Teorema da Fonte fornece condições para que um funcional G-invariante possua uma sequência negativa de pontos críticos (c_k) satifazendo $c_k \to 0$. É natural perguntarmos-nos se uma versão dual do Teorema da Fonte não seria possível para funcionais verificando (H). Não obstante, uma vez que a principal ideia em [83, Theorem 3.18] consiste em aplicar o Teorema da Fonte ao funcional $-I$ para obtermos uma sequência de valores críticos para o funcional I, concluímos que a replicação imediata deste resultado não é possível para funcionais do tipo Szulkin. De fato, se quando I verifica (H) não é imediato que o funcional $-I$ também verifique, assim não podemos aplicar a teoria desenvolvida em [81] concomitantemente aos funcionais I e $-I$.

Visando complementar nosso estudo, ante à ausência de uma versão dual Para o Teorema da Fonte no contexto dos funcionais do tipo Szulkin, debruçamos-nos à investigar a possibilidade de estabelecer um resultado que nos desse o mesmo tipo de informação do Teorema da Fonte dual: encontrar uma sequência de valores críticos negatios (c_k) para um funcional G-invariante I com $c_k \to 0$. Nessa característica, provamos ser válida uma versão do Teorema de Heinz [61, Proposition 2.2], que em sua versão clássica complementa o famoso Teorema de Clark envolvendo teoria de gênero (see [39] para tópicos correlatos). Pudemos notar que em [81], embora uma versão do Teorema de Clark seja estebelecida, não é provada uma versão do resultado devido a Heinz em [61]. Em nosso resultado (Teorema 1.5 na sequência), além de generalizar o resultado devido a Heinz para os funcionais com a estrutura posta em (H), nós consideramos um tipo de ação mais geral do que clássica ação antípoda de $\mathbb{Z}_2 = \{Id, -Id\}$.

Com a técnica introduzida para provarmos o Teorema 1.5 no Capítulo 1, percebemos ser possível complementar um dos resultados densenvolvidos por Szulkin em [81]. Mais precisamente, nossos argumentos permitem provar que a sequência de valores críticos (d_k) dada em [81, Corollary 4.8] é tal que

$$d_k \longrightarrow \infty.$$

Esse o conteúdo do Teorema 1.6 do Capítulo 1.

Uma vez que os resultado apresentados no Capítulo 1 (os quais também constam em [8]) estabelecem novos teoremas minimax para funcionais do tipo Szulkin e que alguns resultados em [81] são melhoradas, nosso estudo pode se configurar como um complemento à teoria proposta por Szulkin em [81].

Como consequência dos teoremas abstratos desenvolvidos, no Capítulo 1 garantimos a existência de uma infinidade de problemas elípticos com simetria e que possuem o funcional energia associado com a forma dada em (H).

Utilizando nossa versão generalizada do Teorema da Fonte, provamos a existência de infinitas soluções para o seguinte problema de inclusão variacional:

$$\begin{cases} -\Delta u + u + \partial F(x,u) \ni u \log u^2, & \text{q.t.p. em} \quad \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases}$$

com $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ uma função N-mensurável e tal que $F(x,t) := \displaystyle\int_0^t f(x,t)\, ds \geq 0$ seja localmente Lipschitz. Como usual, $\partial F(x,t)$ denota o *gradiente generalizado* de F com respeito à variável $t \in \mathbb{R}$ no ponto $x \in \mathbb{R}^N$ (veja [36, 41] para mais detalhes envolvendo a noção de gradiente generalizado).

Tal problema foi inspirado no resultado devido a Ji e Szulkin em [62], no qual, explorando propriedades particulares da não-lineraidade $f(t) = t\log t^2$, estabeleceram a existência de uma infinidade de soluções para o problema

$$-\Delta u + V(x)u = u\log u^2, \quad x \in \mathbb{R}^N, \tag{2}$$

com $V \in C(\mathbb{R}^N, \mathbb{R})$ satisfazendo $\lim_{|x|\to+\infty} V(x) = +\infty$.

A segunda classe de problemas que estudamos é um tipo de perturbação de equações logarítmicas de Schrödinger da forma:

$$\begin{cases} -\Delta u + u = u\log u^2 + \lambda h(x)|u|^{q-2}u \ \text{ em } \ \mathbb{R}^N \\ u \in H^1(\mathbb{R}^N), \end{cases}$$

Nesse caso, utilizamos nossa versão generalizada do Teorema de Heinz para assegurar a existência de uma infinidade de soluções para o problema acima. A necessidade de recorrer à Teoria de Ponto Crítico proposta em [81] dá-se pelo fato de que a condição de crescimento sobre $f(t) = t\log t^2$ não assegura a boa definição do funcional energia associado ao problema sobre o espaço $H^1(\mathbb{R}^N)$ (veja, e.g., [6, 7, 10–13, 62, 69, 79] para mais comentários envolvendo tal sutileza).

Por fim, como aplicação de nosso último teorema do tipo minimax provado no Capítulo 1, mostramos a existência de uma infinidade soluções para a classe de problemas a seguir envolvendo o operador 1-Laplaciano.

$$\begin{cases} -\Delta_1 u = |u|^{p-2}u, \ \text{ em } \ \Omega, \\ u|_\Omega = 0, \ \text{ em } \partial\Omega \end{cases}$$

Aqui $\Omega \subset \mathbb{R}^N$, $N \geq 2$, é um domínio limitado com fronteira suave e $p \in (1, 1^*)$ é uma potência subcrítica. Em um sentido formal, o operador 1-Laplaciano é definido por $\Delta_1 u := \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right)$ (veja [15, 17, 37, 49, 63, 70, 73] e referências relacionadas para uma introdução ao estudo do operador 1-Laplaciano).

Retornando à equação (E_1), considerando $f(t) = t\log t^2$ e $V \equiv 1$, conforme já comentado, dependendo da escolha de $\Omega \subset \mathbb{R}^N$, a equação de Schrödinger

$$(E_2) \qquad -\Delta u + u = u\log u^2, \ \text{ em } \ \Omega$$

pode não ter aplicabilidade imediata do clássico método variacional para funcionais C^1; veja por exemplo as já citadas refrências [6, 7, 10, 12, 13, 62, 79] nas quais o caso Ω ilimitado é abordado.

Quando, por exemplo, tem-se $\Omega = \mathbb{R}^N$, o candidato a funcional energia associado a (E_2) é dado por

$$E(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + 2|u|^2)dx - \frac{1}{2}\int_{\mathbb{R}^N} u^2 \log |u|^2 dx,$$

para $u \in H^1(\mathbb{R}^N)$. Na expressão dada a E está sendo utilizado implicitamente o fato de que

$$\int_0^t s \log s^2\, ds = \frac{1}{2}t^2 \log t^2 - \frac{t^2}{2}.$$

Ocorre que não podemos assegurar que $E \in C^1(H^1(\mathbb{R}^N), \mathbb{R})$. Em verdade, no trabalho [80], encontramos registrado um exemplo de uma função $u_1 \in H^1(\mathbb{R}^N)$ tal que $\displaystyle\int_{\mathbb{R}^N} u_1^2 \log |u_1|^2 dx = -\infty$. Consequentemente, $E(u_1) = \infty$, mostrando que E não está, sequer, bem definido sobre $H^1(\mathbb{R}^N)$. Isso faz com que, além da permeabilidade em aplicações (vide [84]), o estudo das equações logarítmicas torne-se atrativo do ponto de vista matemático.

No sentido de vencer tal dificuldade, a estratégia utilizada nos trabalhos [10–13, 62, 79] - veja também os Capítulos 1, 2 e 3 na sequência - é considerar uma decomposição de $t \log t^2$ da forma:

$$F_2(t) - F_1(t) = \frac{1}{2}t^2 \log t^2 \quad \forall t \in \mathbb{R}, \tag{3}$$

com F_1, $F_2 \in C^1(\mathbb{R})$. Sendo F_2 uma função com crescimento subcrítico e F_1 uma função convexa e par e com $F_1(0) = 0$ (veja o corpo da tese para definição explícita de F_1 e F_2). Vale registrar que a função F_1 satisfaz à seguinte condição de crescimento.

$$|F_1(t)| \leq |t|^r + |t|^p, \quad t \in \mathbb{R},$$

com $r \in (1, 2)$ e $p \in [2, 2^*)$.

Isso nos possibilita escrever $E = \Phi + \Psi$, com

$$\Phi(u) := \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + (V(x))|u|^2 + 1)dx - \int_{\mathbb{R}^N} F_2(u)\, dx$$

e

$$\Psi(u) := \int_{\mathbb{R}^N} F_1(u)\, dx.$$

As condições sobre F_1 e F_2 nos permitem concluir que E verifica (H). Nesse caso um ponto crítico para E é um ponto $u \in H^1(\mathbb{R}^N)$ tal que $E(u) < \infty$ e

$$\begin{aligned}\int_{\mathbb{R}^N} (\nabla u \cdot \nabla(v - u) + 2u(v - u))dx + \int_{\mathbb{R}^N} (F_1(v) - F_1(u))dx -\\ - \int_{\mathbb{R}^N} F_2'(u)(v - u) \geq 0, \quad \forall v \in H^1(\mathbb{R}^N).\end{aligned} \tag{4}$$

Em virtude das mencionadas propriedades de F_1 e F_2 sabemos que $E(v) < \infty$ equivale a $F_1(v) \in L^1(\mathbb{R}^N)$. Assim, tem-se

$$C_0^\infty(\mathbb{R}^N) \subset D(E) = \{v;\ E(v) < \infty\} = \{v;\ \Psi(v) < \infty\}.$$

Com isso, fixada $\phi \in C_0^\infty(\mathbb{R}^N)$, escolhendo $v = u + t\phi$, $t \approx 0^+$, em (4), depois dividindo por t e fazendo $t \to 0$ obtemos

$$\int_{\mathbb{R}^N} (\nabla u \cdot \nabla \phi + 2u\phi)\, dx + \int_{\mathbb{R}^N} F_1'(u)\phi\, dx - \int_{\mathbb{R}^N} F_2'(u)\phi\, dx \geq 0.$$

Substituindo ϕ por $-\phi$ concluímos que um ponto crítico u de E verifica

$$\int_{\mathbb{R}^N} (\nabla u \cdot \nabla \phi + u\phi)\, dx = \int_{\mathbb{R}^N} u \log u^2 \phi\, dx, \ \ \forall \phi \in C_0^\infty(\mathbb{R}^N).$$

A identidade acima, junto à teoria de regularidade para equações elípticas, permite-nos concluir que pontos críticos para E no sentido dos funcionais do tipo Szulkin fornecem soluções clássicas de (E_2). Essa técnica tem sido amplamente explorada no estudo das equações logarítmicas de Schrödinger, no sentindo de reparar a falta de suavidade do funcional, a exemplo dos já supracitados trabalhos [10–13, 62, 79].

Atentando ao procedimento indicado, podemos perceber que os pontos críticos do funcional E devem residir no espaço

$$\left\{ u \in H^1(\mathbb{R}^N);\ \int_{\mathbb{R}^N} F_1(u)\, dx < \infty \right\}.$$

Desde que a função F_1 é convexa e inspirado em Cazenave [42], nos perguntamos se existiria um espaço de Banach (um espaço de Orlicz) contido na coleção acima e sobre o qual o funcional E seja de classe C^1.

Com essa questão em mente, no Capítulo 2, estabelecemos, em verdade, que a função F_1 é uma N-função satisfazendo a denominada *condição* (Δ_2). Com isso o conjunto da forma

$$Z(\Omega) := \left\{ u \in L^1_{loc}(\Omega);\ \int_\Omega F_1(u)\, dx < \infty \right\},$$

com Ω um aberto qualquer de $\mathbb{R}^N$, constitui um espaço de Banach separável e reflexivo; veja o Apêndice C para uma sucinta revisão sobre espaços de Orlicz. Esse resultado envolvendo a função F_1 nos permite atacar a equação do tipo (E_2) via métodos variacionais clássicos, por considerar o funcional E restrito ao espaço

$X = H^1(\mathbb{R}^N) \cap Z(\mathbb{R}^N)$. Como exposto no Capítulo 2, essa restrição permite concluir que $E \in C^1(X, \mathbb{R})$.

Embora as equações logarítmicas tenham sido amplamente estudadas nos últimos anos e vários resultados sobre existência e multiplicidade tenham sido estabelecidos, alguns fatos intrínsecos ao estudo dos problemas elípticos, que recaem em a aplicação do teoria clássica de pontos críticos, não tinham ainda sido estabelecidos para equações logarítmicas de Schrödinger. Citamos aqui, e.g., resultados de multiplicidade à luz do que é feito em [14, 43] utilizando a teoria de *categoria de Lusternik-Schnirelmann.*

No Capítulo 2, introduzindo o novo espaço de funcões associado com F_1 (espaço Z acima), provamos a existência e multiplicidade de soluções para seguinte classe de problemas.

$$\begin{cases} -\varepsilon^2 \Delta u + V(x)u = u \log u^2, & \text{em } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases}$$

com $V : \mathbb{R}^N \longrightarrow \mathbb{R}$ uma função contínua satisfazendo

(V_1): $-1 < \inf_{x \in \mathbb{R}^N} V(x)$;

(V_2): Existem um conjunto aberto e limitado $\Lambda \subset \mathbb{R}^N$ verificando

$$V_0 := \inf_{x \in \Lambda} V(x) < \min_{x \in \partial\Lambda} V(x).$$

O resultado de multiplicidade de solução que provamos ser válido para o problema acima estima inferiormente o número de soluções pela categoria de Lusternik-Schnirelmann do conjunto

$$M := \{x \in \Lambda;\ V(x) = V_0\}$$

em

$$M_\delta := \{x \in \mathbb{R}^N;\ d(x, M) \leq \delta\}, \ \ \delta \approx 0^+.$$

O teorema abstrato que fundamenta nosso resultado de multiplicidade pode ser enunciado como se segue; veja [83, Chapter 5] para uma prova do resultado abaixo e mais detalhes envolvendo a categoria de Lusternik-Schnirelmann.

Teorema 0.0.0.2 *Fixe $V = \psi^{-1}(0)$ uma variedade classe C^1 dada como imagem inversa de um valor regular do funcional $\psi \in C^1(W, \mathbb{R})$, com W um espaço de Banach. Seja $I \in C^1(W, \mathbb{R})$ tal que $I|_V$ é limitado inferiormente. Suponha que I satisfaz a* *condição* $(PS)_c$ *para níveis* $c \in [\inf I|_V, d]$, *então* $I|_V$ *tem ao menos* $cat_{I^d}(I^d)$ *pontos críticos em* $I^d = \{u \in V;\, I(u) \leq d\}$.

É fácil notar que a aplicação do teorema acima só faz sentido no contexto dos funcionais de classe C^1, uma vez que versa sobre pontos críticos para funcionais restrito a variedades de classe C^1. O "approach" por nós introduzido no Capítulo 2 é, portanto, fundamental no sentido de aplicarmos o teorema anterior, porquanto nos permite concluir que o funcional energia associado ao problema é de classe C^1. É válido ainda ressaltar que, diante das condições $(V_1) - (V_2)$ acima, nossos resultados melhoram e estendem os resultados devido a Alves e de Morais [10] e a Alves e Ji [11].

Ainda inspirados pela nova abordagem para estudar equações logarítmicas de Schrödinger introduzida no Capítulo 2, no Capítulo 3 estudamos uma classe de equações logarítmicas sobre domínios exteriores. Mais precisamente, estudamos a existência de solução positiva para a classe de problemas da forma

$$\begin{cases} -\Delta u + u = Q(x)u\log u^2, & \text{em } \Omega, \\ \mathcal{B}u = 0 \ \text{ em } \partial\Omega, \end{cases}$$

com $\Omega \subset \mathbb{R}^N$, $N \geq 3$, um *domínio exterior* (i.e., $\Omega^c = \mathbb{R}^N \setminus \Omega$ é um domínio limitado com fronteira suave). Consideraremos os casos $\mathcal{B}u = u$ e $\mathcal{B}u = \frac{\partial u}{\partial \nu}$.

A principal ideia no estudo do último problema é, no caso Dirichlet ($\mathcal{B}u = u$), adaptar os resultados do importante trabalho de Benci e Cerami [27] e de Alves e de Freitas em [9]. Uma vez mais, faz-se crucial a condição de que o funcional energia associado ao problema seja de classe C^1, dado que os resultados circunstantes em [9,27] fazem uso frequente da regularidade do funcional energia estudado, abordando propriedades e estimativas relacionadas à variedades de classe C^1 (nesse caso específico, à famosa *variedade de Nehari* associada ao problema). No caso Neumann ($\mathcal{B}u = \frac{\partial u}{\partial \nu}$), inspiramos-nos e adpatamos diferentes técnicas desenvolvidas em [4,18,33]. Em nosso caso, nos resultados de compacidade substituimos as sequências de Palais-Smale por sequências de Cerami (veja maiores detalhe na Seção 3.4). Ainda relacianado ao estudo de problemas sobre domínios exteriores, citamos os trabalhos em [2,3,18,21,29,66] no intento de ilustrar o interesse diverso e a relevância dessa classe de problemas.

Os resultados apresentados nos Capítulos 2 e 3 nos propiciaram como fruto os trabalhos em [6,7]. Concomitantemente, tais resultados ilustram como a nova técnica introduzida nos permite obter inéditos e relevantes resultados concernentes ao estudo das equações logarítmicas de Schrödinger.

Para findar a introdução, uma vez exposto o encadeamento teórico de nosso estudo, registramos a seguir alguns aspectos sob os quais o presente texto foi construído.

1º- O texto, naturalmente, pressupõe alguma experiência com os resultados da Análise Funcional e Teoria da Medida e Integração, de modo que, recorrentemente, os resultados clássicos são utilizados tacitamente, ainda que com alguma menção explícita. A experiência com alguns resultados usuais da Teoria das Equações Diferenciais Parciais e da Teoria dos Pontos Críticos podem, e muito, contribuir para o entedimento pleno do texto. No intento de conferir fluidez à leitura, as provas de alguns resultados são, às vezes, apenas referenciadas.

2º- Os apêndices são devotados a tópicos teóricos que permeiam os capítulos, mas que suas respectivas exposições poderiam atribuir algum grau de prolixidade aos temas desenvolvidos. Os apêndices são construídos de modo a apenas listar os resultados de interesse. Nessa perspectiva, apenas as provas não típicas ou as de caráter original são explicitadas nos apêndices.

3º- Informamos que os resultados e conceitos registrados nessa introdução serão reenunciados no momento oportuno durante os capítulos, atenuando-se assim o labor adicional de regressar à introdução para recordar algum resultado de interesse.

Notations

Throughout this text we fix the following notations.

- $H^1_{\mathrm{rad}}(\mathbb{R}^N) := \{u \in H^1(\mathbb{R}^N) : \ u \text{ is radial}\}$.
- $C^\infty_{0,\,\mathrm{rad}}(\mathbb{R}^N) := \{u \in C^\infty_0(\mathbb{R}^N) : \ u \text{ is radial}\}$.
- $L^p(\mathbb{R}^N)$ is the usual Lebesgue space, with norm $\|u\|_p := \left(\int_{\mathbb{R}^N} |u|^p dx\right)^{1/p}$, $1 \le p < 1$, and $\|u\|_\infty := \mathrm{esssup}_{x\in\mathbb{R}^N}|u(x)|$.
- If $\Omega \subset \mathbb{R}^N$ is a measurable set, we simply write $\int_\Omega f$ instead of $\int_\Omega f(x)\,dx$ for any measurable real-values function f defined on Ω.
- If X is a Banach space e $x_0 \in X$, then $B_r(x_0)$ designates the ball centered in x_0 of radius $r > 0$.
- $\operatorname{supp} u$ designates the support of a measurable function $u : \mathbb{R}^N \longrightarrow \mathbb{R}$.
- $\mathrm{int}(A)$ denotes the interior of a set A.
- $\overline{A}$ denotes the closure of a set A.
- ∂A denotes the boundary of a set A.
- $o_n(1)$ denotes a real sequence with $o_n(1) \to 0$
- $o_\varepsilon(1)$ denotes a real parameter that depends on ε satisfying $O_\varepsilon(1) \to 0$, as $\varepsilon \to 0$.
- $C(x_1, ..., x_n)$ denotes a positive constant that depends on $x_1, ..., x_n$.

- $1^* := \dfrac{N}{N-1}$, if $N \geq 2$.
- $2^* := \dfrac{2N}{N-2}$, if $N \geq 3$ and $2^* := \infty$ if either $N = 1$ or $N = 2$.
- i.e.: abbreviation for the Latin expression *id est*.
- e.g.: abbreviation for the Latin expression *exempli gratia*.

CHAPTER 1

Minimax theorems for lower semicontinuous functions and their applications

In 1986, Szulkin [81] generalized the study of Critical Point Theory to a class of lower semicontinuous (l.s.c) functionals $I : X \to (-\infty, +\infty]$ having the following structure

(H_0) $I := \Phi + \Psi$, *with* $\Phi \in C^1(X, \mathbb{R})$ *and* $\Psi : X \to (-\infty, +\infty]$ *is a convex l.s.c. functional and proper, i.e.* $\Psi \not\equiv \infty$.

From now on, a functional $I : X \to (-\infty, +\infty]$ is said to be of Szulkin-type if its structure is given as in (H_0). In the important work [81], Szulkin has established a powerful list of minimax results involving the class (H_0). Generalized versions of the famous Mountain Pass Theorem of Ambrosetti-Rabinowitz [75, Theorem 2.2], the Saddle Point Theorem [75, Theorem 4.6] and classical results of the genus theory has been proved in [81].

However, observing the literature on minimax theorems, we could find some classical results that have not been yet extended for Szulkin-type functionals. For instance, the classical Bartsch's Fountain Theorem, which ensures the existence and multiplicity of critical points for $\mathbb{Z}_2$-symmetric C^1-functionals (see Bartsch [23, Theorem 2.5] and Willem [83, Theorem 3.6]). By exploring the Bartsch's theorem, many authors were interested in finding critical points of real-valued functional Φ

defined on an infinite dimensional Banach space X, which allow to solve wide classes of ordinary or partial differential equations. Besides of the applications in the study of differential equations, several works were focused in establishing generalizations of the Fountain Theorem; see, e.g., [25, 26, 45, 60, 65, 85] for a valuable literature of this subject.

Accounting this questions, we have aimed to solve the following problem:

(Q_1) *Is it possible to prove a Fountain-type Theorem for Szulkin-type functionals?*

In this chapter a complete and positive answer to (Q_1) is given by proving a nonsmooth version of Theorem 2.5 in [23] for Szulkin-type functionals (see Theorem 1.4 below).

Considering the literature related with the Fountain Theorem, a second question that naturally arises in this nonsmooth setting is the following

(Q_2) *Is it possible to prove a dual Fountain-type Theorem for Szulkin-type functionals?*

Indeed, in [25], Bartsch and Willem have proved a dual version of th Fountain Theorem. A careful analysis of the proof of the classical dual Fountain Theorem can be found in [83, Theorem 3.18]. The main basic idea due to Bartsch and Willem consists in applying Theorem 2.5 of [23] to the functional $-\Phi$, with Φ a C^1 functional on X, obtaining a real sequence (c_j) of negative critical values of I such that $c_j \to 0$, as $j \to \infty$. However, when I is a Szulkin-type functional it is easily seen that this procedure cannot be used in general as in the smooth case, because when I is a Szulkin-type functional we do not know, in general, if the functional $-I$ also verifies (H_0).

In order to overcome this difficulty and to give an answer for (Q_2), we have proved a nonsmooth version of a Heinz's Theorem (see [61, Proposition 2.2]) for Szulkin-type functionals. As in the dual Fountain Theorem, this result ensures the existence of a negative sequence (c_j) of critical values converging to 0, as $j \to \infty$.

Finally, we would like to emphasize that, by adapting the arguments used along the proof of the main Theorem 1.5, we are able to show a more precise version of [81, Corollary 4.8]. On the contrary of [81, Corollary 4.8], the conclusion of our result in Theorem 1.6 ensures that the obtained critical levels c_k satisfy $c_k \to \infty$ as $k \to \infty$.

From a theoretical point of view, the results obtained here complete the study made by Szulkin in the seminal paper [81], since new minimax theorems are established.

We would like to register that the results developed in the present chapter are referring to the article [8] due to Alves, da Silva and Molica Bisci.

1.1 Abstract theorems

Throughout this chapter, let $I := \Phi + \Psi$ be a Szulkin-type functional defined on a Banach space $X = (X, \|\cdot\|)$. The *effective domain* of I is defined by

$$D(I) := \{u \in X : I(u) < +\infty\},$$

and so, for a Szulkin-type functional I one has that $D(I) = D(\Psi)$. For each $u \in D(I)$, we say that the *subdifferential* of I at u is the set

$$\partial I(u) := \{\varphi \in X^* : \langle \Phi'(u), v-u\rangle + \Psi(v) - \Psi(u) \geq \langle \varphi, v-u\rangle, \forall v \in X\}. \tag{1.1}$$

For our goals, we will need of the following definition.

Definition 1.1 *Suppose that I is a Szulkin-type functional. Then*

i) a point $u \in X$ is called a critical point of I if $0 \in \partial I(u)$, or more precisely, $u \in D(I)$ and

$$\langle \Phi'(u), v-u\rangle + \Psi(v) - \Psi(u) \geq 0, \quad \forall v \in X,$$

ii) a sequence (u_n) is called a Palais-Smale sequence (briefly (PS) sequence) for I at level $c \in \mathbb{R}$ if $I(u_n) \to c$ and

$$\langle \Phi'(u_n), v-u_n\rangle + \Psi(v) - \Psi(u_n) \geq -\varepsilon_n\|v-u_n\|, \quad \forall v \in X,$$

with $\varepsilon_n \to 0^+$, or equivalently (see [81, Proposition 1.2])

$$\langle \Phi'(u_n), v-u_n\rangle + \Psi(v) - \Psi(u_n) \geq \langle w_n, v-u_n\rangle, \quad \forall v \in X,$$

where $w_n \in X^*$ with $w_n \to 0$ in X^*;

iii) I satisfies the Palais-Smale condition (briefly (PS) condition) at level $c \in \mathbb{R}$ when each (PS) sequence (u_n) at level c has a convergent subsequence. If I verifies the (PS) condition for all level c, we say simply that I satisfies the (PS) condition.

Let us denote by I^c, K and K_c respectively, the following sets

$$I^c := I^{-1}((-\infty, c]) \text{ for every } c \in \mathbb{R},$$

$$K := \{u \in X : u \text{ is a critical point of } I\},$$

and

$$K_c := \{u \in K : I(u) = c\}.$$

In order to prove the main variant of the classical Fountain Theorem given in Theorem 1.4 below, at the beginning of this section, we recall a suitable version of the standard deformation lemma valid for Szulkin-type functionals; see [81, Proposition 2.3]. In addition, in Lemma 1.2 an equivariant version of the aforementioned result has been established. Finally, in the last subsection two abstract results have been proved. More precisely, [61, Proposition 2.2] due to Heinz has been extended to Szulkin-type functionals as well as a new version of [81, Corollary 4.8] is given in Theorem 1.6.

1.1.1 Deformation lemmas and Fountain Theorem

Hereafter, we fix G a compact group that acts isometrically on X; see the Appendix B for a brief of group actions on Banach spaces. The subspace of *invariant elements* of X is defined by

$$Fix(G) := \{u \in X : gu = u \;\; \forall g \in G\}.$$

Example 1.1 Let $Id : X \to X$ be the identity map on X and consider the usual representation
$\mathbb{Z}_2 = \{Id, -Id\}$. Standard computations ensure that the group $\mathbb{Z}_2$ acts isometrically on X.

A subset A of X is said to be *G-invariant* if $gA = A$ for every $g \in G$, where $gA := \{gx : x \in A\}$. Also, when $A \subset X$ is a G-invariant set, a map $\gamma : A \to X$ is called *equivariant map* if

$$\gamma(gx) = g\gamma(x) \;\; \forall x \in A, \forall g \in G.$$

If a functional (not necessarily linear) φ defined on X satisfies $\varphi(gx) = \varphi(x)$ for any $x \in X$ and $g \in G$, we say that φ is a G-invariant functional.

Notation: $\Gamma_G(A) := \{\gamma \in C(A, X) : \gamma \text{ is equivariant}\}$.

By following [83, Section 3.2] and [25], the notion of admissible action is given below.

Definition 1.2 *Let Y be a finite dimensional vector space. Moreover, let us assume that G is a compact topological group that acts diagonally on Y^k, that is*

$$gv = g(v_1, ..., v_k) = (gv_1, ..., gv_k),$$

for every $v = (v_1, ..., v_k) \in Y^k$ and each $g \in G$. The action of G on Y is said to be admissible if, for each equivariant map $\gamma : \partial U \to Y^{k-1}$, where $k \geq 2$ and U is a bounded G-invariant open set of Y^k with $0 \in U$, there is $u \in \partial U$ such that $\gamma(u) = 0$.

For our goals, we will consider a special condition on a decomposition of space X with respect to action of G on X as follows:

(G_0) *G is a compact group that acts isometrically on*

$$X = \overline{\bigoplus_{j\in\mathbb{N}} X_j},$$

where every X_j is a G-invariant subspace of X such that $X_j \cong Y$, being Y a finite dimensional vector space for which the action of G is admissible.

In our theoretical results, we need to deal with the abstract notion of Haar's integral on a compact group G whose the details and related notions can be found in [72]; see the Appendix B for a short review on this subject. Fix $f : G \longrightarrow \mathbb{R}$ an integrable function with respect to a measure μ. We say that μ is a *left invariant* measure if

$$\int_G f(g^{-1}y)d\mu = \int_G f(y)d\mu, \ \forall g \in G, \tag{1.2}$$

for every $f \in \mathcal{L}(G, \mu)$.

Remark 1.1 *When G is a compact group, there is a left invariant positive measure μ such that $\mu(G) = 1$. Such measure is called the Normalized Haar measure. The integral associated with μ is the so called Haar's integral. We also notice that the left invariant Haar measure μ can be extended for X-valued functions (see Appendix B for further details).*

Let $\beta : X \to X$ be a continuous map on X. By the left invariance property of μ, if $\eta : X \to X$ is the map given by

$$\eta(u) := \int_G g\beta(g^{-1}u)d\mu, \ u \in X, \tag{1.3}$$

then $\eta \in \Gamma_G(X)$. This fact will be useful later on.

As usual, by a *deformation* we mean a family of maps of the form

$$\alpha_s := \alpha(s, \cdot) : W \subset X \to X, \ s \in [0, s_0]$$

such that $\alpha_0 \equiv Id|_W$, with $\alpha \in C([0, s_0] \times W, X)$ and $Id|_W$ denotes the restriction of the identity map Id on X to W.

The next result has been proved by Szulkin in [81, Proposition 2.3].

Lemma 1.1 *Let $I = \Phi + \Psi$ be a Szulkin-type functional for which the* (PS) *condition holds and let N be a neighbourhood of K_c. Then, fixed $\varepsilon_0 > 0$, there is $\varepsilon \in (0, \varepsilon_0)$ such that, for each compact set $A \subset X \setminus N$ with*

$$c \leq \sup_{u \in A} I(u) \leq c + \varepsilon,$$

there exist a closed set W, with $A \subset int(W)$, and a deformation $\alpha_s : W \to X$, with $0 \leq s \leq s_0 \approx 0^+$, such that

i) $\|\alpha_s(u) - u\| \leq s, \ \forall u \in W$;

ii) There is a number $\delta = \delta_\varepsilon \approx 0^+$ such that

$$I(\alpha_s(u)) - I(u) \leq s + \delta s \quad \forall u \in W,$$

and

$$I(\alpha_s(u)) - I(u) \leq -3\varepsilon s + \delta s \quad \forall u \in W, \ I(u) \geq c - \varepsilon.$$

Moreover, by ii) it follows that

iii) $I(\alpha_s(u)) - I(u) \leq 2s, \ \forall u \in W$;

iv) $I(\alpha_s(u)) - I(u) \leq -2\varepsilon s, \ \forall u \in W, \ I(u) \geq c - \varepsilon$;

v) $\sup_{u \in A} I(\alpha_s(u)) - \sup_{u \in A} I(u) \leq -2\varepsilon s$.

vi) $I(\alpha_s(u)) - I(u) \leq 0, \ \forall u \in W \cap C$, *for each closed set verifying $C \cap K = \emptyset$.*

We would like to point out that *ii*) is not contained in the statement of [81, Proposition 2.3]. However, the sufficiently small constant $\delta > 0$ in *ii*) explicitly appears along the proof of the cited proposition.

Now, we are able to prove an equivariant version of Lemma 1.1 making use of the next notion that involves a functional $\Psi : X \to (-\infty, +\infty]$ as well as the action of a compact topological group G on X.

Definition 1.3 *Let $\Psi : X \to (-\infty, +\infty]$ be a functional and let G be a compact topological group that acts on X. We say that Ψ is compatible with the action of G on X (briefly G-compatible) if the following inequality holds*

$$\Psi\left(\int_G g^{-1}\beta(gu)d\mu\right) \leq \int_G \Psi(g^{-1}\beta(gu))d\mu, \tag{1.4}$$

for every fixed $u \in X$, $\beta \in C(Gu, X)$, where $Gu := \{gu;\, g \in G\}$ and μ denotes the normalized Haar measure on G.

The inequality in (1.4) is verified in some meaningful cases and some of them are briefly discussed in the next example.

Example 1.2 By using the usual notations, let us restrict our attention to the following cases:

1) Let $\Psi \equiv \|\cdot\| : X \to \mathbb{R}$ be the norm defined on X. Fixed $u \in X$ and a map $\beta \in C(Gu, X)$, let $\eta \in C(G, X)$ be given by $\eta(g) := g^{-1}\beta(gu)$. Next, let (β_n) be a sequence of simple functions with

$$\int_G \beta_n(g)d\mu \to \int_G \eta(g)d\mu \quad \text{and} \quad \int_G \|\beta_n(g)\|d\mu \to \int_G \|\eta(g)\|d\mu. \tag{1.5}$$

Each function β_n can be written as a finite sum:

$$\beta_n = \sum_i \chi_{A_i} v_i \quad \text{where} \quad A_i := \beta_n^{-1}(\{v_i\}) \quad \text{and} \quad v_i \in X.$$

Since μ is the normalized Haar measure on G ($\mu(G) = 1$), we have $\sum_i \mu(A_i) = 1$ and

$$\left\|\int_G \beta_n(g)d\mu\right\| = \left\|\sum_i \mu(A_i)v_i\right\| \leq \sum_i \mu(A_i)\|v_i\| = \int_G \|\beta_n(g)\|d\mu,$$

for every $n \in \mathbb{N}$. Consequently, by using (1.5) it follows that

$$\left\|\int_G \eta(g)d\mu\right\| \leq \int_G \|\eta(g)\|d\mu,$$

that is,

$$\left\| \int_G g^{-1}\beta(gu)d\mu \right\| \le \int_G \left\| g^{-1}\beta(gu) \right\| d\mu$$

So $\|\cdot\|$ is compatible with the action of G on X. In general, the result is still true for an arbitrary convex continuous function $\Psi : X \to \mathbb{R}$.

2) Let us assume that $G := \{g_1, ..., g_k\}$ is a finite group and let $\Psi : X \to (-\infty, +\infty]$ be a convex functional. Since

$$\sum_{i=1}^{k} \mu(\{g_i\}) = 1,$$

for each $u \in X$ and $\beta \in C(Gu, X)$ the integral $\int_G g^{-1}\beta(gu)d\mu$ can be written as a finite convex combination of vectors of X. More precisely, one has

$$\int_G \beta(g)d\mu = \sum_{i=1}^{k} \mu(\{g_i\})v_i,$$

where $v_i := g_i^{-1}\beta(g_i u)$.

Then, since Ψ is convex,

$$\Psi\left(\int_G g^{-1}\beta(gu)d\mu\right) = \Psi\left(\sum_{i=1}^{k} \mu(\{g_i\})v_i\right) \le \sum_{i=1}^{k} \mu(\{g_i\})\Psi(v_i) = \int_G \Psi(g^{-1}\beta(gu))d\mu,$$

i.e. Ψ is compatible with the action of G on X.

The next result (Equivariant Deformation Lemma) is a more general form of Corollary 2.4 in [81]. This preparatory property can be also viewed as a complement of Lemma 5.1 proved by Bereanu and Jebelean in [28].

Lemma 1.2 *Let $I = \Phi + \Psi$ be a Szulkin-type functional for which the* (PS) *condition holds. Assume that Φ and Ψ are G-invariant functionals and Ψ is compatible with the action of the compact topological group G on X. Moreover, suppose that G acts isometrically on X. Under the hypothesis of Lemma 1.1, the same conclusions hold with $\alpha_s : W \to X$ equivariant in A, whenever A is a G-invariant set.*

Proof. Denote by β_s the deformation of Lemma 1.1 and set

$$\alpha_s(u) := \int_G g^{-1}\beta_s(gu)d\mu. \tag{1.6}$$

Thanks to (1.3), we observe that $\alpha_s \in \Gamma_G(A)$. Now, let us prove that the function α_s verifies all the assumptions of Lemma 1.1. More precisely, since *iii*), *iv*) and *v*) are a direct consequence of *ii*), it is enough to show *i*) and *ii*). By Lemma 1.1, Part - *i*), it follows that

$$\begin{aligned}\|\alpha_s(u)-u\| &= \left\| \int_G g^{-1}\beta_s(gu)d\mu - \int_G (g^{-1}g)ud\mu \right\| \\ &\le \int_G \left\| g^{-1}(\beta_s(gu)-gu) \right\| d\mu \\ &\le \int_G s d\mu = s \quad \text{for every } u\in W,\end{aligned} \tag{1.7}$$

i.e. α_s verifies *i*) as claimed.

In order to prove *ii*) let us write $\beta_s(u) = u + h_s(u)$, so that $\alpha_s(u) = u + w_s(u)$, where $w_s(u) = \int_G g^{-1}h_s(gu)d\mu$. Consequently, the Taylor's formula immediately yields

$$I(\alpha_s(u)) = \{\Phi(u) + \langle \Phi'(u), w_s(u)\rangle + r(s)\} + \Psi(\alpha_s(u)), \quad \frac{r(s)}{s} = o_s(1). \tag{1.8}$$

Now, the compatibility condition of Ψ gives

$$I(\alpha_s(u)) \le \int_G (\Phi(u) + \langle \Phi'(u), g^{-1}h_s(gu)\rangle)d\mu + \int_G \Psi(g^{-1}\beta_s(gu))d\mu + \frac{\delta}{2}s, \tag{1.9}$$

for $s \approx 0^+$. Moreover, since

$$\langle \Phi'(u), g^{-1}h_s(gu)\rangle = \langle \Phi'(gu), h_s(gu)\rangle,$$

the G-invariance of Φ and the Taylor's expansion applied to $I(\beta_s(gu))$ give

$$\begin{aligned}I(\alpha_s(u)) &\le \int_G (\Phi(gu) + \langle \Phi'(gu), h_s(gu)\rangle)d\mu + \int_G \Psi(\beta_s(gu))d\mu + \frac{\delta}{2}s \\ &= \int_G (I(\beta_s(gu)) - \rho(s))d\mu + \frac{\delta}{2}s \le \int_G I(\beta_s(gu))d\mu + \delta s.\end{aligned} \tag{1.10}$$

Here, we have used ρ as being the rest in the Taylor's expansion. Finally, by Lemma 1.1, Part - *ii*) and (1.10), it follows that

$$I(\alpha_s(u)) \le \int_G I(gu)d\mu + s + 2\delta s \le I(u) + s + 2\delta s, \tag{1.11}$$

for every $u \in W$. Similarly

$$I(\alpha_s(u)) \le I(u) - 3\varepsilon s + 2\delta s, \quad \text{for every} \quad u\in W \quad \text{and} \quad I(u) \ge c - \varepsilon. \tag{1.12}$$

Inequalities (1.11) and (1.12) ensure that α_s satisfies *ii*) provided that δ is sufficiently small. ■

For the sake of completeness, let us recall now the notion of *homotopy*. Let B be a subset of X and $f, g \in C(B, X)$. As usual, we say that f *is homotopic to* g if there is $h \in C([0,1] \times B, X)$ satisfying

$$h(0,\cdot) \equiv f \quad \text{and} \quad h(1,\cdot) \equiv g. \tag{1.13}$$

The map h is called a *homotopy* between f and g. We will write $f \approx g$ to designate that f is homotopic to g by an equivariant homotopy, i.e., there exists $h \in C([0,1] \times B, X)$ satisfying (1.13) with $h(t,\cdot) \in \Gamma_G(B)$ for any $t \in [0,1]$. It easily seen that $\approx$ is an equivalence relation in $C(B, X)$.

In what follows, for each $k \in \mathbb{N}$, we set

i) $Y_k := \bigoplus_{j=1}^{k} X_j$ and $Z_k := \overline{\bigoplus_{j=k}^{\infty} X_j}$;

ii) $B_k := \{u \in Y_k;\ \|u\| \le \rho_k\}$ and $N_k := \{u \in Z_k;\ \|u\| = r_k\}$, with $\rho_k > r_k > 0$.

Finally, let us recall the Intersection Lemma proved in [83, Lemma 3.4]; see also [25, Theorem 2] for additional comments and remarks.

Lemma 1.3 *Assume that* (G_0) *holds. If* $\gamma \in C(B_k, X) \cap \Gamma_G(B_k)$ *and* $\gamma|\partial B_k \equiv Id|_{\partial B_k}$*, then* $\gamma(B_k) \cap N_k \neq \emptyset$.

We recall in the next result the classical Ekeland's Variational Principle [53, Theorem 1] that will be useful in the sequel.

Theorem 1.3 *Let* (Y, d) *be a complete metric space. Suppose that* $\varphi : Y \to (-\infty, \infty]$ *is a proper lower semicontinuous functional bounded from below. Given* δ*,* $\tau > 0$ *and* $u_0 \in Y$ *such that*

$$\inf_{u \in Y} \varphi(u) \le \varphi(u_0) \le \inf_{u \in Y} \varphi(u) + \delta, \tag{1.14}$$

then, there exists $v_0 \in Y$ *verifying*

i) $\varphi(v_0) \le \varphi(u_0),\ \ d(v_0, u_0) \le 1/\tau$;

ii) $\varphi(v) - \varphi(v_0) \ge -\delta\tau d(v, v_0),\ \forall v \in Y$.

Now, we are ready to show a version of the classical Fountain Theorem due to Bartsch [23] that is valid for Szulkin-type functionals.

Theorem 1.4 *Let* $I = \Phi + \Psi$ *be a Szulkin-type functional for which the* (PS) *condition holds with* $I(0) = 0$*. Assume that* Φ *and* Ψ *are* G*-invariant functionals with* Ψ *compatible with respect to the action of a compact topological group* G *on* X*. Moreover, assume that* (G_0) *holds as well as* *i)* $a_k := \sup_{u \in Y_k, \|u\|=\rho_k} I(u) \leq 0;$

ii) $b_k := \inf_{u \in Z_k, \|u\|=r_k} I(u) \to \infty,$

for every $k \geq 2$. *Finally, set* $c_k := \inf_{\gamma \in \Theta_k} \sup_{u \in B_k} I(\gamma(u)) < \infty$, *where*

$$\Theta_k := \{\gamma \in \Gamma_G(B_k);\ \ \gamma|_{\partial B_k} \equiv Id|_{\partial B_k}\}. \tag{1.15}$$

Then, the functional I *has infinitely many critical points* (u_k) *such that* $I(u_k) = c_k \to \infty$.

Proof. Let us argue by contradiction. In such a case, we may assume that $K_{c_k} = \emptyset$ for some $k \geq 2$. Now, if k is large enough, by Lemma 1.3, one has $c_k \geq b_k > 0$. Thus, we are in position to apply Lemma 1.1 with $N = \emptyset$ and $\varepsilon_0 = c_k$. By fixing $\varepsilon \in (0, c_k)$ given in Lemma 1.1, we will get a contradiction. Indeed, let us define

$$\tilde{\Theta}_k := \left\{\gamma \in \Gamma_G(B_k);\ \gamma|_{\partial B_k} \approx Id|_{\partial B_k} \text{ in } I^{c_k - \frac{\varepsilon}{4}} \text{ and } (I \circ \gamma)|_{\partial B_k} \leq \left(c_k - \frac{\varepsilon}{2}\right)\right\}. \tag{1.16}$$

Thanks to conditions *i)* and *ii)*, if $\gamma \in \Theta_k$ and $u \in \partial B_k$, we derive

$$I(\gamma(u)) = I(u) \leq 0 < c_k - \frac{\varepsilon}{2} < c_k - \frac{\varepsilon}{4}.$$

Hence $\Theta_k \subset \tilde{\Theta}_k$ and

$$\tilde{c}_k := \inf_{\gamma \in \tilde{\Theta}_k} \sup_{u \in B_k} I(\gamma(u)) \leq c_k. \tag{1.17}$$

If $\tilde{c}_k < c_k$, it easily seen that there exists $\gamma_0 \in \tilde{\Theta}_k$ such that

$$m_0 := \sup_{u \in B_k} I(\gamma_0(u)) < c_k.$$

Moreover, by (1.16), there exists a homotopy $H \in C\left([0,1] \times \partial B_k, I^{c_k - \frac{\varepsilon}{4}}\right)$ such that

$$H(0,\cdot) \equiv \gamma_0|_{\partial B_k} \ \text{ and } \ H(1,\cdot) \equiv Id|_{\partial B_k}, \tag{1.18}$$

with $H(t,\cdot)$ equivariant for every $t \in [0,1]$. Since B_k is a ball of radius ρ_k each point $u \in B_k$ can be represented as $u \equiv (s, \tilde{u})$, $s \in [0, \rho_k]$, $\tilde{u} \in \partial B_k$; polar coordinates of u. Hence, if $u \in \partial B_k$ then $u \equiv (\rho_k, u)$. Now, define $\gamma_1 : B_K \to X$ by

$$\gamma_1(s, v) := \begin{cases} \gamma_0(s, v) & s \in \left[0, \dfrac{\rho_k}{2}\right] \\ H\left(\dfrac{2}{\rho_k}s - 1, v\right) & s \in \left[\dfrac{\rho_k}{2}, \rho_k\right]. \end{cases} \tag{1.19}$$

According to (1.18), when $s = \rho_k/2$ it holds $H(2s/\rho_k - 1, \cdot) = H(0, \cdot) \equiv \gamma_0$, which assures that γ_1 is well defined and $\gamma_1 \in \Gamma_G(B_k)$, since γ_0 and $H(t, \cdot)$ are equivariants. By using again (1.18), if $u \in \partial B_k$ one has

$$\gamma_1(u) = H(1, u) = Id|_{\partial B_k}(u),$$

so that $\gamma_1 \in \Theta_k$, and

$$\sup_{u \in B_k} I(\gamma_1(u)) \leq \max\left\{m_0, c_k - \frac{\varepsilon}{4}\right\} < c_k,$$

against the definition of c_k. This contradiction assures that $\tilde{c}_k = c_k$ in (1.17). Consequently, we can work with $\tilde{\Theta}_k$ instead Θ_k.

Now, let us observe that the collection $\tilde{\Theta}_k$ is a (complete) metric subspace of the complete metric space $C(B_k, X)$ endowed by $d(f, g) := \sup_{u \in B_k} \|f(u) - g(u)\|$. Indeed, suppose that $\gamma_n \to \gamma$ in $C(B_k, X)$ with $\gamma_n \in \tilde{\Theta}_k$. The semicontinuity of I yields

$$I(\gamma(u)) \leq \liminf I(\gamma_n(u)) \leq c_k - \frac{\varepsilon}{2}, \quad u \in \partial B_k.$$

Moreover, the action properties give

$$\gamma(gu) = \lim \gamma_n(gu) = g \lim \gamma_n(u) \quad \forall u \in B_k, \ \forall g \in G,$$

so that $\gamma \in \Gamma_G(B_k)$. On the other hand, thanks to the continuity of Φ, it is possible to find a sequence of positive numbers $\tau_n = o_n(1)$ such that

$$\Phi(t\gamma_n(u) + (1-t)\gamma(u)) \leq t\Phi(\gamma_n(u)) + (1-t)\Phi(\gamma(u)) + \tau_n \quad \forall u \in \partial B_k, \ \forall t \in [0,1]. \tag{1.20}$$

More precisely $\tau_n := 2\max\{\tau_n^1, \tau_n^2\}$ with

$$\tau_n^1 := \sup_{u \in B_k, t \in [0,1]} |\Phi(t\gamma_n(u) + (1-t)\gamma(u)) - \Phi(\gamma(u))|$$

and

$$\tau_n^2 := \sup_{u \in B_k} |\Phi(\gamma_n(u)) - \Phi(\gamma(u))|.$$

Inequality (1.20) associated to the convexity of Ψ implies

$$\begin{aligned} I(t\gamma_n(u) + (1-t)\gamma(u)) &\leq tI(\gamma_n(u)) + (1-t)I(\gamma(u)) + \tau_n \\ &\leq c_k - \frac{\varepsilon}{2} + \tau_n \leq c_k - \frac{\varepsilon}{4} \quad \forall u \in \partial B_k, \ \forall t \in [0,1], \end{aligned} \tag{1.21}$$

for n sufficiently large.

Thus $\gamma_n|_{\partial B_k} \approx \gamma|_{\partial B_k}$ via the equivariant homotopy $F(t,\cdot) := t\gamma_n(\cdot) + (1-t)\gamma(\cdot)$. Consequently $\gamma|_{\partial B_k} \approx Id|_{\partial B_k}$, so that $\tilde{\Theta}_k$ is a complete metric subspace of $C(B_k, X)$ as claimed. Hence, the conclusion follows arguing as in [81, Theorem 3.2].

Now, since I is a lower semicontinuous functional, by using [81, Lemma 3.1] and the definition of c_k, we have that the functional $\varphi : \tilde{\Theta}_k \to (-\infty, +\infty]$ defined by

$$\varphi(\gamma) := \sup_{u \in B_k} I(\gamma(u))$$

is lower semicontinuous and bounded from below. Since $\tilde{\Theta}_k$ is a complete metric space, we can apply the classical Ekeland's Variational Principle recalled in Theorem 1.3, to the functional φ with $\delta = \varepsilon$ and $\tau = 1$. Then, we may take $\gamma \in \tilde{\Theta}_k$ such that $\varphi(\gamma) \leq c_k + \varepsilon$, and

$$\varphi(\eta) - \varphi(\gamma) \geq -\varepsilon d(\eta, \gamma) \quad \forall \eta \in \tilde{\Theta}_k. \tag{1.22}$$

It follows that $A := \gamma(B_k)$ is a compact equivariant set with

$$\sup_{v \in A} I(v) = \sup_{u \in B_k} I(\gamma(u)) \leq c_k + \varepsilon,$$

so that A verifies all the assumptions of the equivariant deformation lemma given in Lemma 1.2. Hence, let $\eta := \alpha_s \circ \gamma$, where α_s is the equivariant deformation given in Lemma 1.2 and let us prove that $\eta \in \tilde{\Theta}_k$ for $s \approx 0^+$. Indeed $\eta \in \Gamma_G(B_k)$ and if $u \in \partial B_k$, by $iii)$ and $iv)$ in Lemma 1.1, it follows that

$$\begin{cases} I(\eta(u)) = I(\alpha_s(\gamma(u))) \leq I(\gamma(u)) \leq c_k - \dfrac{\varepsilon}{2}, & I(\gamma(u)) \in \left(c_k - \varepsilon, c_k - \dfrac{\varepsilon}{2}\right] \\ I(\eta(u)) \leq I(\gamma(u)) + 2s \leq c_k - \dfrac{\varepsilon}{2}, & I(u) \leq c_k - \varepsilon, \end{cases} \tag{1.23}$$

so that

$$(I \circ \eta)|_{\partial B_k} \leq c_k - \frac{\varepsilon}{2}.$$

Now, since $\alpha_s \circ \gamma$ can be viewed as an equivariant homotopy such that $(\alpha_s \circ \gamma)|_{\partial B_k} \approx \gamma|_{\partial B_k}$ in $I^{c_k - \frac{\varepsilon}{2}}$, it follows that

$$\eta|_{\partial B_k} \approx (\alpha_s \circ \gamma)|_{\partial B_k} \approx Id|_{\partial B_k} \quad \text{in} \quad I^{c_k - \frac{\varepsilon}{4}},$$

taking into account that $\gamma|_{\partial B_k} \approx Id|_{\partial B_k}$.

Finally, since $\eta \in \tilde{\Theta}_k$, by using $i)$ and $v)$ of Lemma 1.1 and (1.22), one has

$$\begin{aligned} -\varepsilon s &\le \varphi(\eta) - \varphi(\gamma) \\ &= \sup_{u\in B_k} I(\alpha_s(\gamma(u))) - \sup_{u\in B_k} I(\gamma(u)) \le -2\varepsilon s, \end{aligned} \tag{1.24}$$

which is an absurd. Hence, there exists a positive integer k_0 such that $K_{c_k} \neq \emptyset$ for $k \ge k_0$. The proof is complete since, by construction, one clearly has $c_k \ge b_k$. ■

1.1.2 Minimax results involving the G-index theory

Preceding the main results of this subsection, we introduce the notion of the G-index that will be required in our abstract results. The reader can consult [23] for a discussion in a more general situation. Let Σ be the class of subsets of $(X - \{0\})$ that are G-invariant and closed in X. Let us assume that the condition (G_0) holds and let Y be the vector space fixed in that condition.

Definition 1.4 *The G-index of $A \in \Sigma \setminus \{\emptyset\}$ is defined as*

$$\gamma_G(A) := \min\{k \in \mathbb{N} \setminus \{0\} : \exists\, \phi : A \to Y^k \setminus \{0\},\ \phi \in \Gamma_G(A)\}$$

if such integer exists and $\gamma_G(A) := +\infty$ otherwise. Finally, we also set $\gamma_G(\emptyset) := 0$.

Remark 1.2 Note that when $G = \mathbb{Z}_2$ the G-index introduced above coincides with the genus of symmetric subset of $(X - \{0\})$; details and useful remarks on genus theory can be found in [75].

Denote by $\mathcal{C}$ the collection of all nonempty closed and bounded subsets of X. In $\mathcal{C}$ we put the Hausdorff metric d_H given by

$$d_H(A, B) := \max\left\{\sup_{x\in A} d(x, B),\ \sup_{y\in B} d(y, A)\right\},\quad A, B \in \mathcal{C},$$

where d denotes the usual distance on X. It is well known that $(\mathcal{C}, d_H)$ is a complete metric space. Denote by $\mathcal{D}_G$ the subcollection of $\mathcal{C}$ of all nonempty compact G-invariant subset of X. By following the ideas in [81, Section 4] the reader is invited to note that $(\mathcal{D}_G, d_H)$ is a complete metric space; see also [46, Apêndice A] for related computations. By a similar way, we notice that, setting

$$\Gamma_j := \overline{\{A \in \mathcal{D}_G;\ 0 \notin A,\ \gamma_G(A) \ge j\}}^{d_H},$$

the reasoning made in [81] can be adapted to show that the space (Γ_j, d_H) is a complete metric space. The next properties can be proved by using an analogous reasoning as made in [75].

Proposition 1.1 *For every $A, B \in \Sigma$ the following facts hold:*

i) *If there exists $\phi : A \to B$, $\phi \in \Gamma_G(A)$, then $\gamma_G(A) \leq \gamma_G(B)$;*

ii) *$A \subset B$ implies that $\gamma_G(A) \leq \gamma_G(B)$;*

iii) *$\gamma_G(A \cup B) \leq \gamma_G(A) + \gamma_G(B)$;*

iv) *$\gamma_G(\overline{A \setminus B}) \geq \gamma_G(A) - \gamma_G(B)$, since $\gamma_G(B) < \infty$;*

v) *If G is a finite group and A is a compact set, then $\gamma_G(A) < \infty$.*

vi) *If A is a compact set, then we have*

$$\gamma_G(N_\delta(A)) = \gamma_G(A),$$

$\delta \approx 0^+$, where

$$N_\delta(A) := \{x \in X : d(x, A) \leq \delta\}.$$

Proof. The proof of $i) - iv)$ and $vi)$ follows using the same type of argument as made in [75]. To see that $v)$ holds, write $G = \{g_1, ..., g_n\}$ and for each $x \in A$ consider the G-*orbit* $Gx := \{gx;\, g \in G\} = \{g_1x, ..., g_nx\}$. We may fix $\phi = \phi_x : Gx \longrightarrow Y \setminus \{0\}$ an equivariant continuous map (e.g., fix $v_0 \neq 0$ in Y and set $\phi(g_jx) = g_jv_0$). Since Gx is a closed and finite subset of A, we can extend ϕ to $\tilde{\phi} : U \longrightarrow Y \setminus \{0\}$, with $U = U_x$ an equivariant neighborhood of Gx, and $\tilde{\phi} \in \Gamma_G(U)$. By repeating this procedure for each $x \in A$, by the compactness of A it is possible to find $U_1, ..., U_k$ a finite list of equivariant closed sets and equivariant maps $\tilde{\phi}_j : U_j \longrightarrow Y \setminus \{0\}$, $j \in \{1, ..., k\}$, $A \subset \bigcup_j U_j$. Arguing as in [24, §2.3-§2.4], by considering an G-invariant partition of unity subordinate to $\{U_j\}_{1 \leq j \leq k}$, one can obtain $\gamma : A \longrightarrow Y^k \setminus \{0\}$, $\gamma \in \Gamma_G(A)$. So, the item $v)$ holds and the proof is now complete. ■

Finally, in view of the preceding proposition, by following the same idea in [81, Proposition 4.2], we can prove the property below.

Proposition 1.2 *If $A \in \Gamma_j$ is such that $0 \notin A$, then $\gamma_G(A) \geq j$.*

Let A be a compact set of a real Banach space X and $\delta > 0$. Let us recall the notation

$$N_\delta(A) := \{x \in X : d(x, A) \leq \delta\}.$$

The next technical result will be useful in the sequel.

Lemma 1.4 *Let $I = \Phi + \Psi$ be a Szulkin-type functional for which the* (PS) *condition holds. Moreover, let (c_j) be a real sequence such that $c_j \to c \in \mathbb{R}$. Then, given $\delta > 0$, there exists $j_0 \in \mathbb{N}$ such that*

$$K_{c_j} \subset N_\delta(K_c),$$

for every $j \geq j_0$.

Proof. Arguing by contradiction, assume that there exist a subsequence (c_{j_k}) of (c_j), a number $\delta_0 > 0$, and a sequence (u_k) with $u_k \in K_{c_{j_k}}$ such that

$$d(u_k, K_c) > \delta_0, \quad \forall k \in \mathbb{N}. \tag{1.25}$$

The definition of $K_{c_{j_k}}$ immediately yields

$$\langle \Phi'(u_k), v - u_k \rangle + \Psi(v) - \Psi(u_k) \geq 0, \quad \forall v \in X, \tag{1.26}$$

as well as

$$I(u_k) = c_{j_k} \to c,$$

so that (u_k) is a $(\mathrm{PS})_c$ sequence for the functional I. Now, the (PS) condition ensures the existence of $u_0 \in X$ and a subsequence of (u_k), still denoted again by (u_k), such that

$$u_k \to u_0 \quad \text{in} \quad X.$$

Now, taking $v = u_0$ in (1.26), we get $\limsup \Psi(u_k) \leq \Psi(u_0)$. The last inequality in addition to the semicontinuity property of Ψ gives $\lim \Psi(u_k) = \Psi(u_0)$, so that $u_0 \in K_c$. Hence $d(u_k, K_c) \to 0$ as $k \to \infty$, against (1.25). ■

The next result extends [61, Proposition 2.2] to Szulkin-type functionals.

Theorem 1.5 *Let $I = \Phi + \Psi$ be a Szulkin-type functional for which the* (PS) *condition holds and such that $I(0) = 0$. Assume that Φ and Ψ are G-invariant functionals with Ψ compatible with respect to the action of a compact topological group G on X. Moreover, suppose that (G_0) holds and require that the G-index satisfies the following property:*

(G_*) $\qquad$ $\gamma_G(A) < \infty$ *for every compact set* $A \in \Sigma$.

Finally, for every $j \in \mathbb{N}$, set

$$c_j := \inf_{A \in \Gamma_j} \sup_{u \in A} I(u),$$

and assume that the following conditions are verified:

i) $-\infty < c_j$ *for every* $j \in \mathbb{N}$;

ii) Given $j \in \mathbb{N}$, *there exists* $A \in \Sigma$ *such that*

$$\gamma_G(A) \geq j \quad \text{and} \quad \sup_{u \in A} I(u) < 0,$$

where $A \neq \emptyset$ *is a compact set.*

Then, the numbers c_j *are negative critical values of* I *and* $c_j \to 0$ *as* $j \to \infty$.

Proof. We first notice that conditions $i)$ and $ii)$ imply that $-\infty < c_j < 0$. Now, a careful analysis of the arguments in [81, Theorem 4.3] ensures that the sequence (c_j) consists of critical values of I. In fact, the proof of [81, Theorem 4.3] only depends on the properties $i) - vi)$ in Proposition 1.1 with $G = \mathbb{Z}_2$ and where γ_G coincides with the genus of a symmetric set as in Remark 1.2. In view of Proposition 1.1, the argument used in [81, Theorem 4.3] can be adopted in our case. It remains to show that $c_j \to 0$ as $j \to \infty$. To this aim, let us observe that the definition of c_j yields

$$c_j \leq c_{j+1}, \quad \forall j \in \mathbb{N}.$$

Arguing by contradiction, if $c_j \nrightarrow 0$ for $j \to \infty$, there exists $c < 0$ such that $c_j \to c$. The (PS) condition ensures that K_c is compact. Moreover, the assumptions on I yields that K_c is G-invariant and $0 \notin K_c$. Thereby, $K_c \in \Sigma$ and, by following the idea of Lemma 1.4, as $c_j \to c$ and $K_{c_j} \neq \emptyset$, one has that $K_c \neq \emptyset$. By $vi)$ of Proposition 1.1 there is $\delta > 0$ such that $\gamma_G(N_{2\delta}(K_c)) = \gamma_G(K_c)$; note that $N_\delta(K_c) \neq \emptyset$. By (G_*), we can assume that $\gamma_G(K_c) = p$

$$\begin{aligned} \varphi_j : \Gamma_j \quad &\to \quad (-\infty, +\infty] \\ A \quad &\longmapsto \varphi_j(A) := \sup_{u \in A} I(u). \end{aligned}$$

Clearly φ_j is lower semicontinuous functional since I is too. Set

$$\varepsilon_0 := \min\{1, \delta, -c\}$$

and take $\varepsilon \in (0, \varepsilon_0)$ as in Lemma 1.1. Now, let $A_1 \in \Gamma_{j+p}$ be such that

$$c_{j+p} \leq \varphi_{j+p}(A_1) < c_{j+p} + \frac{\varepsilon^2}{2}.$$

Since $c_j \to c$, it follows that, for a convenient $j_0 \in \mathbb{N}$,

$$\varphi_{j+p}(A_1) < c_{j+p} + \frac{\varepsilon^2}{2} \leq c + \frac{\varepsilon^2}{2} \leq c_j + \varepsilon^2 < c_j + \varepsilon < 0,$$

for $j \geq j_0$. Hence, by fixing $j = j_0$, we get $0 \notin A_1$ and $\gamma_G(A_1) \geq j_0 + p$ by Proposition 1.2. If we set $A_2 := \overline{A_1 \setminus N_{2\delta}(K_c)}$ we also have

$$\sup_{u \in A_2} I(u) \leq \sup_{u \in A_1} I(u) < c_{j_0} + \varepsilon^2 < 0,$$

so that $0 \notin A_2$ and $\gamma_G(A_2) \geq (j_0 + p) - p = j_0$ by Proposition 1.1, Part - iv). Consequently $A_2 \in \Gamma_{j_0}$. Now, Theorem 1.3 applied to the function $\varphi_{j_0} : \Gamma_{j_0} \to (-\infty, +\infty]$ (note that Γ_{j_0} is complete) yields the existence of $A \in \Gamma_{j_0}$ such that

$$c_{j_0} \leq \sup_{u \in A} I(u) = \varphi_{j_0}(A) \leq \varphi_{j_0}(A_2) < c_{j_0} + \varepsilon, \;\; d_H(A, A_2) \leq \varepsilon$$

as well as

$$\varphi_{j_0}(B) - \varphi_{j_0}(A) \geq -\varepsilon d_H(A, B) \;\; \forall B \in \Gamma_{j_0}. \tag{1.27}$$

Since Lemma 1.4 gives $K_{c_{j_0}} \subset N_\delta(K_c)$ for $j_0 \approx \infty$, by setting $N = N_\delta(K_c)$ we derive $A \cap N = \emptyset$, taking into account that $\varepsilon < \delta$. These informations ensure that A, N and $K_{c_{j_0}}$ verify the hypothesis of the deformation result given in Lemma 1.1.

Thus by Lemma 1.2 the existence of an equivariant deformation α_s is obtained. In this way, if we set $B := \alpha_s(A)$, on account of Proposition 1.1, Part - i), one has $B \in \Gamma_{j_0}$. Now, combining the properties of α_s with (1.27) we derive the contradiction

$$-2\varepsilon s \geq \varphi(B) - \varphi(A) \geq -\varepsilon s.$$

This completes the proof. ■

Remark 1.3 We emphasize that if G is finite, condition (G_*) in Theorem 1.5 automatically holds; see Proposition 1.1-v).

The last result can be viewed as a complement of Corollary 4.8 proved by Szulkin in [81].

Theorem 1.6 *Let $I = \Phi + \Psi$ be a Szulkin-type functional for which the* (PS) *condition holds and such that $I(0) = 0$. Assume that Φ and Ψ are G-invariant functionals with Ψ compatible with respect to the action of a compact topological group G on X. Moreover, suppose that (G_0) holds and require that the G-index satisfies (G_*).*

Finally, assume that there exist subspaces Y, Z of X such that $X = Y \oplus Z$, $\dim Y < \infty$, Z is closed and

i) There are numbers $r, \rho > 0$ such that $I|_{\partial B_r(0) \cap Z} \geq \rho$;

ii) For each positive integer k there is a k-dimensional subspace X_k of X such that $I(u) \to -\infty$ as $\|u\| \to \infty$ with $u \in X_k$.

Then I has infinitely many critical values. Furthermore, if I^{-c_0} has no critical points for some $c_0 > 0$, then there exists a sequence (c_j) of critical values of I with $c_j \to \infty$ as $j \to \infty$.

In order to prove Theorem 1.6 some notations are introduced. To this aim, let us fix $c_0 > 0$ such that I^{-c_0} has no critical points and set $M_k := \overline{B}_{R_k}(0) \cap X_k$ with $R_k > r$ and $I|_{\partial M_k} \leq -c_0$. Now, let us define the following sets

$$\mathcal{F} := \{\eta \in \Gamma_G(M_k);\ \eta|_{\partial M_k} \approx Id|_{\partial M_k} \text{ in } I^{-c_0} \text{ by an equivariant homotopy}\},$$

for each $j \in \mathbb{N}$ and $k \geq j$,

$$\tilde{\Lambda}_j^k := \left\{ \begin{array}{c} \eta(M_k \setminus U) : \eta \in \mathcal{F},\, U \text{ is } G\text{-invariant and open in } M_k,\, U \cap \partial M_k = \emptyset, \\ \text{with } \gamma_G(W) \leq k - j, \text{ for } W \in \Sigma,\, W \subset U. \end{array} \right\}$$

and

$$\tilde{\Lambda}_j := \bigcup_{k \geq j} \tilde{\Lambda}_j^k.$$

Finally, for each $j \in \mathbb{N}$, we fix

$$\Lambda_j := \{A \subset X : A \text{ is compact, } G\text{-invariant and for each open } U \supset A, \text{ there is } A_0 \in \tilde{\Lambda}_j,\ A_0 \subset U\}.$$

and

$$c_j := \inf_{A \in \Lambda_j} \sup_{u \in A} I(u).$$

By applying the same arguments used in [81, Theorem 4.4, Lemma 4.6] we can prove that Λ_j verifies the properties $i) - v)$ below (note that, in view of the Proposition 1.1, the arguments in [81, Theorem 4.4] can be applied to the G-index γ_G).

Lemma 1.5 *The sets Λ_j defined above satisfy the following claims:*

i) (Λ_j, d_H) *is a complete metric space;*

ii) $c_j \geq \rho$, *for all* $j > \dim Y$;

iii) $\Lambda_{j+1} \subset \Lambda_j$;

iv) *Let* $A \in \Lambda_j$ *and* W *be a closed* G*-invariant set containing* A *in its interior. Moreover, if* $\alpha : W \to X$ *is an equivariant mapping such that*

$$\alpha|_{W\cap I^{-c_0}} \approx Id|_{W\cap I^{-c_0}}$$

by an equivariant homotopy, then $\alpha(A) \in \Lambda_j$;

v) *For each compact* B *with* $B \in \Sigma$, $\gamma_G(B) \leq p$, $I|_B > -c_0$, *there exists a number* $\delta_0 > 0$ *such that* $A \setminus int(N_\delta(B)) \in \Lambda_j$, *for* $A \in \Lambda_{j+p}$, $\delta \in (0, \delta_0)$.

Part - *v)* in Lemma 1.5 is different with respect to the statement of [81, Lemma 4.6]. However, the main assertion is a direct consequence of the arguments proved there.

Proof of Theorem 1.6. The first part of the proof can be derived by using similar arguments given in [81, Corollary 4.8]. Hence, it remains to show that $c_j \to \infty$ as $j \to \infty$. Now, by Lemma 1.5, Part - *iii)*, it follows that

$$c_j \leq c_{j+1} \quad \forall j \in \mathbb{N}.$$

Thus, if $c_j \nrightarrow \infty$, by *ii)* of last lemma, there exists $c > 0$ such that $c_j \to c$. Arguing as in the proof of Theorem 1.5, we deduce that K_c is a compact G-invariant set with $0 \notin K_c$ and $K_c \neq \emptyset$. Hence, for a convenient $\delta > 0$, by condition (G_*), one has $\gamma_G(N_{2\delta}(K_c)) = \gamma_G(K_c) =: p \in \mathbb{N}$. Now, set $\varepsilon_0 := \min\{1, \delta\}$, take $\varepsilon \in (0, \varepsilon_0)$ as in Lemma 1.1 and define

$$\begin{aligned} \varphi_j : \Lambda_j &\to (-\infty, +\infty] \\ A &\longmapsto \varphi(A) := \sup_{u\in A} I(u). \end{aligned}$$

Clearly φ_j is a lower semicontinuous functional that is bounded from below for every $j \in \mathbb{N}$. Hence, let $A_1 \in \Lambda_{j+p}$ be such that

$$\varphi_{j+p}(A_1) < c_{j+p} + \frac{\varepsilon^2}{2}.$$

Consequently, for some $j_0 \in \mathbb{N}$,

$$\varphi_{j+p}(A_1) < c_j + \varepsilon,$$

for $j \geq j_0$. Now, if $A_2 := \overline{A_1 \setminus \mathrm{int}(N_{2\delta}(K_c))}$, by Part - v) of Lemma 1.5 we have $A_2 \in \Lambda_{j_0}$ and $\varphi_{j_0}(A_2) \leq \varphi_{j_0}(A_1)$. Moreover, by Theorem 1.3, there exists $A \in \Lambda_{j_0}$ such that

$$\varphi_{j_0}(A) \leq \varphi_{j_0}(A_2) < c_{j_0} + \varepsilon \quad d_H(A, A_2) \leq \varepsilon$$

as well as

$$\varphi_{j_0}(B) - \varphi_{j_0}(A) \geq -\varepsilon d_H(B, A) \quad \forall B \in \Lambda_{j_0}. \tag{1.28}$$

If we set $N := N_\delta(K_c)$, Lemma 1.4 implies that $K_{c_{j_0}} \subset N$ if $j_0 \approx \infty$. The definition of ε_0 yields $A \cap N = \emptyset$ and

$$c_{j_0} \leq \sup_{u \in A} I(u) < c_{j_0} + \varepsilon.$$

Then, we can apply Lemma 1.2 to obtain an equivariant deformation α_s. If we set $B := \alpha_s(A)$, by Part - vi) of Lemma 1.1 and Part - iv) of Lemma 1.5, one has $B \in \Lambda_{j_0}$. Finally, a contradiction is achieved by replacing B in (1.28) and arguing as in the proof of Theorem 1.5. ■

1.2 Some Applications to elliptic problems

In this section we illustrate how the abstract results of the previous section can be applied to establish the existence of infinitely many solutions for some classes of elliptic problems.

1.2.1 A logarithmic variational inclusion problem

We start this subsection by recalling some concepts related to the critical point theory for locally Lipschitz functions required in the sequel. Additional comments and remarks about this subject can be found in the Appendix A (we also refer the texts in [34, 36, 40, 41, 71]).

Let $\varphi \in C(X, \mathbb{R})$ be a locally Lipschitz function (briefly $\varphi \in \mathrm{Lip}_{\mathrm{loc}}(X, \mathbb{R})$). The *generalized directional derivative* of φ at u along the direction $v \in X$ is defined by

$$\varphi^{\circ}(u; v) := \limsup_{w \to u,\, t \to 0^+} \frac{\varphi(w + tv) - \varphi(w)}{t}.$$

The *generalized gradient* of the function $\varphi \in \mathrm{Lip}_{\mathrm{loc}}(X, \mathbb{R})$ in u is the set

$$\partial\varphi(u) = \{\phi \in X^* : \varphi^{\circ}(u; v) \geq \langle \phi, v \rangle \,, \forall\, v \in X\}.$$

By a *critical point* of $\varphi \in \mathrm{Lip}_{\mathrm{loc}}(X, \mathbb{R})$, we mean a point $u \in X$ is if $0 \in \partial\varphi(u)$. If, in addition, the functional $\varphi \in \mathrm{Lip}_{\mathrm{loc}}(X, \mathbb{R})$ is convex, then the generalized gradient of φ at u is given by

$$\partial\varphi(u) := \{\phi \in X^* : \varphi(v) - \varphi(u) \geq \langle \phi, v - u \rangle \,, \forall\, v \in X\}, \tag{1.29}$$

i.e., the set $\partial\varphi(u)$ coincides with the subdifferential of φ at u *in the sense of the convex analysis*.

In this subsection we study the existence of infinitely many solutions for the logarithmic inclusion problem

$$(P_1) \qquad \begin{cases} -\Delta u + u + \partial G(x, u) \ni u \log u^2, & \text{in } \mathbb{R}^N \\ u \in H^1(\mathbb{R}^N), \end{cases}$$

where $G(x, t) := \int_0^t g(x, s)ds$ is a convex locally Lipschitz function with $G(x, \cdot) \geq 0$ for every $x \in \mathbb{R}^N$. The notation $\partial G(x, t)$ designates the generalized gradient of G with respect to the variable t.

We also require that the nonlinear term g is a N-measurable function that satisfies the following technical conditions:

(f_1) There is a nonnegative and radial function $h \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ such that

$$|f(x,t)| \leq h(x)|t|, \quad \forall x \in \mathbb{R}^N \quad \text{and} \quad \forall t \in \mathbb{R}.$$

(f_2) $g(x,-t) = -g(x,t)$ and $f(|x|,t) = g(x,t)$ for all $x \in \mathbb{R}^N$ and $t \in \mathbb{R}$.

(f_3) There is $C > 0$ such that for any $\eta_t \in \partial G(x,t)$ it holds

$$G(x,u) - \frac{1}{2}\eta_t t \geq -Ch(x), \;\; \text{a.e } x \in \mathbb{R}^N, \, \forall\, t \in \mathbb{R}.$$

Example 1.7 (A function satisfying $(f_1)-(f_3)$) : Consider

$$G(x,t) := h(x)\int_0^t H(|s|-a)s\,ds,$$

where $a > 0$, $h \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ is nonnegative and radial and H is the Heaviside function, i.e.,

$$H(t) := \begin{cases} 0, & t \leq 0 \\ 1, & t > 0. \end{cases}$$

In this case, we notice that

$$\partial G(x,t) = h(x)\begin{cases} \{s\} & |s| > a, \\ [-a,0] & s = -a, \\ [0,a] & s = a, \\ \{0\} & |s| < a. \end{cases}$$

Direct computations ensure that $(f_1)-(f_3)$ are verified.

Now, consider the energy functional associated to problem (P_1) given by

$$I(u) := \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u|^2 + |u|^2) + \int_{\mathbb{R}^N} G(x,u) - \int_{\mathbb{R}^N} L(u), \;\; u \in H^1(\mathbb{R}^N),$$

where

$$L(t) := -\frac{t^2}{2} + \frac{t^2\log t^2}{2}, \quad \forall t \in \mathbb{R}.$$

Hereafter, we make use of the approach given in [10, 11, 62] to decompose I as a sum of a C^1 functional and a convex lower semicontinuous functional. To this aim, fixed $\delta > 0$ sufficiently small, we set

$$F_1(s) := \begin{cases} 0 & s = 0 \\ -\frac{1}{2}s^2 \log s^2 & 0 < |s| < \delta \\ -\frac{1}{2}s^2(\log \delta^2 + 3) + 2\delta|s| - \frac{\delta^2}{2} & |s| \geq \delta \end{cases}$$

and

$$F_2(s) := \begin{cases} 0 & s = 0 \\ -\frac{1}{2}s^2 \log\left(\frac{s^2}{\delta^2}\right) + 2\delta|s| - \frac{3}{2}s^2 - \frac{\delta^2}{2} & |s| \geq \delta \end{cases}$$

for every $s \in \mathbb{R}$. Therefore

$$F_2(s) - F_1(s) = \frac{1}{2}s^2 \log s^2 \quad \forall s \in \mathbb{R},$$

and

$$I(u) = \frac{1}{2}\|u\|^2 + \int_{\mathbb{R}^N} G(x,u) + \int_{\mathbb{R}^N} F_1(u) - \int_{\mathbb{R}^N} F_2(u) \quad u \in H^1(\mathbb{R}^N), \tag{1.30}$$

where $\|\cdot\|$ denotes the norm in $H^1(\mathbb{R}^N)$ induced by the inner product given by

$$\langle u, v\rangle := \int_{\mathbb{R}^N} (\nabla u \cdot \nabla v + 2uv), \quad \forall u, v \in H^1(\mathbb{R}^N).$$

According to [10, Section 2] and [62, Section 2] the functions F_1 and F_2 satisfy the following conditions:

(A_1) F_1 is an even function with $F_1'(s)s \geq 0$ and $F_1 \geq 0$. Moreover $F_1 \in C^1(\mathbb{R},\mathbb{R})$ and convex provided that $\delta \approx 0^+$;

(A_2) $F_2 \in C^1(\mathbb{R},\mathbb{R})$ and for each $p \in (2, 2^*)$, there exists $C = C_p > 0$ such that

$$|F_2'(s)| \leq C|s|^{p-1} \quad \forall s \in \mathbb{R}.$$

Now, by (A_1) and (A_2), it is easily seen that I is a Szulkin-type functional with

$$\Phi(u) := \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} F_2(u)$$

and

$$\Psi(u) := \int_{\mathbb{R}^N} F_1(u) + \int_{\mathbb{R}^N} G(x,u).$$

We notice that $\Psi = \Psi_1 + \Psi_2$, where

$$\Psi_1(u) := \int_{\mathbb{R}^N} F_1(u) \quad \text{and} \quad \Psi_2(u) := \int_{\mathbb{R}^N} G(x,u).$$

Direct arguments and [10, Lemma 2.1] ensure the validity of the next result.

Lemma 1.6 *Let* $\Psi_1 : H^1(\mathbb{R}^N) \to (-\infty, +\infty]$ *be the functional defined above. Then*

i) $D(I) = D(\Psi_1)$, *that is* $I(u) < \infty$ *if and only if* $\Psi_1(u) < \infty$.

ii) Let $\Omega \subset \mathbb{R}^N$ *be a bounded domain with regular boundary. Then the functional*

$$\tilde{\Psi}_1(u) = \int_\Omega F_1(u) \tag{1.31}$$

belongs to $C^1(H^1(\Omega), \mathbb{R})$.

Moreover, according to [36], the structural conditions on the function G assure that the functional $\Psi_2 : H^1(\mathbb{R}^N) \to \mathbb{R}$ is convex and lower semicontinuous as well as $\Psi_2 \in \mathrm{Lip}_{\mathrm{loc}}(H^1(\mathbb{R}^N), \mathbb{R})$.

From now on, for each $u \in H^1(\mathbb{R}^N)$, let us consider the functional φ_1^u defined by

$$\langle \varphi_1^u, v\rangle := \int_{\mathbb{R}^N} F_1'(u)v, \quad \forall v \in C_0^\infty(\mathbb{R}^N). \tag{1.32}$$

If

$$\|\varphi_1^u\| := \sup_{v \in C_0^\infty(\mathbb{R}^N),\, \|v\| \le 1} \langle \varphi_1^u, v\rangle < \infty,$$

then φ_1^u can be extended to a continuous linear functional on $H^1(\mathbb{R}^N)$.

Moreover, if $\tilde{I} : H^1(\mathbb{R}^N) \to (-\infty, +\infty]$ denotes the functional given by

$$\tilde{I}(u) := \frac{1}{2}\|u\|^2 + \int_{\mathbb{R}^N} F_1(u) - \int_{\mathbb{R}^N} F_2(u),$$

then $\tilde{I}$ is a Szulkin-type functional and $I = \tilde{I} + \Psi_2$.

By [10, Lemma 2.2 and Corollary 2.1] the following lemma holds.

Lemma 1.7 *If* $u \in D(\tilde{I})$ *and* $\|\varphi_1^u\| < \infty$ *then there is a unique functional in* $\partial\tilde{I}(u)$, *denoted by* $\tilde{I}'(u)$, *such that*

$$\tilde{I}'(u)(v) = \langle \Phi'(u), v\rangle + \int_{\mathbb{R}^N} F_1'(u)v \quad \forall v \in C_0^\infty(\mathbb{R}^N). \tag{1.33}$$

Furthermore, $F_1'(u)u \in L^1(\mathbb{R}^N)$, *and*

$$\tilde{I}'(u)(u) = \int_{\mathbb{R}^N} (|\nabla u|^2 + |u|^2) - \int_{\mathbb{R}^N} u^2 \log u^2, \tag{1.34}$$

as well as

$$\tilde{I}(u) - \frac{1}{2}\tilde{I}'(u)(u) = \frac{1}{2}\int_{\mathbb{R}^N} |u|^2. \tag{1.35}$$

Remark 1.4 Lemma 1.7 remains valid if we take $\tilde{J} := \tilde{I}|_{H^1_{\mathrm{rad}}(\mathbb{R}^N)}$. Indeed, the arguments used in [10, Lemma 2.2 and of Corollary 2.1] can be adapted to the radial space $H^1_{\mathrm{rad}}(\mathbb{R}^N)$ by taking $\{\varphi^u_1\} \subset H^1_{\mathrm{rad}}(\mathbb{R}^N)$ and

$$\langle \varphi^u_1, v\rangle = \int_{\mathbb{R}^N} F'_1(u)v \quad v \in C^\infty_{0,\mathrm{rad}}(\mathbb{R}^N).$$

The notion of solution for problem (P_1) requires some comments. To this aim, let us define the functions

$$\underline{g}(x,t) := \lim_{r\downarrow 0} \operatorname{essinf}\{g(x,s) : |s-t| < r\} \tag{1.36}$$

and

$$\overline{g}(x,t) := \lim_{r\downarrow 0} \operatorname{esssup}\{g(x,s) : |s-t| < r\}. \tag{1.37}$$

According to [36, Section 2] if $G(x,t) = \displaystyle\int_0^t g(x,s)\,ds$, then

$$\partial G(x,t) = [\underline{g}(x,t), \overline{g}(x,t)].$$

The above remark makes sense to the following notion.

Definition 1.5 *A function $u \in H^1(\mathbb{R}^N)$ is said to be a solution of (P_1) if $u^2 \log u^2 \in L^1(\mathbb{R}^N)$ and there exists $\rho \in L^2(\mathbb{R}^N)$ such that*

$$\rho(x) \in [\underline{g}(x,u(x)), \overline{g}(x,u(x))] \qquad a.e \ in \quad \mathbb{R}^N$$

and

$$\int_{\mathbb{R}^N} (\nabla u \cdot \nabla \phi + u\phi) + \int_{\mathbb{R}^N} \rho\phi = \int_{\mathbb{R}^N} u \log u^2 \phi, \ \ \forall \phi \in C^\infty_0(\mathbb{R}^N). \tag{1.38}$$

A proof of the next technical result can be found in [16, Lemma 4.1].

Lemma 1.8 *The functions $\underline{g}$ and $\overline{g}$ are N-measurable functions, $\Psi_2 \in \mathrm{Lip}_{\mathrm{loc}}(L^2(\mathbb{R}^N), \mathbb{R})$ and*

$$\partial \Psi_2(u) \subseteq \partial G(x,u) = [\underline{g}(x,u(x)), \overline{g}(x,u(x))], \tag{1.39}$$

for every $u \in L^2(\mathbb{R}^N)$.

The inclusion in (1.39) has the following meaning: for each $\eta \in \partial\Psi_2(u)$ there is a function $\tilde{\eta} \in L^2(\mathbb{R}^N)$ such that

i) $\eta(v) = \displaystyle\int_{\mathbb{R}^N} \tilde{\eta} v \ \ \forall v \in L^2(\mathbb{R}^N)$;

ii) $\tilde{\eta}(x) \in [\underline{g}(x,u(x)), \overline{g}(x,u(x))]$ a.e. in $\mathbb{R}^N$.

Our next step is to prove that the critical points of I in the sense given in Definition A.1 are solutions of (P_1).

Lemma 1.9 *Every critical point of the functional I is a solution of (P_1).*

Proof. Suppose that $u \in D(I)$ is a critical point of I, that is

$$\begin{aligned}\int_{\mathbb{R}^N} (\nabla u \cdot \nabla(v-u) + 2u(v-u)) + \int_{\mathbb{R}^N} (G(x,v) - G(x,u)) \\ \geq \int_{\mathbb{R}^N} F_2'(u)(v-u) - \int_{\mathbb{R}^N} (F_1(v) - F_1(u)),\end{aligned} \tag{1.40}$$

for every $v \in H^1(\mathbb{R}^N)$. The last sentence means that the functional $-\Phi'(u)$ belongs to $\partial\Psi(u)$. Hence, by choosing $v = u + t\phi$, $t > 0$, $\phi \in C_0^\infty(\mathbb{R}^N)$, we find

$$\int_{\mathbb{R}^N} \frac{1}{t}(G(x, u+t\phi) - G(x,u)) + \int_{\mathbb{R}^N} \frac{1}{t}(F_1(u+t\phi) - F_1(u)) \geq \langle -\Phi'(u), \phi\rangle, \tag{1.41}$$

which is equivalent to

$$\frac{1}{t}[\Psi_2(u+t\phi) - \Psi_2(u)] + \int_{\mathbb{R}^N} \frac{1}{t}(F_1(u+t\phi) - F_1(u)) \geq \langle -\Phi'(u), \phi\rangle. \tag{1.42}$$

As Ψ_2 is convex, when $t \to 0^+$, the Lemmas A.4 and 1.6 imply that

$$\Psi_2^\circ(u,\phi) + \int_{\mathbb{R}^N} F_1'(u)\phi \geq \langle -\Phi'(u), \phi\rangle. \tag{1.43}$$

Replacing ϕ with $-\phi$ in (C.3) and by using Lemma A.4 it follows that

$$\Psi_2^\circ(u,-\phi) - \langle \Phi'(u), \phi\rangle \geq \int_{\mathbb{R}^N} F_1'(u)\phi. \tag{1.44}$$

Then, according to the notation introduced in (1.32), one has

$$\Psi_2^\circ(u,-\phi) - \langle \Phi'(u), \phi\rangle \geq \langle \varphi_1^u, \phi\rangle. \tag{1.45}$$

The following claim will be crucial in the rest of the proof.

Claim 1.1 $\displaystyle\sup_{\phi \in C_0^\infty(\mathbb{R}^N),\, \|\phi\| \leq 1} \Psi_2^\circ(u,\phi) < \infty.$

Indeed, by Lemma 1.8, for each $\phi \in C_0^\infty(\mathbb{R}^N)$ with $\|\phi\| \leq 1$, there is $\tilde{\eta}_\phi \in L^2(\mathbb{R}^N)$ such that $\tilde{\eta}_\phi(x) \in [\underline{g}(x,u(x)), \overline{g}(x,u(x))]$ and

$$\Psi_2^\circ(u,\phi) = \int_{\mathbb{R}^N} \tilde{\eta}_\phi \phi.$$

Now, by (f_1), there exists a constant $C := C(u,h) > 0$, independent of ϕ, such that

$$\left|\int_{\mathbb{R}^N} \tilde{\eta}_\phi \phi\right| \leq C\|\phi\|.$$

The above inequality ensures our assertion.

Now, Claim 1.1 in addition to inequality (1.45) ensures that

$$\sup_{\phi\in C_0^\infty(\mathbb{R}^N),\, \|\phi\|\leq 1} \langle \varphi_1^u, \phi\rangle < \infty.$$

Consequently, the classical Hahn-Banach's extension theorem ensures that the functional φ_1 admits an extension, still denoted by φ_1, to a continuous linear functional on $H^1(\mathbb{R}^N)$. Moreover, Lemma A.1, inequality (C.3) and the density of $C_0^\infty(\mathbb{R}^N)$ in $H^1(\mathbb{R}^N)$ yield

$$\langle -\Phi'(u) - \varphi_1^u, v\rangle \leq \Psi_2^\circ(u,v) \quad \forall v \in H^1(\mathbb{R}^N), \tag{1.46}$$

that is,

$$-\Phi'(u) - \varphi_1^u \in \partial\Psi_2(u). \tag{1.47}$$

Thus, there exists $\varphi_2 \in \partial\Psi_2(u)$ such that $-\Phi'(u) - \varphi_1^u = \varphi_2$. Now, by Lemma 1.8, there exists $\rho \in L^2(\mathbb{R}^N)$ such that $\rho(x) \in [\underline{g}(x,u(x)), \overline{g}(x,u(x))]$ a.e. in $\mathbb{R}^N$ and

$$\langle \varphi_2, v\rangle = \int_{\mathbb{R}^N} \rho v, \quad \forall v \in H^1(\mathbb{R}^N).$$

Hence

$$\langle -\Phi'(u), v\rangle = \langle \varphi_1^u, v\rangle + \int_{\mathbb{R}^N} \rho v \quad \forall v \in H^1(\mathbb{R}^N).$$

Taking $v = \phi \in C_0^\infty(\mathbb{R}^N)$ in the above equation, one has

$$\int_{\mathbb{R}^N} \rho\phi + \int_{\mathbb{R}^N} F_1'(u)\phi = \langle -\Phi'(u), \phi\rangle \quad \forall \phi \in C_0^\infty(\mathbb{R}^N), \tag{1.48}$$

which completes the proof. ■

Next, we cite an important result due to Kobayashi-Ôtani that generalizes the Principle of Symmetric Criticality due to Palais (see [83, Theorem 1.28]) and it is a key point in the arguments used in the sequel.

Theorem 1.8 *Let X be a reflexive Banach space and let G be a compact topological group that acts isometrically on X. If $I = \Phi + \Psi$ is a Szulkin-type functional with Φ and Ψ being G-invariant, then*

$$0 \in \partial(I|_Z)(u) \implies 0 \in \partial I(u), \tag{1.49}$$

for any $u \in Z := Fix(G)$.

An exhaustive proof of Theorem 1.8 is given in [64, Theorem 3.16].

The main result of this subsection reads as follows.

Theorem 1.9 *The functional I has a sequence of critical points (u_n) such that $I(u_n) \to \infty$ as $n \to \infty$. Hence, the problem (P_1) has infinitely many nontrivial solutions.*

The proof of Theorem 1.9 is divided into several preliminary results. To this goal, let $O(N)$ be the orthogonal group in $\mathbb{R}^N$. So, by using a standard change of variable, it is easy to check that the functional I is $O(N)$-invariant. Moreover, the space of invariant elements of $H^1(\mathbb{R}^N)$ under the natural action of $O(N)$ coincides with the subspace $H^1_{\text{rad}}(\mathbb{R}^N)$ of radial functions of $H^1(\mathbb{R}^N)$. The classical Symmetric Criticality Principle recalled in Theorem 1.8 ensures that the critical points of $J := I|_{H^1_{\text{rad}}(\mathbb{R}^N)}$ are also critical points of the functional I. We notice that Theorem 1.9 can be proved by using Theorem 1.4 due to the $\mathbb{Z}_2$-invariant of the even functional J; see Example 1.1 for related topics. A key ingredient along the proof of Theorem 1.9 is the Sobolev compact embedding

$$H^1_{\text{rad}}(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N), \quad \forall p \in (2, 2^*). \tag{1.50}$$

See [83, Corollary 1.26] for additional comments and remarks.

Let us prove the following technical result.

Lemma 1.10 *Let (u_n) be a (PS) sequence for the functional J at a level c and let $\varphi_1^{(n)} := \varphi_1^{u_n}$ as in (1.32). Then, $\|\varphi_1^{(n)}\| < \infty$ for any $n \in \mathbb{N}$ and there is a unique $w_n \in \partial J(u_n)$, which will be denoted by $J'(u_n)$, such that:*

i) For some $\varphi_2^{(n)} \in \partial\Psi_2(u_n)$ one has

$$J'(u_n)(v) = \langle \varphi_2^{(n)}, v\rangle + \langle \varphi_1^{(n)}, v\rangle + \langle \Phi'(u_n), v\rangle, \quad \forall v \in H^1_{rad}(\mathbb{R}^N).$$

ii) $J'(u_n)u_n = o_n(1)\|u_n\|$ with

$$J'(u_n)(u_n) \leq \Psi_2^\circ(u_n, u_n) + \int_{\mathbb{R}^N} F_1'(u_n)u_n + \langle \Phi'(u_n), u_n\rangle, \quad \forall n \in \mathbb{N}.$$

Proof. Let (u_n) be a (PS) for the functional J. Then

$$\Psi_2(v) - \Psi_2(u_n) + \int_{\mathbb{R}^N} (F_1(v) - F_1(u_n)) \geq \langle -\Phi'(u_n), v - u_n\rangle + \langle w_n, v - u_n\rangle, \quad v \in H^1_{\text{rad}}(\mathbb{R}^N), \tag{1.51}$$

with $w_n \in (H^1_{\rm rad}(\mathbb{R}^N))'$, and $w_n \to 0$. Set $\phi \in C^\infty_{0,\,\rm rad}(\mathbb{R}^N)$, and take $v := u_n + t\phi$, with $t > 0$. By Lemma A.4 it follows that

$$\Psi_2^\circ(u_n, \phi) + \int_{\mathbb{R}^N} F_1'(u_n)\phi \geq \langle -\Phi'(u_n), \phi\rangle + \langle w_n, \phi\rangle \quad \forall \phi \in C^\infty_{0,\,\rm rad}(\mathbb{R}^N), \tag{1.52}$$

as $t \to 0^+$. Since

$$\langle \varphi_1^{(n)}, \phi\rangle = \int_{\mathbb{R}^N} F_1'(u_n)\phi \quad \phi \in C^\infty_{0,\,\rm rad}(\mathbb{R}^N),$$

arguing as in the proof of Lemma 1.9, one has

$$\sup_{\phi \in C^\infty_{0,\,\rm rad}(\mathbb{R}^N),\, \|\phi\| \leq 1} \langle \varphi_1^{(n)}\ \phi\rangle < \infty. \tag{1.53}$$

Therefore, the functional φ_1^n can be extended to the whole $H^1_{\rm rad}(\mathbb{R}^N)$. By using (1.52), again as in Lemma 1.9, we get

$$-\Phi'(u_n) - \varphi_1^{(n)} + w_n \in \partial\Psi_2(u_n). \tag{1.54}$$

Consequently, by setting $J'(u_n) := w_n$, one has

$$J'(u_n) = \varphi_2^{(n)} + \varphi_1^{(n)} + \Phi'(u_n), \tag{1.55}$$

for some $\varphi_2^{(n)} \in \partial\Psi_2(u_n)$. Hence part $i)$ has been proved. In order to show part $ii)$, let us observe that

$$J'(u_n)(u_n) = \langle w_n, u_n\rangle = o_n(1)\|u_n\|,$$

as $J'(u_n) \to 0$. Hence, by choosing $v := u_n + tu_n$ in (1.51), we have

$$J'(u_n)(u_n) \leq \frac{1}{t}[\Psi_2(u_n + tu_n) - \Psi_2(u_n)] + \int_{\mathbb{R}^N} \frac{1}{t}[F_1(u_n + tu_n) - F_1(u_n)] + \langle \Phi'(u_n), u_n\rangle. \tag{1.56}$$

Since F_1 is convex, the map

$$t \longmapsto \frac{F_1(u_n + tu_n) - F_1(u_n)}{t}, \quad t > 0$$

is monotone and

$$\frac{F_1(u_n + tu_n) - F_1(u_n)}{t} \to F_1'(u_n)u_n,$$

as $t \to 0^+$. Now, Lemma 1.7 and (1.53) yields $F_1'(u_n)u_n \in L^1(\mathbb{R}^N)$ and

$$\int_{\mathbb{R}^N} \frac{F_1(u_n + tu_n) - F_1(u_n)}{t} \to \int_{\mathbb{R}^N} F_1'(u_n)u_n,$$

by using the classical Lebesgue's Dominated Convergence Theorem. In conclusion, as $t \to 0$ in (1.56), by Lemma A.4, it follows that

$$J'(u_n)(u_n) \leq \Psi_2^\circ(u_n, u_n) + \int_{\mathbb{R}^N} F_1'(u_n)u_n + \langle \Phi'(u_n), u_n \rangle.$$

This completes the proof. ■

A consequence of Lemma 1.10 is the following result that will be useful in order to prove that any (PS) sequence for the functional J is bounded; see Lemma 1.12.

Lemma 1.11 *Let (u_n) be a (PS) sequence for the functional J at level c. Then*

$$\int_{\mathbb{R}^N} |u_n|^2 \leq M + o_n(1)\|u_n\|, \quad n \geq n_0 \tag{1.57}$$

for some $M > 0$ and $n_0 \in \mathbb{N}$.

Proof. Since $J(u_n) \to c$, there is $n_0 \in \mathbb{N}$ such that

$$J(u_n) \leq c + 1, \quad n \geq n_0. \tag{1.58}$$

By setting $\tilde{J} = \tilde{I}|_{H^1_{\mathrm{rad}}(\mathbb{R}^N)}$, i.e.

$$\tilde{J}(u) = \frac{1}{2}\|u\|^2 + \int_{\mathbb{R}^N} F_1(u) - \int_{\mathbb{R}^N} F_2(u) \quad u \in H^1_{\mathrm{rad}}(\mathbb{R}^N),$$

we can write $J = \tilde{J} + \Psi_2|_{H^1_{\mathrm{rad}}(\mathbb{R}^N)}$. By Lemmas 1.7 and 1.10 Part - *ii*), one has

$$J'(u_n)(u_n) \leq \tilde{J}'(u_n)(u_n) + \Psi_2^\circ(u_n, u_n)$$

as well as

$$J(u_n) - \frac{1}{2}J'(u_n)(u_n) \geq \frac{1}{2}\int_{\mathbb{R}^N} |u_n|^2 + \left(\Psi_2(u_n) - \frac{1}{2}\Psi_2^\circ(u_n, u_n)\right). \tag{1.59}$$

Now, gathering $J'(u_n)u_n = o_n(1)\|u_n\|$ with (1.58) and (1.59), we get

$$c + 1 + o_n(1)\|u_n\| \geq \frac{1}{2}\int_{\mathbb{R}^N} |u_n|^2 + \left(\Psi_2(u_n) - \frac{1}{2}\Psi_2^\circ(u_n, u_n)\right), \quad \forall n \geq n_0.$$

In order to finish the proof, it is enough to show that there is $M > 0$ (independent of n) such that

$$\left(\Psi_2(u_n) - \frac{1}{2}\Psi_2^\circ(u_n, u_n)\right) \geq -M, \quad \forall n \in \mathbb{N}. \tag{1.60}$$

Bearing in mind the above computations, we employ Lemma 1.8 to obtain

$$\Psi_2(u_n) - \frac{1}{2}\Psi_2^\circ(u_n, u_n) = \int_{\mathbb{R}^N} G(x, u_n) - \frac{1}{2}\int_{\mathbb{R}^N} \eta^{(n)} u_n,$$

where $\eta^{(n)} \in L^2(\mathbb{R}^N)$ and $\eta^{(n)}(x) \in [\underline{g}(x, u_n(x)), \overline{g}(x, u_n(x))]$ a.e. in $\mathbb{R}^N$. Finally, the condition (f_3) yields

$$\int_{\mathbb{R}^N} G(x, u_n) - \frac{1}{2}\int_{\mathbb{R}^N} \eta^{(n)} u_n \geq -C\int_{\mathbb{R}^N} h(x) \geq -M,$$

for some $M = M_h > 0$. This completes the proof. ∎

Let us recall now the so-called logarithmic Sobolev inequality proved in [10, p. 144], as well as [62, Sentence (2.4)] and the references therein. More precisely, for each $b > 0$, one has

$$\int_{\mathbb{R}^N} u^2 \log u^2 \leq \frac{b^2}{\pi}\|\nabla u\|_2^2 + (\log \|u\|_2^2 - N(1 + \log b))\|u\|_2^2 \tag{1.61}$$

for every $u \in H^1(\mathbb{R}^N)$.

An immediate consequence of (1.61) is given below.

Corollary 1.1 *There is $C > 0$ such that*

$$\int_{\mathbb{R}^N} u^2 \log u^2 \leq \frac{1}{2}\|\nabla u\|_2^2 + C(\log \|u\|_2^2) + 1)\|u\|_2^2,$$

for every $u \in H^1(\mathbb{R}^N)$.

The following results involve the notion of (PS) condition and will be proved as consequences of Corollary 1.1.

Lemma 1.12 *If (u_n) is a (PS) sequence for the functional J at level $c \in \mathbb{R}$, then (u_n) is bounded.*

Proof. By Lemma 1.11 and Corollary 1.1, for each $r \in (0, 1)$ there is $C_1 > 0$ such that

$$\frac{1}{2}\int_{\mathbb{R}^N} u_n^2 \log u_n^2 \leq \frac{1}{4}\|u\|^2 + C_1(1 + \|u_n\|^{1+r}).$$

Since $J(u_n) \to c$, there is $n_0 \in \mathbb{N}$ such that

$$c + 1 \geq J(u_n) \geq \frac{1}{2}\|u_n\|^2 - \frac{1}{2}\int_{\mathbb{R}^N} u_n^2 \log u_n^2, \quad n \geq n_0.$$

Then

$$c + 1 \geq \frac{1}{4}\|u_n\|^2 - C_1(1 + \|u_n\|^{1+r}),$$

for every $n \geq n_0$. The proof is complete. ∎

Lemma 1.13 *The functional J satisfies the* (PS) *condition.*

Proof. Let (u_n) be a (PS) sequence for J at level c. By Lemma 1.12, the sequence (u_n) is bounded. Consequently, the embedding (1.50) yields

i) $u_n \rightharpoonup u_0$ in $H^1_{\text{rad}}(\mathbb{R}^N)$;

ii) $u_n \to u_0 \in L^p(\mathbb{R}^N)$ with $p \in (2, 2^*)$;

iii) $\|u_n\| \to M$ and $u_n(x) \to u_0(x)$ a.e. in $\mathbb{R}^N$.

As (u_n) is a (PS) sequence, we have that

$$\langle u_n, v - u_n\rangle + \Psi(v) - \Psi(u_n) \geq -\varepsilon_n\|v - u_n\| + \int_{\mathbb{R}^N} F_2'(u_n)(v - u_n), \quad \forall v \in H^1_{\text{rad}}(\mathbb{R}^N), \tag{1.62}$$

with $\varepsilon_n \to 0^+$. If we take $v := u_0$ in (1.62), the boundedness of (u_n) and the subcritical growth of F_2 immediately give

$$\langle u_n, u_0 - u_n\rangle + \Psi(u_0) - \Psi(u_n) \geq o_n(1). \tag{1.63}$$

Hence, the lower semicontinuity property of Ψ combined with inequality (1.63) leads to

$$\|u_0\|^2 \geq \lim \|u_n\|^2 = M^2, \tag{1.64}$$

on account of $i), ii)$ and $iii)$. In conclusion $u_n \to u_0$ in $H^1_{\text{rad}}(\mathbb{R}^N)$. ■

In order to prove that J satisfies the hypotheses of the Fountain Theorem 1.4, a suitable splitting of the Sobolev space $H^1_{\text{rad}}(\mathbb{R}^N)$ is necessary. To this aim, we first observe that by [67, Proposition 1.a.9 and Section 1.b, p. 8] and [62, Section 5] the next property holds.

Lemma 1.14 *Let A be a dense subset of $H^1(\mathbb{R}^N)$, then $H^1(\mathbb{R}^N)$ has an orthonormal hilbertian basis that is constituted by elements of A.*

Thanks to Lemma 1.14 the following result holds.

Corollary 1.2 *The space $H^1(\mathbb{R}^N)$ has an orthonormal hilbertian basis constituted by elements of $C_0^\infty(\mathbb{R}^N)$. Consequently, there exists a sequence $(v_j) \subset C_0^\infty(\mathbb{R}^N)$ such that*

$$H^1(\mathbb{R}^N) = \overline{\bigoplus_{j\in\mathbb{N}} X_j} \quad \textit{with} \quad X_j = \textit{span}\{v_j\}, \tag{1.65}$$

and $\langle v_i, v_j\rangle = 0$, for every $i \neq j$.

Moreover, the same conclusion holds if we replace $H^1(\mathbb{R}^N)$ and $C_0^\infty(\mathbb{R}^N)$ by $H^1_{\text{rad}}(\mathbb{R}^N)$ and $C^\infty_{0,\text{rad}}(\mathbb{R}^N)$ respectively.

From now on, let us consider

$$H^1_{\rm rad}(\mathbb{R}^N) = \overline{\bigoplus_{j\in\mathbb{N}} X_j} \tag{1.66}$$

and set

$$Y_k := \bigoplus_{j=1}^{k} X_j \quad \text{as well as} \quad Z_k := \overline{\bigoplus_{j=k}^{\infty} X_j}, \tag{1.67}$$

for every $k \in \mathbb{N}$.

Since the action of $\mathbb{Z}_2$ on $H^1_{\rm rad}(\mathbb{R}^N)$ satisfies (G_0) with $X_j \cong \mathbb{R} =: V$ we only need to prove that the functional J satisfies the Parts - $i)$ and $ii)$ of Theorem 1.4.

To this aim, let us briefly recall the next fact.

Lemma 1.15 *Let β_k defined by*

$$\beta_k := \sup_{u\in Z_k, \|u\|=1} \|u\|_p. \tag{1.68}$$

Then $\beta_k \to 0$.

See [83, Lemma 3.8] as well as the proof of Proposition 3.7 in [62] for additional comments and remarks.

Taking into account Lemma 1.15, we are able to prove that the functional J satisfies the Fountain geometry.

Lemma 1.16 *The functional J verifies*

$i)$ $\displaystyle\sup_{u\in Y_k, \|u\|=\rho_k} J(u) \leq 0;$

$ii)$ $\displaystyle\inf_{u\in Z_k, \|u\|=r_k} J(u) \to \infty.$

Proof. We first recall that

$$J(u) = \frac{1}{2}\|u\|^2 + \int_{\mathbb{R}^N} G(x,u) + \int_{\mathbb{R}^N} F_1(u) - \int_{\mathbb{R}^N} F_2(u), \quad \forall\, u \in H^1_{\rm rad}(\mathbb{R}^N).$$

Part - $i)$ By (f_1) one has

$$|G(x,s)| \leq B|s|^2, \;\; \forall x \in \mathbb{R}^N \quad \text{and} \quad \forall s \in \mathbb{R},$$

for some constant $B > 0$. Now, by definition, since $Y_k \subset C^\infty_{0,\rm rad}(\mathbb{R}^N)$ it follows that $Y_k \subset D(J)$ for each $k \in \mathbb{N}$. Hence

$$J(u) \leq \frac{1}{2}\|u\|^2 + B\|u\|_2^2 - \frac{1}{2}\int_{\mathbb{R}^N} u^2 \log u^2, \tag{1.69}$$

for every $u \in Y_k$.
If we take $v := \dfrac{u}{\|u\|}$ for $u \neq 0$, it follows that

$$
\begin{aligned}
J(u) &\le \frac{1}{2}\|u\|^2 \left(1 + B - \int_{\mathbb{R}^N} v^2 \log(v^2\|u\|^2)\right) \\
&= \frac{1}{2}\|u\|^2 \left(1 + B - \int_{\mathbb{R}^N} v^2 \log v^2 - \log(\|u\|^2) \int_{\mathbb{R}^N} v^2\right),
\end{aligned} \tag{1.70}
$$

for every $u \in Y_k$. As $\dim Y_k < \infty$, all the norms on Y_k are equivalent. Hence, if $\|u\| = \rho_k \approx \infty$, one gets

$$1 + B - \int_{\mathbb{R}^N} v^2 \log v^2 - \log(\|u\|^2) \int_{\mathbb{R}^N} v^2 \le 0.$$

Then

$$\sup_{u \in Y_k, \|u\| = \rho_k} J(u) \le 0,$$

so that $i)$ is verified.

Part - $ii)$ By (A_2) for every $s \in \mathbb{R}$,

$$|F_2(s)| \le C|s|^p, \quad p \in (2, 2^*),$$

for some $C > 0$. Hence

$$J(u) \ge \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} F_2(u) \ge \frac{1}{2}\|u\|^2 - \beta_k^p C\|u\|^p,$$

for every $u \in Z_k$. Moreover, by Lemma 1.15 one has $\beta_k \to 0$. Then, by choosing

$$r_k := (pC\beta_k^p)^{\frac{1}{2-p}},$$

it follows that $r_k \to \infty$ and

$$J(u) \ge \left(\frac{1}{2} - \frac{1}{p}\right) r_k^2.$$

In conclusion

$$\inf_{u \in Z_k,\, \|u\| = r_k} J(u) > 0,$$

for k sufficiently large. ■

Conclusion of the proof of Theorem 1.9. First of all, we emphasize that, for every $k \in \mathbb{N}$, the minimax levels

$$c_k := \inf_{\gamma \in \Theta_k} \sup_{u \in B_k} J(\gamma(u))$$

are finite. Indeed, if we take $\tilde{\gamma} := Id|_{B_k}$, by using the classical inequality

$$|t^2 \log t^2| \leq C(|t| + |t|^p), \;\; p > 2 \quad \text{and} \quad \forall t \in \mathbb{R},$$

we infer that there exists $C_1 > 0$ such that

$$J(\tilde{\gamma}(u)) \leq |J(u)| \leq \frac{1}{2}\|u\|^2 + B\|u\|_2^2 + C_1(\|u\|_1 + \|u\|_p^p), \tag{1.71}$$

for every $u \in B_k \subset Y_k$. The equivalence of the norms in Y_k in addition to (1.71) guarantee that

$$c_k = \inf_{\gamma \in \Theta_k} \sup_{u \in B_k} J(\gamma(u)) \leq \sup_{u \in B_k} J(\tilde{\gamma}(u)) < \infty.$$

Finally, we would like to point out that if $u \in H^1(\mathbb{R}^N)$ is a critical point of I, then there exists $\rho \in L^2(\mathbb{R}^N)$ with

$$\rho(x) \in [\underline{g}(x, u(x)), \overline{g}(x, u(x))] \;\; \text{a.e. in } \mathbb{R}^N,$$

such that

$$\int_{\mathbb{R}^N} (\nabla u \cdot \nabla \phi + u\phi) + \int_{\mathbb{R}^N} \rho(x)\phi = \int_{\mathbb{R}^N} u^2 \log u\phi, \quad \forall \phi \in C_0^\infty(\mathbb{R}^N).$$

Therefore, by elliptic regularity theory, there is $r \geq 1$ such that $u \in H^1(\mathbb{R}^N) \cap W^{2,r}_{\text{loc}}(\mathbb{R}^N)$ and

$$-\Delta u + u + \rho(x) = u \log u^2 \;\; \text{a.e. in } \mathbb{R}^N.$$

In conclusion

$$\Delta u - u + u \log u^2 \in [\underline{g}(x, u(x)), \overline{g}(x, u(x))] \;\; \text{a.e. in } \mathbb{R}^N.$$

1.2.2 A concave perturbation of logarithmic equation

In this subsection we study the existence of solutions for the following class of problems

$$(P_2) \qquad \begin{cases} -\Delta u + u = u \log u^2 + \lambda h(x)|u|^{q-2}u, \;\; \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases}$$

where λ is a positive parameter, $q \in (1, 2)$ and $h : \mathbb{R}^N \to \mathbb{R}$ is chosen as in the condition (f_1) above. By using the same notations of the previous subsection, the energy functional associated to (P_2) is given by

$$I_\lambda(u) := \frac{1}{2}\|u\|^2 + \int_{\mathbb{R}^N} F_1(u) - \int_{\mathbb{R}^N} F_2(u) - \frac{\lambda}{q}\int_{\mathbb{R}^N} h(x)|u|^q, \;\; \forall\, u \in H^1(\mathbb{R}^N). \tag{1.72}$$

Note that I_λ is a Szulkin-type functional, with $I_\lambda(u) = \Phi(u) + \Psi(u)$, where

$$\Phi(u) := \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} F_2(u) - \frac{\lambda}{q}\int_{\mathbb{R}^N} h|u|^q$$

and

$$\Psi(u) := \int_{\mathbb{R}^N} F_1(u).$$

In the sequel, we say that a function $u \in H^1(\mathbb{R}^N)$ is a *solution* of (P_2) if $u^2 \log u^2 \in L^1(\mathbb{R}^N)$ and

$$\int_{\mathbb{R}^N} (\nabla u \nabla \phi + u\phi) = \int_{\mathbb{R}^N} (u \log u^2 \phi + \lambda h(x)|u|^{q-2}u\phi), \quad \forall \phi \in C_0^\infty(\mathbb{R}^N). \tag{1.73}$$

By Part - *ii*) of Lemma 1.6 it is possible to see that any critical point of the Szulkin-type functional I_λ is a solution of (P_2); see also [10, Lemma 2.1]. Moreover, if $J_\lambda := I_\lambda|_{H^1_{\text{rad}}(\mathbb{R}^N)}$, again by Theorem 1.8, the critical points of J_λ are also critical points of the functional I_λ.

The main result this subsection reads as follows.

Theorem 1.10 *There exists $\lambda_0 > 0$ such that, for $\lambda \in (0, \lambda_0)$, the functional J_λ has infinitely many critical points (u_n) with $J_\lambda(u_n) \to 0$ as $n \to \infty$. Hence, for $\lambda \in (0, \lambda_0)$, the problem (P_2) has infinitely many nontrivial solutions.*

In order to prove Theorem 1.10, let us introduce a modified functional $\tilde{J}_\lambda$ which will be crucial in our approach. However, let us start by proving the following technical result.

Proposition 1.3 *If $\lambda \approx 0^+$, then there is a function*

$$g(t) := \frac{1}{2}t^2 - Bt^p - C\lambda t^q, \ \ t > 0,$$

with $p \in (2, 2^*)$ and $B, C > 0$, that attains a nonnegative maximum and

$$J_\lambda(u) \geq g(\|u\|), \quad \forall u \in H^1(\mathbb{R}^N).$$

Proof. Since $F_1 \geq 0$, we have that, for every $u \in H^1(\mathbb{R}^N)$

$$\begin{aligned} J_\lambda(u) \geq & \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} F_2(u) - \frac{\lambda}{q}\int_{\mathbb{R}^N} h(x)|u|^q \\ \geq & \frac{1}{2}\|u\|^2 - C_1\|u\|^p - \lambda C_2\|u\|^q \\ =: & \, g(\|u\|), \end{aligned}$$

for some $C_1 = C(p) > 0$ and $C_2 = C(h,q) > 0$. Here, we have chosen $g(t) := \frac{1}{2}t^2 - C_1 t^p - \lambda C_2 t^q$. Moreover, if $\lambda \approx 0^+$ it is clearly seen that the function g attains a nonnegative maximum. ∎

Now, fix R_0, R_1 and R_2 positive constants satisfying:

(g_1) $g|_{[0,R_0]} \leq 0$ and $g(R_0) = 0$;

(g_2) $g|_{[R_0,R_2]} \geq 0$, $g|_{[R_2,\infty)} \leq 0$ and $g(R_2) = 0$, where $R_0 < R_1 < R_2$ and R_1 is the point in which g attains its maximum value; note that $g(t) \to -\infty$, as $t \to \infty$.

Moreover, take $\eta \in C^\infty([0,\infty))$ such that the following condition holds:

(η_1) η is a nonnegative and non-increasing function such that

$$\eta|_{[0,R_0]} \equiv 1 \quad \text{and} \quad \eta|_{[R_2,\infty)} \equiv 0.$$

Set $\varphi(u) := \eta(\|u\|)$. Arguing as in [74], let us consider the energy functional

$$\tilde{J}_\lambda(u) := \frac{1}{2}\|u\|^2 + \int_{\mathbb{R}^N} F_1(u) - \varphi(u)\int_{\mathbb{R}^N} F_2(u) - \frac{\lambda}{q}\int_{\mathbb{R}^N} h(x)|u|^q, \tag{1.74}$$

for every $u \in H^1_{\text{rad}}(\mathbb{R}^N)$.

Lemma 1.17 *Let $\tilde{J}_\lambda$ be the functional given in (1.74). Then, the following facts hold:*

i) $\tilde{J}_\lambda \in (H_0)$ *with* $\tilde{J}_\lambda = \tilde{\Phi}_\lambda + \tilde{\Psi}$ *and* $\tilde{\Psi} = \Psi|_{H^1_{\text{rad}}(\mathbb{R}^N)}$;

ii) If $\tilde{J}_\lambda(u) < 0$ *then* $\|u\| < R_0$ *and* $\tilde{J}_\lambda(u) = J_\lambda(u)$;

iii) Let (u_n) *be a* $(\text{PS})_c$ *sequence for* $\tilde{J}_\lambda$ *with* $c < 0$ *then* (u_n) *is a* $(\text{PS})_c$ *sequence for* J_λ;

iv) If $u \in B_{R_0}(0)$ *is a critical point of* $\tilde{J}_\lambda$ *then* u *is a critical point of* J_λ.

Proof. Part - *i)* immediately follows by (η_1) and the definition of $\tilde{J}_\lambda$. Moreover, if $\lambda \approx 0^+$ then

$$\tilde{g}(t) := \frac{1}{2}t^2 - \lambda C_2 t^q \geq 0$$

for every $t \geq R_2$ and $\tilde{J}_\lambda(\|u\|) \geq \tilde{g}(\|u\|)$. Hence, Part - *ii)* holds. The rest of the proof is an easy consequence of *i)* and *ii)*. ∎

By using the above notations and results we are able to prove Theorem 1.10.

Proof of Theorem 1.10. - By Lemma 1.17 it is sufficient to show that $\tilde{J}_\lambda$ has a sequence of critical points (u_n) with $u_n \in B_{R_0}(0)$ for every $n \in \mathbb{N}$. This will be done by showing that $\tilde{J}_\lambda$ satisfies the hypotheses of Theorem 1.5. To this aim, we first notice that $\tilde{J}_\lambda$ is even and $\tilde{J}_\lambda(0) = 0$. Therefore, we can apply Theorem 1.5 with $G = \mathbb{Z}_2$. In this way, $\gamma_G = \gamma$ is the genus of a symmetric closed set; see Remark 1.2. Moreover, $\tilde{J}_\lambda$ is a coercive functional and consequently any $(\mathrm{PS})_c$ sequence for $\tilde{J}_\lambda$ is bounded. If (u_n) is a $(\mathrm{PS})_c$ sequence for $\tilde{J}_\lambda$, with $c < 0$, then Lemma 1.17 ensures that (u_n) is also a $(\mathrm{PS})_c$ sequence for J_λ. Finally, arguing as in Lemma 1.13, it easily seen that $\tilde{J}_\lambda$ satisfies the $(\mathrm{PS})_c$ condition for $c < 0$. It remains to show that $\tilde{J}_\lambda$ satisfies $i)$ and $ii)$ of Theorem 1.5.

Part - $i)$ Since $\tilde{J}_\lambda$ satisfies

$$\tilde{J}_\lambda(u) \geq g(\|u\|) \quad \forall u \in H^1(\mathbb{R}^N)$$

and $\tilde{J}_\lambda(u) \geq 0$ for every $\|u\| \geq R_2$, we conclude that $\tilde{J}_\lambda$ is bounded from below. Consequently

$$c_j := \inf_{A \in \Gamma_j} \sup_{u \in A} \tilde{J}_\lambda(u) > -\infty.$$

Part - $ii)$ For each $k \in \mathbb{N}$, let us consider Y_k and Z_k as in (1.67). In this case $\dim Y_k < \infty$ and $Y_k \subset C_0^\infty(\mathbb{R}^N)$. Bearing in mind that

$$F_1(u) < \infty, \ \ \forall u \in Y_k,$$

we infer that $Y_k \subset D(\tilde{J}_\lambda)$ for any $k \in \mathbb{N}$. As $\tilde{J}_\lambda \equiv J_\lambda$ in B_{R_0}, one has

$$\tilde{J}_\lambda(u) = \frac{1}{2}\|u\|^2 - \frac{1}{2}\int_{\mathbb{R}^N} u^2 \log u^2 - \frac{\lambda}{q}\int_{\mathbb{R}^N} h(x)|u|^q.$$

Moreover, if $\delta \approx 0^+$

$$|t|^2|\log t^2| \leq C_1(|t|^{2-\delta} + |t|^{2+\delta}), \quad \forall t \in \mathbb{R},$$

for some $C_1 = C_1(\delta) > 0$. Consequently

$$\tilde{J}_\lambda(u) \leq \frac{1}{2}\|u\|^2 + C\int_{\mathbb{R}^N} (|u|^{2-\delta} + |u|^{2+\delta}) - \frac{\lambda}{q}\int_{\mathbb{R}^N} h(x)|u|^q,$$

for every $u \in B_{R_0}$. Now, if $u \in Y_k$ then $u \in L^r(\mathbb{R}^N)$ for every $r \in [1, 2)$. Since all the norms on Y_k are equivalent, one has

$$\tilde{J}_\lambda(u) \leq \frac{1}{2}\|u\|^2 + C_2(\|u\|^{2-\delta} + \|u\|^{2+\delta}) - C\|u\|^q, \tag{1.75}$$

for some constant $C_2 > 0$. Now, for each $k \in \mathbb{N}$, fix $A := S_\rho(0) \cap Y_k$ with $\rho \approx 0^+$. Then A is a closed and symmetric set with $\gamma(A) = k$. By choosing δ such that $2 - \delta > q$, on account of (1.75), it follows that

$$\sup_{u \in A} \tilde{J}_\lambda(u) < 0.$$

The proof is now complete. ■

1.2.3 A problem involving the 1-Laplacian operator with subcritical growth

In this subsection we study the existence of infinitely many solutions for the following problem

$$(P_3) \qquad \begin{cases} -\Delta_1 u = |u|^{p-2}u, & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ (with $N \geq 2$) is a bounded domain with smooth boundary $\partial\Omega$ and $p \in (1, 1^*)$. In order to simplify the notation, we set $q := p/(p-1)$.

Several classes of problem involving the 1-Laplacian operator in a similar configuration of (P_3) have been studied in last years. Here we refer [17, 57, 58].

From now on we denote by $\mathcal{M}(\Omega, \mathbb{R}^N)$ (briefly $\mathcal{M}(\Omega)$) the space of the vector Radon measures on Ω and by $BV(\Omega)$ the space of the functions $u : \Omega \to \mathbb{R}$ of bounded variation, i.e.,

$$BV(\Omega) := \{u \in L^1(\Omega) : \; Du \in \mathcal{M}(\Omega)\},$$

where Du denotes the distributional derivative of $u \in L^1(\Omega)$. It is well known that $u \in BV(\Omega)$ if, and only if, $u \in L^1(\Omega)$ and

$$\int_\Omega |Du| = \sup \left\{ \int_\Omega u \mathrm{div}\phi : \; \phi \in C_0^1(\Omega, \mathbb{R}^N), \quad \text{and} \quad \|\phi\|_\infty \leq 1) \right\} < +\infty.$$

Moreover $BV(\Omega)$ is a Banach space endowed by the norm

$$\|u\|_{BV(\Omega)} := \int_\Omega |Du| + \int_{\partial\Omega} |u| d\mathcal{H}^{N-1},$$

where, as usual, $\mathcal{H}^{N-1}$ denotes the $(N-1)$-dimensional Hausdorff measure. We also recall that the continuous embedding

$$BV(\Omega) \hookrightarrow L^r(\Omega), \;\; r \in [1, 1^*] \tag{1.76}$$

is compact provided that $r \in [1, 1^*)$; see [20, 22, 63] for advanced theoretical results on the subject.

According to Kawohl and Schuricht in [63], as well as Degiovanni in [48], the notion of solution for problem (P_3) can be formulated as follows.

Definition 1.6 *We say that a function $u \in BV(\Omega)$ is a solution of (P_3) if there exists $z \in L^\infty(\Omega, \mathbb{R}^N)$ with $\|z\|_\infty \leq 1$, such that*

$$\begin{cases} -\int_\Omega u\mathrm{div}z = \int_\Omega |Du| + \int_{\partial\Omega} |u| d\mathcal{H}^{N-1}, \quad \mathrm{div}z \in L^q(\Omega), \\ -\mathrm{div}z = |u|^{p-2}u \quad a.e. \ in \ \ \Omega, \end{cases}$$

where $q := p/(p-1)$.

Remark 1.5 Notice that the vector field z in the preceding definition gives the *formal sense* for div $\left(\frac{\nabla u}{|\nabla u|}\right)$. More precisely, the map z replaces $Du/|Du|$ when the expression $Du/|Du|$ is undetermined.

Now, let us consider the energy functional $I : L^p(\Omega) \to (-\infty, +\infty]$ given by

$$I(u) = \Phi(u) + \Psi(u), \tag{1.77}$$

where

$$\Phi(u) := -\frac{1}{p}\int_\Omega |u|^p$$

and

$$\Psi(u) := \begin{cases} \int_\Omega |Du| + \int_{\partial\Omega} |u| d\mathcal{H}^{N-1} & u \in BV(\Omega) \\ \infty & u \in L^p(\Omega) \setminus BV(\Omega) \end{cases},$$

for every $u \in L^p(\Omega)$.

It is easily seen that $\Phi \in C^1(L^p(\Omega), \mathbb{R})$ as well as Ψ is a convex and lower semicontinuous functional, so that I is a Szulkin-type functional. Consequently $D(I) = BV(\Omega)$ and, for each fixed $u \in BV(\Omega)$, the subdifferential $\partial\Psi(u)$ can be identified as a subset of $L^q(\Omega)$.

The next results will be crucial in the sequel.

Lemma 1.18 *If $u \in BV(\Omega)$ and $\partial\Psi(u) \neq \emptyset$ then $u \in L^\infty(\Omega)$.*

Proof. We first notice that $L^{1^*}(\Omega) \hookrightarrow L^p(\Omega)$, so that $L^q(\Omega) \hookrightarrow L^N(\Omega)$. Consequently, if $w \in \partial\Psi(u) \subset L^q(\Omega)$, one has that $w \in L^N(\Omega)$. The conclusion is achieved by arguing as in [48, Proposition 3.3]. ■

Lemma 1.19 *If $u \in BV(\Omega)$ then, for each $w \in \partial\Psi(u)$, there exists $z \in L^\infty(\Omega, \mathbb{R}^N)$ with $\|z\|_\infty \le 1$, such that*

$$\begin{cases} w = -\text{div} z \in L^q(\Omega) \\ -\displaystyle\int_\Omega u\text{div} z = \int_\Omega |Du| + \int_{\partial\Omega} |u| d\mathcal{H}^{N-1}. \end{cases}$$

Proof. Let us define

$$\tilde{\Psi}(u) := \begin{cases} \displaystyle\int_\Omega |Du| + \int_{\partial\Omega} |u| d\mathcal{H}^{N-1} & u \in BV(\Omega) \\ \infty & u \in L^{1^*}(\Omega) \setminus BV(\Omega) \end{cases},$$

and take $w \in \partial\Psi(u) \subset L^q(\Omega)$. Then $w \in L^N(\Omega)$ and

$$\tilde{\Psi}(v) - \tilde{\Psi}(u) = \Psi(v) - \Psi(u) \ge \int_\Omega w(v-u), \qquad \forall v \in BV(\Omega) = D(\tilde{\Psi}),$$

so that $w \in \partial\tilde{\Psi}(u)$. The conclusion follows by [63, Proposition 4.23]. ■

The next result connects critical points of the energy functional I with solutions of (P_3).

Lemma 1.20 *If $u \in BV(\Omega)$ is a critical point of the functional I then $u \in L^\infty(\Omega)$. Moreover, the function u is a solution of (P_3) in the sense of Definition 1.6.*

Proof. Let $u \in BV(\Omega)$ be a critical point of I. Then

$$-\Phi'(u) \in \partial\Psi(u) \subset L^q(\Omega).$$

Thereby, there exists $w \in \partial\Psi(u)$ such that

$$-\Phi'(u) = w \ \text{ in } \ L^q(\Omega).$$

Consequently, Lemma 1.19 and the definition of Φ yield the existence of $z \in L^\infty(\Omega, \mathbb{R}^N)$, with $\|z\|_\infty \le 1$, such that $-\text{div} z = w$ in $L^q(\Omega)$ and

$$\begin{cases} -\displaystyle\int_\Omega u\text{div} z = \int_\Omega |Du| + \int_{\partial\Omega} |u| d\mathcal{H}^{N-1}, \ \text{div} z \in L^q(\Omega) \\ -\text{div} z = |u|^{p-2}u \ \text{ a.e. in } \ \Omega. \end{cases}$$

Moreover, Lemma 1.18 ensures that $u \in L^\infty(\Omega)$. The proof is now complete. ■

By Lemmas 1.19 and 1.20 we are able to prove the main result of this subsection.

Theorem 1.11 *The functional I has infinitely many critical points (u_n) with $I(u_n) \to \infty$ as $n \to \infty$. Hence, problem (P_3) has infinitely many nontrivial solutions.*

Proof. Hereafter, we are going to prove that I verifies the assumptions of Theorem 1.6 with $Y = \{0\}$. We first prove that I satisfies the compactness (PS) condition. To this end, let (u_n) be a (PS) sequence for I. So, let $c \in \mathbb{R}$ such that

$$I(u_n) \to c,$$

and

$$\Psi(v) - \Psi(u_n) \geq \int_\Omega |u_n|^{p-2}u_n(v - u_n) + \int_\Omega w_n(v - u_n), \quad \forall v \in BV(\Omega),$$

where $w_n \in L^q(\Omega)$ and $w_n \to 0$ in $L^q(\Omega)$. The last inequality gives

$$|u_n|^{p-2}u_n + w_n \in \partial\Psi(u_n), \quad \forall n \in \mathbb{N}.$$

Hence, Lemma 1.19 yields

$$\Psi(u_n) = \int_\Omega |Du_n| + \int_{\partial\Omega} |u_n| d\mathcal{H}^{N-1} = \int_\Omega |u_n|^p + \int_\Omega w_n u_n, \quad \forall n \in \mathbb{N}.$$

If we set

$$A(u_n) := \Psi(u_n) - \int_\Omega |u_n|^p + \int_\Omega w_n u_n = 0,$$

the classical Hölder's inequality leads to

$$\begin{aligned} c + 1 &\geq I(u_n) - \frac{1}{r}A(u_n) \\ &\geq \left(1 - \frac{1}{r}\right)\Psi(u_n) + \left(\frac{1}{r} - \frac{1}{p}\right)\|u_n\|^p_{L^p(\Omega)} - \frac{1}{r}\|w_n\|_{L^q(\Omega)}\|u_n\|_{L^p(\Omega)} \\ &\geq C_1\|u_n\|_{BV(\Omega)} + C_2\left(\|u_n\|^p_{L^p(\Omega)} - \|u_n\|_{L^p(\Omega)}\right), \end{aligned}$$

for some $r < p$ and n large enough. Since the real function $h(t) := t^p - t$, for every $t \geq 0$, is bounded from below, the last inequality clearly implies that $\sup_{n\in\mathbb{N}} \|u_n\|_{BV(\Omega)} < \infty$.

Therefore the (PS) condition is verified, since the embedding $BV(\Omega) \hookrightarrow L^p(\Omega)$ is compact. Now, if $u \in BV(\Omega)$ is a critical point of I then

$$|u|^{p-2}u \in \partial\Psi(u).$$

Consequently, by Lemma 1.19, it follows that

$$\int_\Omega |u|^p = \int_\Omega |Du| + \int_{\partial\Omega} |u| d\mathcal{H}^{N-1}.$$

Thereby, by setting

$$B(u) = \int_\Omega |Du| + \int_{\partial\Omega} |u| d\mathcal{H}^{N-1} - \int_\Omega |u|^p,$$

one has

$$I(u) = I(u) - \frac{1}{p}B(u) = \left(1 - \frac{1}{p}\right)\|u\|_{BV(\Omega)} \geq 0,$$

for every $u \in L^p(\Omega)$. Hence, the set I^{-c} has no critical points for any $c > 0$.

Finally, let us prove that the functional I satisfies conditions $i)$ and $ii)$ of Theorem 1.6.

Part - $i)$ Without loss of generality we can suppose $u \in BV(\Omega)$, otherwise $I(u) = \infty$. Now, if $u \in BV(\Omega)$, the embedding $BV(\Omega) \hookrightarrow L^p(\Omega)$ immediately yields

$$I(u) \geq C\|u\|_{L^p(\Omega)} - \frac{1}{p}\|u\|^p_{L^p(\Omega)},$$

for some constant $C > 0$. Since $p > 1$, if $\|u\|_{L^p(\Omega)} = r \approx 0^+$, we also have

$$I(u) \geq \rho,$$

for some $\rho > 0$. Thus, condition $i)$ of Theorem 1.6 is proved with $Z = L^p(\Omega)$.

Part - $ii)$ For each $k \in \mathbb{N}$, let us consider X_k be a k-dimensional subspace of $C_0^\infty(\Omega)$. Since all the norms are equivalent on X_k, it easily seen that

$$I(u) \leq C_k\|u\|_{L^p(\Omega)} - \frac{1}{p}\|u\|^p_{L^p(\Omega)} \quad \forall u \in X_k,$$

for a convenient $C_k > 0$. Thus

$$I(u) \to -\infty, \quad \text{as} \quad \|u\|_{L^p(\Omega)} \to \infty \, \text{and} \, u \in X_k.$$

The proof is now complete. ■

CHAPTER 2

Existence of multiple solutions for a Schrödinger logarithmic equation via Lusternik-Schnirelman category theory

In the current chapter we are interested in the following problem

$$(P_\varepsilon) \qquad \begin{cases} -\varepsilon^2 \Delta u + V(x)u = u \log u^2, \text{ in } \mathbb{R}^N, \\ \quad u \in H^1(\mathbb{R}^N), \end{cases}$$

where $V : \mathbb{R}^N \longrightarrow \mathbb{R}$ is a continuous function satisfying

(V_1): $-1 < \inf_{x \in \mathbb{R}^N} V(x)$;

(V_2): There exists an open and bounded set $\Lambda \subset \mathbb{R}^N$ satisfying

$$V_0 := \inf_{x \in \Lambda} V(x) < \min_{x \in \partial\Lambda} V(x).$$

We emphisize that, without lost of generality, we will assume throughout this chapter that $0 \in \Lambda$ and $V_0 = V(0)$.

Before presenting the main results concerning with the study of problem (P_ε), we would like to mention some interesting aspects related to the equation

$$(E_1) \qquad -\varepsilon^2 \Delta u + V(x)u = u \log u^2, \quad x \in \mathbb{R}^N,$$

under different assumptions on V and ε.

It is natural to apply variatonal methods to look by solutions of (E_1). The usual variational framework lead us to consider the energy functional

$$E_\varepsilon(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + (V(\varepsilon x))|u|^2)dx - \int_{\mathbb{R}^N} F(u)dx, \tag{2.1}$$

with

$$F(t) = \int_0^t s\log s^2\, ds = \frac{1}{2}t^2\log t^2 - \frac{t^2}{2}.$$

However, it is well known that the functional E_ε is not well defined, e.g., on $H^1(\mathbb{R}^N)$ because there exist functions $u \in H^1(\mathbb{R}^N)$ such that $\displaystyle\int_{\mathbb{R}^N} u^2\log u^2 = -\infty$, which gives the possibility that $E_\varepsilon(u) = \infty$.

In the literature there is a broad list of works that provide different techniques to carry out this difficulty referring to the study of equation (E_1) via variational methods. Here we refer the works [10–13, 44, 62, 79]. The main point in those works consists in to use alternatives critical point theories for nonsmooth functionals. Although the frameworks introduced in those works allows us to get solutions for (E_1), some questions involving critical points for C^1-functionals cannot be explored in those works (we would like to cite, e.g., the existence of multiple solutions for (E_1) via the Lusternik-Schnirelmann's category; see [83, Chapater 5]).

Motivated by the above fact, we intent to prove the existence of multiple solution for (P_ε) by relating the multiplicity of solution with the category of Lusternik-Schnirelmann of the set

$$M := \{x \in \Lambda;\ V(x) = V_0\}$$

in the set

$$M_\delta := \{x \in \mathbb{R}^N;\ d(x, M) \leq \delta\},\ \ \delta \approx 0^+.$$

We would like to mention that this type of information is a novelty for logarithmic Schrödinger equations. In our search, we have not found any article that relates the multiplicity of solution for equations of (E_1)-type with the Lusternik-Schnirelmann's category.

The main result to be proved in this chapter is the following.

Theorem 2.1 *If the conditions $(V_1)-(V_2)$ hold and $\delta > 0$ is small enough, then there is $\varepsilon_3 > 0$, such that, for $\varepsilon \in (0, \varepsilon_3)$, the following items are valid:*

i) (P_ε) has at least $\frac{cat_{M_\delta}(M)}{2}$ positive solutions , if $cat_{M_\delta}(M)$ is an even number;

ii) (P_ε) *has at least* $\frac{cat_{M_\delta}(M)+1}{2}$ *positive solutions, if* $cat_{M_\delta}(M)$ *is an odd number.*

In order to prove the preceding theorem, we will introduce a new reflexive and separable Banach space in which the functional E_ε in (2.1) is a C^1-functional. Such technique enable us to adapt some results valid in the classical Critical Point Theory. We also mention that, in view of conditions $(V_1) - (V_2)$ above, the results presented throughout this chapter improve the results of Alves and de Morais Filho [10] and Alves and Ji [11] on the existence and concentration of positive solutions for (P_ε).

Note that, by the change of variable $u(x) = v(x/\varepsilon)$, the problem (P_ε) is equivalent to the problem

$$(S_\varepsilon) \qquad \begin{cases} -\Delta v + V(\varepsilon x)v = v\log v^2, \ \text{ in } \ \mathbb{R}^N, \\ \ v \in H^1(\mathbb{R}^N), \end{cases}$$

We will explore this fact in our computations.

We would like to mention that the results developed in the present chapter have been published in the paper [7].

2.1 Variational framework on the logarithmic equation

In this section we present the main tools requested to our variational approach. We start by recalling the decomposition of the nonlinearity $f(t) = t\log t^2$ explored in Chapter 1, which is an important step in order to overcome the lack of smoothness of energy functional associated with (S_ε). Finally, taking into account the conditions $(V_1) - (V_2)$ mentioned above and motivated by [11, 51], we introduce an auxiliary problem that is a crucial tool in our study to obtain the existence of solution for (S_ε).

2.1.1 Basics on the logarithmic equation

Let us start by presenting a convenient decomposition of the function

$$F(t) = \int_0^t s\log s^2\, ds = \frac{1}{2}t^2\log t^2 - \frac{t^2}{2},$$

which has been explored in Section 1.2, as well as in a lot of works (see, e.g., [10–12, 62, 79]).

Fixed $\delta > 0$ sufficiently small, we set

$$F_1(s) := \begin{cases} 0, & s = 0 \\ -\frac{1}{2}s^2 \log s^2, & 0 < |s| < \delta \\ -\frac{1}{2}s^2(\log \delta^2 + 3) + 2\delta|s| - \frac{\delta^2}{2}, & |s| \geq \delta \end{cases} \tag{2.2}$$

and

$$F_2(s) := \begin{cases} 0, & |s| < \delta \\ \frac{1}{2}s^2 \log\left(\frac{s^2}{\delta^2}\right) + 2\delta|s| - \frac{3}{2}s^2 - \frac{\delta^2}{2}, & |s| \geq \delta \end{cases}$$

for every $s \in \mathbb{R}$. Hence,

$$F_2(s) - F_1(s) = \frac{1}{2}s^2 \log s^2, \quad \forall s \in \mathbb{R}. \tag{2.3}$$

By direct computations, one can verifies that F_1 and F_2 verify the properties $(P_1)-(P_4)$ below:

(P_1) F_1 is an even function with $F_1'(s)s \geq 0$ and $F_1 \geq 0$. Moreover $F_1 \in C^1(\mathbb{R}, \mathbb{R})$ and it is also convex if $\delta \approx 0^+$.

(P_2) $F_2 \in C^1(\mathbb{R}, \mathbb{R}) \cap C^2((\delta, +\infty), \mathbb{R})$ and for each $p \in (2, 2^*)$, there exists $C = C_p > 0$ such that

$$|F_2'(s)| \leq C|s|^{p-1}, \quad \forall s \in \mathbb{R}.$$

(P_3) $s \mapsto \frac{F_2'(s)}{s}$ is a nondecreasing function for $s > 0$ and a strictly increasing function for $s > \delta$.

(P_4) $\lim\limits_{s \to \infty} \dfrac{F_2'(s)}{s} = \infty$.

We recall below the definition of a N-function, which plays a special role in the sequel.

Definition 2.1 *A continuous function $\Phi : \mathbb{R} \to [0, +\infty)$ is a N-function if:*

(i) Φ *is convex.*

(ii) $\Phi(t) = 0 \Leftrightarrow t = 0$.

(iii) $\lim\limits_{t \to 0} \dfrac{\Phi(t)}{t} = 0$ *and* $\lim\limits_{t \to \infty} \dfrac{\Phi(t)}{t} = +\infty$.

(iv) Φ *is an even function.*

Associated with each N-function we have the conjugate function $\tilde{\Phi}$ that is given by the Legendre's transformation of Φ, more precisely,

$$\tilde{\Phi}(t) = \max_{t\geq 0}\{st - \Phi(t)\} \text{ for } s \geq 0.$$

See the Appendix C for further details involving N-functions.

An important step in our study is the fact that the function F_1 is a N-function. More precisely, the following result is valid.

Proposition 2.1 *The function F_1 is a N-function. Furthermore, it holds that F_1, $\tilde{F}_1 \in (\Delta_2)$. Equivalently, there exists $l \in (1,2)$ such that*

$$1 < l \leq \frac{F_1'(s)s}{F_1(s)} \leq 2, \quad \forall s > 0. \tag{2.4}$$

Proof. See the Proposition C.2 in Appendix C. ■

The last proposition allows us to conclude that the space

$$L^{F_1}(\mathbb{R}^N) = \left\{u \in L^1_{loc}(\mathbb{R}^N) \; ; \int_{\mathbb{R}^N} F_1\left(|u|\right) dx < +\infty\right\}$$

is a reflexive and separable Banach space. In a more precise description, $L^{F_1}(\mathbb{R}^N)$ is the *Orlicz space associated with the N-function* F_1. On $L^{F_1}(\mathbb{R}^N)$, we will consider the usual Luxemburg norm

$$||u||_{F_1} = \inf\left\{\lambda > 0 \; ; \int_\Omega F_1\left(\frac{|u|}{\lambda}\right) \leq 1\right\}.$$

The study of problem (S_ε) lead us to work in the space

$$H_\varepsilon := \left\{u \in H^1(\mathbb{R}^N); \int_{\mathbb{R}^N} V(\varepsilon x)|u|^2 dx < \infty\right\}.$$

In the sequel, in order to avoid the points $u \in H^1(\mathbb{R}^N)$ that verify $F_1(u) \notin L^1(\mathbb{R}^N)$, we will restrict the functional E_ε given in (2.1) to the space $X_\varepsilon := H_\varepsilon \cap L^{F_1}(\mathbb{R}^N)$, which will be denoted by I_ε, that is, $I_\varepsilon \equiv E_\varepsilon|_{X_\varepsilon}$. Hereafter, let us consider on X_ε the norm

$$||\cdot||_\varepsilon := ||\cdot||_{H_\varepsilon} + ||\cdot||_{F_1},$$

where

$$||u||_{H_\varepsilon} := \left(\int_{\mathbb{R}^N} (|\nabla u|^2 + (V(\varepsilon x) + 1)|u|^2)\right)^{1/2}, \quad u \in H_\varepsilon.$$

In view of the Proposition 2.1, $(X_\varepsilon, ||\cdot||_\varepsilon)$ is a reflexive and separable Banach space. In this way, from the conditions on F_1 and V, one has $I_\varepsilon \in C^1(X_\varepsilon, \mathbb{R})$ with

$$I'_\varepsilon(u)v = \int_{\mathbb{R}^N} (\nabla u \nabla v + (V(\varepsilon x) + 1)uv + \int_{\mathbb{R}^N} F'_1(u)v - \int_{\mathbb{R}^N} F'_2(u)v, \quad \forall v \in X_\varepsilon.$$

Note also that, as a natural consequence of the definition of $||\cdot||_\varepsilon$, the embedding $X_\varepsilon \hookrightarrow H^1(\mathbb{R}^N)$ and $X_\varepsilon \hookrightarrow L^{F_1}(\mathbb{R}^N)$ are continuous.

2.1.2 The auxiliary problem

From now on, we fix $b_0 \approx 0^+$ and $a_0 > \delta$ in a such way that $(\inf_{\mathbb{R}^N} V + 1) > 2b_0$ and $\frac{F'_2(a_0)}{a_0} = b_0$. Using these notations, we set

$$\overline{F}'_2(s) := \begin{cases} F'_2(s), & 0 \le s \le a_0; \\ b_0 s & s \ge a_0. \end{cases}$$

Now, consider t_1, $t_2 > 0$ with $a_0 \in (t_1, t_2)$ and $h \in C^1([t_1, t_2])$ verifying

(h_1): $h(t) \le \overline{F}'_2(t)$, $t \in [t_1, t_2]$;

(h_2): $h(t_i) = \overline{F}'_2(t_i)$ and $h'(t_i) = \overline{F}''_2(t_i)$, $i \in \{1, 2\}$;

(h_3): $\frac{h(t)}{t}$ is a nondecreasing function.

Remark 2.1 The existence of a such function h is assured by using the results in [5, Appendix A].

In the building of the function h, it is considered that, besides of the properties $(P_2) - (P_4)$ above, the function F_2 belongs to $C^2((\delta, +\infty), \mathbb{R})$.

Define

$$\tilde{F}'_2(s) := \begin{cases} \overline{F}'_2(s), & t \notin [t_1, t_2]; \\ h(t), & t \in [t_1, t_2]. \end{cases}$$

Denote by χ_Λ the characteristic function of the set Λ and let $g_2 : \mathbb{R}^N \times [0, \infty) \longrightarrow \mathbb{R}$ given by

$$g_2(x, t) := \chi_\Lambda(x) F'_2(t) + (1 - \chi_\Lambda(x)) \tilde{F}'_2(t).$$

On account that F'_2 is an odd function, we can extend the definition of g_2 to $\mathbb{R}^N \times \mathbb{R}$ by setting $g_2(x, t) = -g_2(x, -t)$, for each $t \le 0$ and $x \in \mathbb{R}^N$.

Hereafter, we will study the existence of solution for the following auxiliary problem

$$(\tilde{S}_\varepsilon) \qquad \begin{cases} -\Delta u + (V(\varepsilon x)+1)u = g_2(\varepsilon x, u) - F_1'(u), \text{ in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N) \cap L^{F_1}(\mathbb{R}^N). \end{cases}$$

Setting

$$\Lambda_\varepsilon := \{x \in \mathbb{R}^N;\, \varepsilon x \in \Lambda\},$$

we see that if u is a positive solution of $(\tilde{S}_\varepsilon)$ satisfying

$$0 < u(x) < t_1, \quad \forall x \in (\mathbb{R}^N - \Lambda_\varepsilon), \tag{2.5}$$

then u is a solution of (S_ε). Have this in mind, we will study the existence of positive solutions for (S_ε) by looking for solutions of $(\tilde{S}_\varepsilon)$ that satisfy (2.5).

From the definition of g_2, it is possible to prove the following properties:

$$(A_1): \begin{cases} i): \; g_2(x,t) \le b_0|t| + C|t|^{p-1}, \quad t \ge 0,\, x \in \mathbb{R}^N; \\ ii): \; g_2(x,t) \le F_2'(t), \quad x \in \mathbb{R}^N; \\ iii): \; g_2(x,t) \le b_0 t, \quad t \ge 0,\, x \in \left(\mathbb{R}^N - \Lambda\right); \\ iv): \; \frac{1}{2}|t|^2 + [F_2(t) - \frac{1}{2}F_2'(t)t + \frac{1}{2}G_2'(\varepsilon x,t)t - G_2(\varepsilon x,t)] \ge 0, \;\; \forall t \in \mathbb{R},\, x \in \mathbb{R}^N. \end{cases}$$

Associated with $(\tilde{S}_\varepsilon)$ we have the following functional

$$J_\varepsilon(u) := \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + (V(\varepsilon x)+1)|u|^2) + \int_{\mathbb{R}^N} F_1(u) - \int_{\mathbb{R}^N} G_2(\varepsilon x, u), \;\; \forall u \in X_\varepsilon,$$

where $G_2(x,t) := \displaystyle\int_0^t g_2(x,s)\,ds$. The conditions on g_2 ensures that $J_\varepsilon \in C^1(X_\varepsilon, \mathbb{R})$, and thereby, critical points of J_ε are weak solutions of $(\tilde{S}_\varepsilon)$.

2.2 Existence of solution for the auxiliary problem

In this section we will establish the existence of solution for $(\tilde{S}_\varepsilon)$. We start by showing that J_ε satisfies the geometric configuration of the Mountain Pass Theorem (see [19]).

Lemma 2.1 *Given $\varepsilon > 0$, the functional J_ε satisfies*

i) There exist $r, \rho > 0$ such that $J_\varepsilon(u) \ge \rho$ for any $u \in X_\varepsilon$, $||u||_\varepsilon = r$.

ii) There exits $v \in X_\varepsilon$ with $||v||_\varepsilon > r$ satisfying $J_\varepsilon(v) < 0 = J_\varepsilon(0)$.

Proof. $i)$: From (A_1), one has that $G_2(\varepsilon x, t) \leq F_2(t)$, and so,

$$J_\varepsilon(u) \geq \frac{1}{2}||u||^2_{H_\varepsilon} + \int_{\mathbb{R}^N} F_1(u) - \int_{\mathbb{R}^N} F_2(u).$$

Gathering (C.3) with (C.6) (note that m can be chosen equal to 2) and using (P_2), there is $r \approx 0^+$ such that

$$J_\varepsilon(u) \geq \frac{1}{2}||u||^2_{H_\varepsilon} + ||u||^2_{F_1} - D||u||^p_\varepsilon \geq C||u||^2_\varepsilon - D||u||^p_\varepsilon,$$

for some C, $D > 0$. The last inequality gives the desired condition, because $p > 2$.

$ii)$: Fix $u \in O_\varepsilon := \{u \in X_\varepsilon;\ |\mathrm{supp}(|u|) \cap \Lambda_\varepsilon| > 0\}$. Note that, for each $x \in \mathbb{R}^N$ we can write

$$F_1(t) = \chi_{\Lambda_\varepsilon}(x)F_1(t) + (1 - \chi_{\Lambda_\varepsilon}(x))F_1(t).$$

Therefore, from the definition of g_2,

$$J_\varepsilon(tu) \leq \frac{t^2}{2}||u||^2_{H_\varepsilon} - \frac{1}{2}\int_{\mathbb{R}^N} \chi_{\Lambda_\varepsilon}|tu|^2 \log|tu|^2 + \frac{1}{2}\int_{[t|u| \leq t_1]} (1 - \chi_{\Lambda_\varepsilon})|tu|^2 \log|tu|^2 + \\ + \int_{[t|u|>t_1]} (1 - \chi_{\Lambda_\varepsilon})[F_1(tu) - \tilde{F}_2(tu)].$$

Recalling that $X_\varepsilon \hookrightarrow L^2(\mathbb{R}^N)$, there is $C > 0$ independent of t such that

$$\int_{[t|u|>t_1]} |tu|^2 \leq C,$$

and so,

$$|[t|u| > t_1]| \leq \frac{C}{t_1^2}t^2 =: C_1 t^2.$$

By the definition of F_1,

$$F_1(t) \leq At^2 + B, \quad t \geq 0,$$

with A, $B > 0$. Then,

$$\int_{[t|u|>t_1]} (1 - \chi_{\Lambda_\varepsilon})F_1(t|u|) \leq Dt^2,$$

for a convenient $D > 0$. Since $\tilde{F}_2 \geq 0$, we find

$$J_\varepsilon(tu) \leq t^2[\ \frac{1}{2}||u||^2_{H_\varepsilon} - \int_{\mathbb{R}^N} \chi_{\Lambda_\varepsilon}|u|^2 \log|u|^2 - \log t \left(\int_{\mathbb{R}^N} \chi_{\Lambda_\varepsilon}|u|^2 + \int_{[t|u| \leq t_1]} (\chi_{\Lambda_\varepsilon} - 1)|u|^2 \right) \\ + \int_{[t|u| \leq t_1]} (1 - \chi_{\Lambda_\varepsilon})|u|^2 \log|u|^2 + D\ ].$$

By the Lebesgue Dominated Convergence Theorem, we have

$$\int_{[t|u|\leq t_1]} (\chi_{\Lambda_\varepsilon} - 1)|u|^2 \longrightarrow 0, \text{ as } t \to +\infty.$$

Note also that, since $u \in O_\varepsilon$ and $u \in L^{F_1}(\mathbb{R})$, it holds

$$\int_{\mathbb{R}^N} \chi_{\Lambda_\varepsilon}|u|^2 > 0$$

and

$$\frac{1}{2}\int_{[t|u|\leq t_1]} (1-\chi_{\Lambda_\varepsilon})|u|^2 \log|u|^2 \leq \int_{\mathbb{R}^N} F_2(u)\,dx < \infty.$$

Combining all of the above information we derive that

$$J_\varepsilon(tu) \to -\infty, \text{ as } t \to \infty,$$

and the proof is finished by taking $v = tu$ with t large enough. ■

For the next lemma, we have adapted the reasoning employed in [12, Lemma 3.1]. However, taking into account that in our case the functional J_ε is on X_ε, which has a different topology of $H^1(\mathbb{R}^N)$, it was necessary to develop new estimates that are not found in [12].

In the sequel, we will need of the following logarithmic inequality (see [50, pg 153])

$$\int_{\mathbb{R}^N} |u|^2 \log\left(\frac{|u|}{||u||_2}\right) \leq C||u||_2 \log\left(\frac{||u||_{2^*}}{||u||_2}\right), \quad \forall u \in L^2(\mathbb{R}^N)\cap L^{2^*}(\mathbb{R}^N),$$

for some positive constant C. As an immediate consequence,

$$\int_{\Lambda_\varepsilon} |u|^2 \log\left(\frac{|u|}{||u||_{L^2(\Lambda_\varepsilon)}}\right) \leq C||u||_{L^2(\Lambda_\varepsilon)} \log\left(\frac{||u||_{L^{2^*}(\Lambda_\varepsilon)}}{||u||_{L^2(\Lambda_\varepsilon)}}\right), \quad \forall u \in L^2(\Lambda_\varepsilon)\cap L^{2^*}(\Lambda_\varepsilon). \tag{2.6}$$

Lemma 2.2 *Let (v_n) be a $(PS)_c$ sequence for J_ε. Then, the sequence (v_n) is bounded in X_ε.*

Proof. Let (v_n) be a $(PS)_c$ sequence for J_ε. Then,

$$J_\varepsilon(v_n) - \frac{1}{2}J'_\varepsilon(v_n)v_n \leq (c+1) + o_n(1)||v_n||_\varepsilon, \tag{2.7}$$

for large n.

On the other hand, observe that

$$J_\varepsilon(v_n) - \frac{1}{2}J'_\varepsilon(v_n)v_n = \int_{\mathbb{R}^N}(F_1(v_n) - \frac{1}{2}F'_1(v_n)v_n) + \int_{\mathbb{R}^N}(\frac{1}{2}G'_2(\varepsilon x, v_n)v_n - G_2(\varepsilon x, v_n)) =$$
$$= \frac{1}{2}\int_{\mathbb{R}^N}|v_n|^2 + \int_{\mathbb{R}^N}[F_2(v_n) - \frac{1}{2}F'_2(v_n)v_n + \frac{1}{2}G'_2(\varepsilon x, v_n)v_n - G_2(\varepsilon x, v_n)], \tag{2.8}$$

because

$$\int_{\mathbb{R}^N}[(F_1(v_n) - \frac{1}{2}F'_1(v_n)v_n) + (\frac{1}{2}F'_2(v_n)v_n - F_2(v_n))] = \frac{1}{2}\int_{\mathbb{R}^N}|v_n|^2.$$

Consequently,

$$J_\varepsilon(v_n) - \frac{1}{2}J'_\varepsilon(v_n)v_n \geq \frac{1}{2}\int_{\Lambda_\varepsilon}|v_n|^2 + \int_{\{\Lambda_\varepsilon^c\cap[|v_n|>t_1]\}}(\frac{1}{2}|v_n|^2 + F_2(v_n) - \frac{1}{2}F'_2(v_n)v_n) +$$
$$+ \int_{\{\Lambda_\varepsilon^c\cap[|v_n|>t_1]\}}(\frac{1}{2}G'_2(\varepsilon x, v_n)v_n - G_2(\varepsilon x, v_n)).$$

From $(A_1) - iv)$,

$$J_\varepsilon(v_n) - \frac{1}{2}J'_\varepsilon(v_n)v_n \geq \frac{1}{2}\int_{\Lambda_\varepsilon}|v_n|^2$$

and so, from (2.7),

$$(c+1) + o_n(1)||v_n||_\varepsilon \geq \frac{1}{2}\int_{\Lambda_\varepsilon}|v_n|^2. \tag{2.9}$$

Recall that there are constants A, $B > 0$ such that

$$F_1(t) \leq A|t|^2 + B, \ \ \forall t \in \mathbb{R}.$$

This together with (2.9) leads to

$$\int_{\Lambda_\varepsilon}F_1(v_n) \leq C_\varepsilon + ||v_n||_\varepsilon, \tag{2.10}$$

for some $C_\varepsilon > 0$. Thanks to (2.6),

$$\frac{1}{2}\int_{\Lambda_\varepsilon}|v_n|^2\log|v_n|^2 \leq C||v_n||_{L^2(\Lambda_\varepsilon)}\log\left(\frac{||v_n||_{L^{2^*}(\Lambda_\varepsilon)}}{||v_n||_{L^2(\Lambda_\varepsilon)}}\right) + ||v_n||^2_{L^2(\Lambda_\varepsilon)}\log(||v_n||_{L^2(\Lambda_\varepsilon)}) =$$
$$= (||v_n||^2_{L^2(\Lambda_\varepsilon)} - C||v_n||_{L^2(\Lambda_\varepsilon)})\log(||v_n||_{L^2(\Lambda_\varepsilon)}) + C||v_n||_{L^2(\Lambda_\varepsilon)}\log\left(||v_n||_{L^{2^*}(\Lambda_\varepsilon)}\right).$$

that combines with the embedding $X_\varepsilon \hookrightarrow H_\varepsilon$ to give

$$\int_{\Lambda_\varepsilon}|v_n|^2\log|v_n|^2 \leq (2||v_n||^2_{L^2(\Lambda_\varepsilon)} - 2C||v_n||_{L^2(\Lambda_\varepsilon)})\log(||v_n||_{L^2(\Lambda_\varepsilon)}) + \tilde{C}||v_n||_\varepsilon\left|\log(\tilde{C}||v_n||_\varepsilon)\right|,$$

for some convenient $\tilde{C} > 0$ independent of ε. In order to get the last inequality, we have explored the fact that the function $t \mapsto \log t$, $t > 0$, is increasing. Now, using the fact that given $r \in (0, 1)$ there is $A > 0$ satisfying

$$|t \log t| \leq A(1 + |t|^{1+r}), \ \ t \geq 0,$$

we obtain, by gathering this inequality with (2.9), the inequalities below

$$||v_n||_{L^2(\Lambda_\varepsilon)} \log(||v_n||_{L^2(\Lambda_\varepsilon)}) \leq A(1 + ||v_n||^{1+r}_{L^2(\Lambda_\varepsilon)})$$

and

$$||v_n||^2_{L^2(\Lambda_\varepsilon)} \log(||v_n||^2_{L^2(\Lambda_\varepsilon)}) \leq A(1 + (||v_n||^2_{L^2(\Lambda_\varepsilon)})^{1+r}) \leq \tilde{A}(1 + ||v_n||_{L^2(\Lambda_\varepsilon)})^{1+r}).$$

From these information, modifying A if necessary, we arrive at

$$\int_{\Lambda_\varepsilon} |v_n|^2 \log |v_n|^2 \leq A(1 + ||v_n||^{1+r}_\varepsilon). \tag{2.11}$$

As (v_n) is a $(PS)_c$ sequence for J_ε,

$$(c+1) \geq J_\varepsilon(v_n) = \frac{1}{2}||v_n||^2_{H_\varepsilon} + \int_{\Lambda^c_\varepsilon} F_1(v_n) - \int_{\Lambda_\varepsilon} |v_n|^2 \log |v_n|^2 - \int_{\Lambda^c_\varepsilon} G_2(\varepsilon x, v_n)$$

for large n. From (A_1),

$$G_2(\varepsilon x, t) \leq \frac{b_0}{2} t^2, \ \ \forall x \in \Lambda^c_\varepsilon,$$

then

$$(c+1) + A(1 + ||v_n||^{1+r}_\varepsilon) \geq C||v_n||^2_{H_\varepsilon} + \int_{\Lambda^c_\varepsilon} F_1(v_n),$$

for some $C > 0$, and so, by (2.10),

$$D_\varepsilon + ||v_n||_\varepsilon + A(1 + ||v_n||^{1+r}_\varepsilon) \geq \tilde{C}\left(||v_n||^2_{H_\varepsilon} + \int_{\mathbb{R}^N} F_1(v_n)\right), \tag{2.12}$$

where $D_\varepsilon := (C_\varepsilon + c + 1) > 0$ and $\tilde{C} := \min\{C, 1\}$. From now on in this proof, we fix $r \in (0, 1)$ so that $1 + r < l$, where l is the number obtained in (C.6).

Suppose that $||v_n||_{F_1} \leq 1$. Employing (C.3) in (2.12), and modifying $\tilde{C}$ if necessary, one gets

$$D_\varepsilon + ||v_n||_\varepsilon + A(1 + ||v_n||^{1+r}_\varepsilon) \geq \tilde{C}(||v_n||_{H_\varepsilon} + ||v_n||_{F_1})^2 = \tilde{C}||v_n||^2_\varepsilon. \tag{2.13}$$

Otherwise, if $||v_n||_{F_1} > 1$, we have two possibilities: $||v_n||_{H_\varepsilon} > 1$ or $||v_n||_{H_\varepsilon} \leq 1$. When $||v_n||_{H_\varepsilon} > 1$, in the same way of the preceding case we obtain

$$D_\varepsilon + ||v_n||_\varepsilon + A(1 + ||v_n||_\varepsilon^{1+r}) \geq C_l ||v_n||_\varepsilon^l. \tag{2.14}$$

If it occurs $||v_n||_{F_1} > 1$ and $||v_n||_{H_\varepsilon} \leq 1$, using the definition $|| \cdot ||_\varepsilon$ in (2.12), we find

$$\tilde{D}_\varepsilon + ||v_n||_{F_1} + C_r ||v_n||_{F_1}^{1+r} \geq \tilde{C} ||v_n||_{F_1}^l. \tag{2.15}$$

The proof is completed by combining (2.13)-(2.15). ■

Next, we present an important property of the (PS) sequences whose the proof can be found in [11] and that is a crucial tool in order to prove that J_ε satisfies the (PS) condition in the space X_ε.

Lemma 2.3 *Let (v_n) be a $(PS)_c$ sequence for J_ε. Then, given $\tau > 0$ there is $R > 0$ such that*

$$\limsup_{n\to\infty} \int_{B_R^c(0)} (|\nabla v_n|^2 + (V(\varepsilon x) + 1)|v_n|^2) < \tau.$$

Proof. See [11, Lemma 3.4] or [56, Lemma 3.3] for a similar result. ■

Corollary 2.1 *The functional J_ε satisfies the (PS) condition.*

Proof. Let (v_n) be a $(PS)_c$ sequence for J_ε. Without loss of generality we may assume that $v_n \rightharpoonup v$ in X_ε for some $v \in X_\varepsilon$. Moreover, arguing as in [5, Section 2], we also have $J_\varepsilon'(v) = 0$, and so, $J_\varepsilon'(v)v = 0$, i.e.,

$$||v||_{H_\varepsilon}^2 + \int_{\mathbb{R}^N} F_1'(v)v = \int_{\mathbb{R}^N} G_2'(\varepsilon x, v)v. \tag{2.16}$$

As the embedding $X_\varepsilon \hookrightarrow L^q(B_R(0))$ is compact for each $R > 0$ and $p \in [2, 2^*)$, the growth condition on G_2' (see (A_1)) together with the Lemma 2.3 yields

$$\int_{\mathbb{R}^N} G_2'(\varepsilon x, v_n)v_n \longrightarrow \int_{\mathbb{R}^N} G_2'(\varepsilon x, v)v.$$

Taking into account this information and using the fact that (v_n) is (PS) sequence, we find

$$||v_n||_{H_\varepsilon}^2 + \int_{\mathbb{R}^N} F_1'(v_n)v_n = \int_{\mathbb{R}^N} G_2'(\varepsilon x, v_n)v_n + o_n(1).$$

The last equality combined with (2.16) implies that

$$||v_n||_{H_\varepsilon}^2 + \int_{\mathbb{R}^N} F_1'(v_n)v_n = ||v||_{H_\varepsilon}^2 + \int_{\mathbb{R}^N} F_1'(v)v + o_n(1),$$

from where it follows that

$$||v_n||^2_{H_\varepsilon} \to ||v||^2_{H_\varepsilon} \tag{2.17}$$

and

$$\int_{\mathbb{R}^N} F_1'(v_n)v_n \longrightarrow \int_{\mathbb{R}^N} F_1'(v)v, \tag{2.18}$$

and so, $v_n \to v$ in H_ε. It remains to show that $v_n \to v$ in $L^{F_1}(\mathbb{R}^N)$. Note that, since $F_1'(t)t \geq 0$, the convergence in (2.18) means that

$$F_1'(v_n)v_n \to F_1'(v)v \quad \text{in} \quad L^1(\mathbb{R}^N).$$

This fact associated with (C.6) and Lebesgue's Dominated Convergence Theorem shows that, going to a subsequence if necessary,

$$F_1(v_n) \to F_1(v) \quad \text{in} \quad L^1(\mathbb{R}^N).$$

Finally, using that $F_1 \in (\Delta_2)$, we deduce that

$$\int_{\mathbb{R}^N} F_1(|v_n - v|) \longrightarrow 0,$$

showing that $v_n \to v$ in $L^{F_1}(\mathbb{R}^N)$, which finishes the proof. ■

The main result of this section reads as follows

Theorem 2.2 *For each $\varepsilon > 0$ the functional J_ε has a nontrivial critical point u_ε. Consequently, $(\tilde{S}_\varepsilon)$ has a nontrivial solution.*

Proof. By Lemma 2.1 and Corollary 2.1, we see that the functional J_ε satisfies the assumptions of the Mountain Pass Theorem found in [19, Theorem 2.1], then the mountain pass level given by

$$c_\varepsilon := \inf_{\gamma \in \Gamma_\varepsilon} \max_{t \in [0,1]} J_\varepsilon(\gamma(t))$$

with

$$\Gamma_\varepsilon := \{\gamma \in C([0,1], X_\varepsilon);\ \gamma(0) = 0 \ \text{ and } \ J_\varepsilon(\gamma(1)) < 0\},$$

is a critical point of J_ε. ■

From now on, otherwise mentioned, the notation u_ε designates the solution of $(\tilde{S}_\varepsilon)$ given in the preceding theorem.

2.3 The Nehari manifold and the existence of positive solution for (P_ε)

In this section we will prove that the Nehari set associated with J_ε, namely

$$\mathcal{N}_\varepsilon := \{u \in X_\varepsilon - \{0\};\ J'_\varepsilon(u)u = 0\},$$

is a C^1-manifold and that critical points of $J_\varepsilon|_{\mathcal{N}_\varepsilon}$ are critical points of J_ε in the usual sense. Furthermore, by studying the behavior of levels c_ε as $\varepsilon \to 0^+$, we will prove some properties related with $\mathcal{N}_\varepsilon$ that allows us to prove that the solutions u_ε of $(\tilde{S}_\varepsilon)$ are solutions of (S_ε) for $\varepsilon \approx 0^+$.

2.3.1 Main properties of $\mathcal{N}_\varepsilon$

First of all, set

$$\Psi_\varepsilon(u) := J_\varepsilon(u) - \frac{1}{2}\int_{\mathbb{R}^N} |u|^2 - \left[\int_{\mathbb{R}^N} [F_2(u) - \frac{1}{2}F'_2(u)u + \frac{1}{2}G'_2(\varepsilon x, u)u - G_2(\varepsilon x, u)]\right].$$

Accordingly to (2.8),

$$\mathcal{N}_\varepsilon = \Psi_\varepsilon^{-1}(\{0\}).$$

We start our study with the following result

Proposition 2.2 *There exists $\beta > 0$, such that*

$$||u||_\varepsilon \geq ||u||_{H_\varepsilon} \geq \beta, \ \ \forall u \in \mathcal{N}_\varepsilon,$$

for all $\varepsilon > 0$.

Proof. For each $u \in \mathcal{N}_\varepsilon$,

$$\int_{\mathbb{R}^N} (|\nabla u|^2 + (V(\varepsilon x) + 1)|u|^2) + \int_{\mathbb{R}^N} F'_1(u)u = \int_{\mathbb{R}^N} G'_2(\varepsilon, u)u.$$

Therefore, from (A_1),

$$\int_{\mathbb{R}^N} (|\nabla u|^2 + (\alpha_0 + 1 - b_0)|u|^2) \leq C\int_{\mathbb{R}^N} |u|^p, \tag{2.19}$$

where $\alpha_0 = \inf_{\mathbb{R}^N} V$. The number b_0 has been chosen so that $\alpha_0 + 1 - b_0 > 0$, then the expression

$$||u||_0^2 := \int_{\mathbb{R}^N} (|\nabla u|^2 + (\alpha_0 + 1 - b_0)|u|^2)$$

defines a norm on $H^1(\mathbb{R}^N)$. Setting $H = (H^1(\mathbb{R}^N), ||\cdot||_0)$, one sees that the embedding $H \hookrightarrow L^p(\mathbb{R}^N)$ is continuous. From (2.19),

$$M \leq ||u||_0^{p-2},$$

for a convenient $M > 0$ that is independent of ε. The last inequality yields

$$0 < \beta := M^{\frac{1}{(p-2)}} \leq ||u||_0 \leq ||u||_{H_\varepsilon} \leq ||u||_\varepsilon.$$

■

For the sake of completeness, we would like to mention that repeating the ideas found in [11, Lemma 3.6 and Remark 3.1], it can be proved the following lemma

Lemma 2.4 *For each* $u \in O_\varepsilon = \{u \in X_\varepsilon; |supp(|u|) \cap \Lambda_\varepsilon| > 0\}$, *there is a unique* $t_u > 0$ *such that* $t_u u \in \mathcal{N}_\varepsilon$. *Reciprocally, if* $u \in \mathcal{N}_\varepsilon$, *then* $u \in O_\varepsilon$.

In the next proposition we prove that $\mathcal{N}_\varepsilon$ is a C^1-manifold for each $\varepsilon > 0$.

Proposition 2.3 $\mathcal{N}_\varepsilon$ *is a* C^1*-manifold for each* $\varepsilon > 0$.

Proof. In the sequel we will prove that for all $u \in \mathcal{N}_\varepsilon$ we must have $\Psi'_\varepsilon(u)u \neq 0$. Assume by contradiction that there is $u \in \mathcal{N}_\varepsilon$ with $\Psi'_\varepsilon(u)u = 0$, i.e.,

$$0 = -\int_{\mathbb{R}^N} |u|^2 - \left[\int_{\mathbb{R}^N} (\frac{1}{2}F_2'(u)u - \frac{1}{2}F_2''(u)u^2) + \int_{\mathbb{R}^N} (\frac{1}{2}G_2''(\varepsilon x, u)u^2 - \frac{1}{2}G_2'(\varepsilon x, u)u)\right].$$

Using that $G_2' \equiv F_2'$ in Λ_ε, we find

$$0 = -\int_{\Lambda_\varepsilon} |u|^2 - \left[\int_{\Lambda_\varepsilon^c} (|u|^2 + \frac{1}{2}F_2'(u)u - \frac{1}{2}F_2''(u)u^2) + \int_{\Lambda_\varepsilon^c} (\frac{1}{2}G_2''(\varepsilon x, u)u^2 - \frac{1}{2}G_2'(\varepsilon x, u)u)\right]. \tag{2.20}$$

By the definition of F_2,

$$F_2'(s) := \begin{cases} 0, & s \in [0, \delta]; \\ s \log\left(\frac{s^2}{\delta^2}\right) + 2\delta - 2s, & |s| \geq \delta, \end{cases}$$

and so,

$$t^2 + \frac{1}{2}F_2'(t)t - \frac{1}{2}F_2''(t)t^2 = \delta t > 0, \quad t \geq \delta,$$

leading to

$$|u|^2 + \frac{1}{2}F_2'(u)u - \frac{1}{2}F_2''(u)u^2 \geq 0, \quad \text{a.e } x \in \Lambda_\varepsilon^c.$$

Using this information and the fact that $G_2'(\varepsilon x,t) \equiv F_2'(t)$, for $x \in \Lambda_\varepsilon^c$ and $t \leq t_1$ in (2.20), we arrive at

$$\int_{\Lambda_\varepsilon} |u|^2 \leq -\int_{\Lambda_\varepsilon^c \cap [t_1<|u|<t_2]} (\frac{1}{2}G_2''(\varepsilon x,u)u^2 - \frac{1}{2}G_2'(\varepsilon x,u)u) - \int_{\Lambda_\varepsilon^c \cap [|u|\geq t_2]} (\frac{1}{2}G_2''(\varepsilon x,u)u^2 - \frac{1}{2}G_2'(\varepsilon x,u)u).$$

As $G_2'(\varepsilon x,u) = h(u)$ for $x \in \Lambda_\varepsilon^c$ and $u(x) \in (t_1,t_2)$, (h_3) gives

$$G_2''(\varepsilon x,u)u^2 - \frac{1}{2}G_2'(\varepsilon x,u)u = \frac{1}{2}(h'(u)u - h(u))u \geq 0, \text{ a.e } x \in \Lambda_\varepsilon^c \cap [t_1 < |u| < t_2].$$

Note also that, by the definition of $\overline{F}_2'$,

$$G_2''(\varepsilon x,u)u^2 - \frac{1}{2}G_2'(\varepsilon x,u)u = 0, \text{ a.e } x \in \Lambda_\varepsilon^c \cap [|u| \geq t_2].$$

Gathering the above information, we derive that $u = 0$, a.e. $x \in \Lambda_\varepsilon$. Hence, inasmuch as $u \in \mathcal{N}_\varepsilon$, we get

$$||u||_{H_\varepsilon}^2 + \int_{\mathbb{R}^N} F_1'(u)u = \int_{\Lambda_\varepsilon^c} G_2'(\varepsilon x,u)u \leq b_0 \int_{\mathbb{R}^N} |u|^2$$

that leads to $u \equiv 0$, which is absurd because $u \in \mathcal{N}_\varepsilon$, showing the desired result. ■

In view of the last proposition, we can establish the notion of critical point for $J_\varepsilon|_{\mathcal{N}_\varepsilon}$. Recall that $u \in \mathcal{N}_\varepsilon$ is a critical point of J_ε constrained to $\mathcal{N}_\varepsilon$ when

$$||J_\varepsilon'(u)||_* := \min_{\lambda \in \mathbb{R}} ||J_\varepsilon'(u) - \lambda \Psi_\varepsilon'(u)|| = 0. \quad \text{(See [83, Proposition 5.2])}$$

By a $(PS)_c$ sequence associated with $J_\varepsilon|_{\mathcal{N}_\varepsilon}$, we mean a sequence (u_n) in $\mathcal{N}_\varepsilon$ such that

$$J(u_n) \to c \text{ and } ||J'(u_n)||_* \to 0.$$

From now on, we say that $J_\varepsilon|_{\mathcal{N}_\varepsilon}$ satisfies the (PS) condition when each $(PS)_c$ sequence for $J_\varepsilon|_{\mathcal{N}_\varepsilon}$ has a convergent subsequence, for any $c \in \mathbb{R}$.

The next proposition relates critical points of $J_\varepsilon|_{\mathcal{N}_\varepsilon}$ with critical points of J_ε in X_ε.

Proposition 2.4 *Let $u \in \mathcal{N}_\varepsilon$ be a critical point of J_ε constrained to $\mathcal{N}_\varepsilon$. Then u is a critical point of J_ε on X_ε.*

Proof. If $u \in \mathcal{N}_\varepsilon$ is a critical point of $J_\varepsilon|_{\mathcal{N}_\varepsilon}$, then

$$J_\varepsilon'(u) = \lambda \Psi' \varepsilon(u),$$

for some $\lambda \in \mathbb{R}$. Consequently,

$$0 = J'_\varepsilon(u)u = \lambda \Psi'_\varepsilon(u)u.$$

Since $u \in \mathcal{N}_\varepsilon$, the arguments explored in the proof of Proposition 2.3 yields $\Psi'_\varepsilon(u)u \neq 0$. Hence, the above equality guarantees that $\lambda = 0$ and the proof is over. ■

We finish this subsection by proving that $J_\varepsilon|_{\mathcal{N}_\varepsilon}$ satisfies the (PS) condition.

Proposition 2.5 *$J_\varepsilon|_{\mathcal{N}_\varepsilon}$ satisfies the (PS) condition.*

Proof. Let (u_n) be an arbitrary $(PS)_c$ sequence for $J_\varepsilon|_{\mathcal{N}_\varepsilon}$. Then,

$$J_\varepsilon(u_n) \to c \quad \text{and} \quad J'_\varepsilon(u_n) = \lambda_n \Psi'_\varepsilon(u_n) + o_n(1),$$

for some sequence of real numbers (λ_n). Taking into account that $J_\varepsilon(u_n) \to c$ and $J'_\varepsilon(u_n)u_n = 0$, repeating the same reasoning of the proof of Lemma 2.2, one has that (u_n) is a bounded sequence. By Corollary 2.1, it suffices to show that (u_n) is a $(PS)_c$ sequence for J_ε. Aiming this fact, we will prove that

$$\lambda_n \to 0. \tag{2.21}$$

Note that (u_n) satisfies

$$0 = J'_\varepsilon(u_n)u_n = \lambda_n \Psi'_\varepsilon(u_n)u_n + o_n(1).$$

Arguing as in the proof of Proposition 2.3, it is possible to show that if $|\Psi'_\varepsilon(u_n)u_n| = o_n(1)$, then

$$\int_{\Lambda_\varepsilon} |u_n|^2 \leq o_n(1) \Rightarrow \int_{\Lambda_\varepsilon} |u_n|^2 = o_n(1).$$

This combined with the boundedness of (u_n) leads to

$$\int_{\Lambda_\varepsilon} |u_n|^p = o_n(1).$$

Consequently,

$$||u_n||^2_{H_\varepsilon} + \int_{\mathbb{R}^N} F'_1(u_n)u_n = \int_{\Lambda_\varepsilon} F'_2(u_n)u_n + \int_{\Lambda^c_\varepsilon} G'_2(\varepsilon x, u_n) \leq o_n(1) + b_0 \int_{\mathbb{R}^N} |u_n|^2,$$

which combines with (C.6) to give

$$\int_{\mathbb{R}^N} (|\nabla u_n|^2 + (V(\varepsilon x) + 1)|u_n|^2) + \int_{\mathbb{R}^N} F_1(u_n) \leq o_n(1).$$

The above inequality implies that $u_n \to 0$ in X_ε, which contradicts Proposition 2.3. Thereby, (2.21) is true and the proof is completed. ■

2.3.2 Existence of positive solution for (P_ε)

For the goals of this section, we will consider the following autonomous problem

$$(P_0) \qquad \begin{cases} -\Delta u + V_0 u = u \log u^2, & \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N) \cap L^{F_1}(\mathbb{R}^N). \end{cases}$$

The energy functional related to the (P_0) is given by

$$J_0(u) := \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + (V_0+1)|u|^2) + \int_{\mathbb{R}^N} F_1(u) - \int_{\mathbb{R}^N} F_2(u).$$

It is well known (see [10, 11, 79]) that (P_0) has a positive ground state solution u_0, which satisfies

$$c_0 := \inf_{u \in \mathcal{N}_0} J_0(u) = J_0(u_0),$$

where $\mathcal{N}_0$ is the Nehari set associated with J_0, i.e.,

$$\mathcal{N}_0 := \left\{ u \in H^1(\mathbb{R}^N) \cap L^{F_1}(\mathbb{R}^N);\ J_0(u) = \frac{1}{2}\int_{\mathbb{R}^N} |u|^2 \right\}.$$

Hereafter, we fix

$$X = \left(H^1(\mathbb{R}^N) \cap L^{F_1}(\mathbb{R}^N),\ (||\cdot||_{H^1(\mathbb{R}^N)} + ||\cdot||_{L^{F_1}(\mathbb{R}^N)}) \right), \tag{2.22}$$

where $||\cdot||_{H^1(\mathbb{R}^N)}$ denotes the usual norm in $H^1(\mathbb{R}^N)$.

The level c_0 can be characterized by

$$c_0 = \inf_{u \in \mathcal{N}_0} J_0(u) = \inf_{u \in (X - \{0\})} \max_{t \geq 0} J_0(tu).$$

In the next lemma we prove that the solution u_ε obtained in Theorem 2.2 is a ground state solution of $(\tilde{S}_\varepsilon)$, and we study the behavior of levels c_ε, as $\varepsilon \to 0^+$. By a ground state solution we mean a solution of least energy of $(\tilde{S}_\varepsilon)$, that is, a solution verifying

$$\inf_{u \in \mathcal{N}_\varepsilon} J_\varepsilon(u) = J_\varepsilon(u_\varepsilon).$$

Lemma 2.5 *The following properties hold:*

i) *There is $\gamma_0 > 0$ such that $c_\varepsilon \geq \gamma_0$ for all $\varepsilon > 0$.*

ii) $c_\varepsilon = \inf_{u \in \mathcal{N}_\varepsilon} J_\varepsilon(u)$ *for all $\varepsilon > 0$.*

iii) $\limsup_{\varepsilon \to 0} c_\varepsilon \leq c_0$.

Proof. *i*) Note that

$$J_\varepsilon(u) \geq \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u|^2 + (\alpha_0+1)|u|^2) + \int_{\mathbb{R}^N} F_1(u) - \int_{\mathbb{R}^N} F_2(u),$$

with $\alpha_0 = \inf_{\mathbb{R}^N} V$. Arguing as Lemma 2.1-*i*), we find $r_0 \approx 0^+$ and $\gamma_0 > 0$ independent of ε such that

$$J_\varepsilon(u) \geq \rho_0, \quad \forall u \in X_\varepsilon, ||u||_\varepsilon = r_0.$$

By the definition of c_ε, we derive $c_\varepsilon \geq \gamma_0$.

ii) By Lemma 2.4 we know that $u \in O_\varepsilon$ for each $u \in \mathcal{N}_\varepsilon$. In this way, using the same ideas of Theorem 2.1-*ii*), there is t_0 such that $J_\varepsilon(t_0u) < 0$. Setting $\eta : [0,1] \longrightarrow X_\varepsilon$ given by $\eta(t) := t(t_0u)$, it follows that $\eta \in \Gamma_\varepsilon$, and so,

$$c_\varepsilon \leq \max_{t\in[0,1]} J_\varepsilon(\eta(t)) \leq \max_{s\geq 0} J_\varepsilon(su) \leq J_\varepsilon(u).$$

The above inequality shows that

$$c_\varepsilon \leq \inf_{u\in\mathcal{N}_\varepsilon} J_\varepsilon(u).$$

The reverse inequality follows by observing that

$$\inf_{u\in\mathcal{N}_\varepsilon} J_\varepsilon(u) \leq J_\varepsilon(u_\varepsilon) = c_\varepsilon.$$

iii): Let $u_0 \in \mathcal{N}_0$ be a positive ground state solution of (P_0), i.e,

$$J_0(u_0) = c_0 \quad \text{and} \quad J_0'(u_0) = 0.$$

For each $R > 0$, set $\phi_R(x) := \phi(\frac{1}{R}x)$, where $\phi \in C_0^\infty(\mathbb{R}^N)$ is such that $\phi(x) = 1$, for $x \in B_1(0)$, and $\phi(x) = 0$, for $x \in B_2^c(0)$. Then, putting $u_R := \phi_R u_0$, it is easy to check that

$$u_R \to u_0 \text{ in } H^1(\mathbb{R}^N) \quad \text{as} \quad R \to \infty.$$

Since $0 \leq u_R \leq u_0$, the Lebesgue Dominated Convergence Theorem ensures that

$$\int_{\mathbb{R}^N} F_1(u_R) \longrightarrow \int_{\mathbb{R}^N} F_1(u_0), \text{ as } R \to \infty.$$

By the last two limits we can infer that $u_R \to u_0$ in X.

Given $R > 0$, from the definition of u_R, one can see that $u_R \in O_\varepsilon$ for each $\varepsilon > 0$, since $u_0 > 0$ and $0 \in \Lambda_\varepsilon$. So, thanks to preceding item, we find $t_\varepsilon > 0$ in such way that

$$c_\varepsilon \leq \max_{t\geq 0} J_\varepsilon(tu_R) = J_\varepsilon(t_\varepsilon u_R).$$

Our next step is to show that, for some $\varepsilon_0 > 0$, the family $(t_\varepsilon)_{0<\varepsilon<\varepsilon_0}$ is bounded. In fact, as $t_\varepsilon u_R \in \mathcal{N}_\varepsilon$,

$$\int_{\mathbb{R}^N}(|\nabla u_R|^2+(V(\varepsilon x)+1)|u_R|^2) = \frac{1}{t_\varepsilon}\int_{\Lambda_\varepsilon} F_2'(t_\varepsilon u_R)u_R+\frac{1}{t_\varepsilon}\int_{\Lambda_\varepsilon^c} \tilde{F}_2'(t_\varepsilon u_R)u_R-\frac{1}{t_\varepsilon}\int_{\mathbb{R}^N} F_1'(t_\varepsilon u_R)u_R.$$

Considering that $u_R \equiv 0$ in $B^c_{2R}(0)$ and $V(\varepsilon x) \to V(0) = V_0$, we have

$$\int_{\mathbb{R}^N}(|\nabla u_R|^2 + (V(\varepsilon x) + 1)|u_R|^2) \longrightarrow \int_{\mathbb{R}^N}(|\nabla u_R|^2 + (V_0 + 1)|u_R|^2),$$

as $\varepsilon \to 0$, for each $R > 0$. On the other hand, if $t_\varepsilon \to \infty$ as $\varepsilon \to 0$, the following claim holds:

Claim 2.1

$$\left(\frac{1}{t_\varepsilon}\int_{\Lambda_\varepsilon} F_2'(t_\varepsilon u_R)u_R + \frac{1}{t_\varepsilon}\int_{\Lambda_\varepsilon^c} \tilde{F}_2'(t_\varepsilon u_R)u_R - \frac{1}{t_\varepsilon}\int_{\mathbb{R}^N} F_1'(t_\varepsilon u_R)u_R\right) \longrightarrow \infty.$$

First of all, the limit $\chi_{\Lambda_\varepsilon}(x) \to 1$ as $\varepsilon \to 0^+$ together with (A_1) guarantees that

$$\frac{1}{t_\varepsilon}\int_{\Lambda_\varepsilon^c} \tilde{F}_2'(t_\varepsilon u_R)u_R = o_\varepsilon(1).$$

Thereby, in order to get the Claim 2.1, it suffices to show that

$$A_\varepsilon := \left(\frac{1}{t_\varepsilon}\int_{\Lambda_\varepsilon} F_2'(t_\varepsilon u_R)u_R - \frac{1}{t_\varepsilon}\int_{\mathbb{R}^N} F_1'(t_\varepsilon u_R)u_R\right) \longrightarrow \infty.$$

Observe that, by (2.3),

$$\begin{aligned} A_\varepsilon =& \int_{\mathbb{R}^N} |u_R|^2 + \int_{\mathbb{R}^N} |u_R|^2 \log(t_\varepsilon|u_R|)^2 - \frac{1}{t_\varepsilon}\int_{\Lambda_\varepsilon^c} F_2'(t_\varepsilon u_R)u_R = \\ =& \log(t_\varepsilon)^2\int_{\mathbb{R}^N} |u_R|^2 - \frac{1}{t_\varepsilon}\int_{\Lambda_\varepsilon^c} F_2'(t_\varepsilon u_R)u_R + C_R, \end{aligned}$$

with $C_R = \displaystyle\int_{\mathbb{R}^N}(|u_R|^2 + |u_R|^2 \log|u_R|^2)$. From the definition of F_2,

$$\frac{1}{t_\varepsilon}F_2'(t_\varepsilon u_R)u_R = u_R^2 \log(t_\varepsilon|u_R|)^2 - \log\delta^2 u_R^2 + \frac{2\delta}{t_\varepsilon}u_R - 2u_R^2,$$

and so,

$$\frac{1}{t_\varepsilon}\int_{\Lambda_\varepsilon^c} F_2'(t_\varepsilon u_R)u_R \le \int_{\Lambda_\varepsilon^c} u_R^2 \log(t_\varepsilon|u_R|)^2 + \frac{2\delta}{t_\varepsilon}\int_{\mathbb{R}^N} u_R + B_R,$$

with $B_R := -\log\delta^2\displaystyle\int_{\mathbb{R}^N} u_R^2$. From this and using that $t_\varepsilon \to \infty$ as $\varepsilon \to 0$, one finds

$$A_\varepsilon \ge \log(t_\varepsilon)^2\int_{\mathbb{R}^N} |u_R|^2 - \int_{\Lambda_\varepsilon^c} u_R^2 \log(t_\varepsilon|u_R|)^2 + o_\varepsilon(1) + D_R,$$

where $D_R = C_R - B_R$. Therefore,

$$A_\varepsilon \geq \log(t_\varepsilon)^2 \int_{\Lambda_\varepsilon} |u_R|^2 - \int_{\Lambda_\varepsilon^c} u_R^2 \log |u_R|^2 + o_\varepsilon(1) + D_R,$$

from where it follows that

$$A_\varepsilon \to \infty \ \text{ as } \ \varepsilon \to 0^+,$$

showing the Claim 2.1.

As a byproduct of the Claim 2.1, we get that $(t_\varepsilon)_{0<\varepsilon<\varepsilon_0}$ is bounded, for some $\varepsilon_0 > 0$. Now, take $t_R > 0$ such that $J_0(t_R u_R) = \max_{t\geq 0} J_0(tu_R)$. Note that

$$J_\varepsilon(t_\varepsilon u_R) - J_0(t_\varepsilon u_R) = \frac{t_\varepsilon^2}{2} \int_{\mathbb{R}^N} (V(\varepsilon x) - V_0)|u_R|^2 + \int_{\Lambda_\varepsilon^c} (F_2(t_\varepsilon u_R) - \tilde{F}_2(t_\varepsilon u_R)).$$

Using that u_R has compact support, $u_R \to u_0$ in X as $R \to \infty$ and the Lebesgue's Dominated Convergence Theorem, we arrive at

$$J_\varepsilon(t_\varepsilon u_R) - J_0(t_\varepsilon u_R) = o_\varepsilon(1),$$

$$\limsup_{\varepsilon\to 0} c_\varepsilon \leq \limsup_{\varepsilon\to 0} J_\varepsilon(t_\varepsilon u_R) \leq J_0(t_R u_R). \tag{2.23}$$

The choose of t_R gives $t_R \to 1$ (see [11, Lemma 3.7]), and then,

$$J_0(t_R u_R) \to J_0(u_0) = c_0, \ \text{ as } \ R \to \infty.$$

The result is a direct consequence of the limit above and (2.23). ■

Now, we are ready to prove the existence of positive ground state solution for $(\tilde{S}_\varepsilon)$.

Proposition 2.6 *Given $\varepsilon > 0$ the problem $(\tilde{S}_\varepsilon)$ has a positive ground state solution.*

Proof. Let u_ε be the solution of $(\tilde{S}_\varepsilon)$ given in Theorem 2.2. For $v \in X_\varepsilon$, set $v^+ := \max\{v, 0\}$ and $v^- := \max\{0, -v\}$. Therefore, either $u_\varepsilon^+ = 0$ or $u_\varepsilon^- = 0$, otherwise we would have u_ε^+, $u_\varepsilon^- \in \mathcal{N}_\varepsilon$ and $J_\varepsilon(u_\varepsilon) = J_\varepsilon(u_\varepsilon^+) + J_\varepsilon(u_\varepsilon^-) \geq 2c_\varepsilon$, which contradicts $J_\varepsilon(u_\varepsilon) = c_\varepsilon$. Thereby, since g is odd, we may assume that u_ε is a nonnegative solution of $(\tilde{S}_\varepsilon)$. By an analogous reasoning as used in the proof of [11, Theorem 3.1] and [44, Section 3.1], using a suitable version of maximum principle ([82, Theorem 1]), we deduce that u_ε is positive in whole $\mathbb{R}^N$. ■

Our next result improves [11, Lemma 3.9] and it is an essential step in order to get a solution for (S_ε).

Lemma 2.6 *Let (u_n) be a nonnegative sequence with $u_n \in X_{\varepsilon_n}$, $J_{\varepsilon_n}(u_n) = c_{\varepsilon_n}$, $J'_{\varepsilon_n}(u_n) = 0$ and $\varepsilon_n \to 0$. Then, there exits a sequence $(y_n) \subset \mathbb{R}^N$ such that $w_n(x) := u_n(x + y_n)$ has a convergent subsequence, $\sup_{n\in\mathbb{N}} ||w_n||_\infty < \infty$ and*

$$w_n(x) \to 0 \quad as \quad |x| \to \infty \quad uniformly\ in \quad n \in \mathbb{N}. \tag{2.24}$$

Furthermore, for some $y_0 \in \Lambda$, the following limit holds $\lim\limits_{n\to+\infty}(\varepsilon_n y_n) = y_0$.

Proof. To begin with, note that (u_n) is a bounded sequence in the space X given in (2.22). Indeed, by the assumptions and employing Lemma 2.5-*iii*), (u_n) must satisfy

$$J_{\varepsilon_n}(u_n) \le M_1 \ \text{ and } \ J'_{\varepsilon_n}(u_n)u_n = 0, \quad \forall n \in \mathbb{N},$$

for some positive M_1. By following closely the arguments of Lemma 2.2, we find, instead of (2.9),

$$M_1 \ge \frac{1}{2}\int_{\Lambda_{\varepsilon_n}} |u_n|^2.$$

Hence, by the same ideas explored in the proof of Lemma 2.2, there are a $M_1, M_2 > 0$ such that

$$\int_{\Lambda_{\varepsilon_n}} |u_n|^2 \log |u_n|^2 \le M_2(1 + ||v_n||^{1+r}_{H_{\varepsilon_n}})$$

and,

$$M_1 + M_2(1 + ||v_n||^{1+r}_{H_{\varepsilon_n}}) \ge C||u_n||^2_{H_{\varepsilon_n}} + \int_{\Lambda^c_\varepsilon} F_1(u_n) \ge C||u_n||^2_{H_{\varepsilon_n}}, \quad \forall n \in \mathbb{N},$$

for some $C > 0$ and $0 < r < 1$, which shows the boundedness of $(||u_n||_{H_{\varepsilon_n}})$ in $\mathbb{R}$. Now, the conditions on V ensure that (u_n) is bounded in $H^1(\mathbb{R}^N)$. Since

$$\int_{\mathbb{R}^N} F_1(u_n) = J_{\varepsilon_n}(u_n) - \frac{1}{2}||u_n||^2_{H_{\varepsilon_n}} + \int_{\mathbb{R}^N} G_2(\varepsilon_n x, u_n),$$

we infer that

$$\sup_{n\in\mathbb{N}} \int_{\mathbb{R}^N} F_1(u_n) < \infty,$$

proving the boundedness of (u_n) in X. For some $r, \lambda > 0$ and a sequence (y_n) it holds

$$\limsup_{n\to+\infty} \int_{B_r(y_n)} |u_n|^2 \ge \lambda > 0. \tag{2.25}$$

Otherwise, using a concentration-compactness principle due to Lions ([83, Lemma 1.21]), we would have

$$u_n \to 0 \ \text{ in } \ L^p(\mathbb{R}^N) \quad \forall p \in (2, 2^*),$$

then

$$\int_{\mathbb{R}^N} G_2'(\varepsilon_n x, u_n)u_n = o_n(1) \text{ and } \int_{\mathbb{R}^N} G_2(\varepsilon_n x, u_n) = o_n(1).$$

From the assumptions in the statement we get $J'_{\varepsilon_n}(u_n)u_n = 0$. This associated with the last equality give

$$o_n(1) = ||u_n||^2_{H_{\varepsilon_n}} + \int_{\mathbb{R}^N} F_1'(u_n)u_n.$$

The above limit together with (C.6) ensures that

$$||u_n||^2_{H_{\varepsilon_n}} + \int_{\mathbb{R}^N} F_1(u_n) \to 0,$$

which permits to conclude that $J_{\varepsilon_n}(u_n) = c_{\varepsilon_n} \to 0$, contradicting Lemma 2.5-i).

From now on, set $w_n := u_n(\cdot + y_n)$. The boundedness of (u_n) and (2.25) yield that (w_n) is a bounded sequence in X, and so, we may assume that there is $w \in X - \{0\}$ such that

$$w_n \rightharpoonup w \text{ in } X.$$

Our next step is proving that $(\varepsilon_n y_n)$ is a bounded sequence in $\mathbb{R}^N$. This fact is a direct consequence of the claim below.

Claim 2.2 *It holds* $\lim_{n\to+\infty} d(\varepsilon_n y_n, \overline{\Lambda}) = 0$, *with d being the usual distance between $\varepsilon_n y_n$ and $\overline{\Lambda}$ in $\mathbb{R}^N$.*

The proof of the claim follows the same ideas of [11, Claim 3.1], however for the reader's convenience we will write its proof. Arguing by contradiction, if the claim is not true, there exist some subsequence of $(\varepsilon_n y_n)$, still denoted by itself, and $\gamma > 0$ satisfying

$$d(\varepsilon_n y_n, \overline{\Lambda}) \geq \gamma, \quad \forall n \in \mathbb{N}.$$

Then, for some $r > 0$,

$$B_r(\varepsilon_n y_n) \subset \Lambda^c, \quad \forall n \in \mathbb{N}.$$

Now, for each $j \in \mathbb{N}$, we fix $v_j = \phi_j w$, with ϕ_j defined as in Lemma 2.5-iii). So, we know that $v_j \to w$ in X. For each j fixed, a simple change of variable leads to

$$\int_{\mathbb{R}^N} (\nabla w_n \nabla v_j + (V(\varepsilon_n x + \varepsilon_n y_n))w_n v_j) + \int_{\mathbb{R}^N} F_1'(w_n)v_j = \int_{\mathbb{R}^N} G_2'(\varepsilon_n x, w_n)v_j. \tag{2.26}$$

Writing

$$\int_{\mathbb{R}^N} G_2'(\varepsilon_n x, w_n)v_j = \int_{B_{\frac{r}{\varepsilon_n}}(0)} G_2'(\varepsilon_n x, w_n)v_j + \int_{B^c_{\frac{r}{\varepsilon_n}}(0)} G_2'(\varepsilon_n x, w_n)v_j$$

and using (A_1), we find

$$\int_{\mathbb{R}^N} G_2'(\varepsilon_n x, w_n)v_j \leq b_0 \int_{B_{\frac{r}{\varepsilon_n}}(0)} w_n v_j + \int_{B^c_{\frac{r}{\varepsilon_n}}(0)} F_2'(w_n)v_j,$$

and so

$$\int_{\mathbb{R}^N} (\nabla w_n \nabla v_j + C w_n v_j) + \int_{\mathbb{R}^N} F_1'(w_n)v_j \leq \int_{B^c_{\frac{r}{\varepsilon_n}}(0)} F_2'(w_n)v_j, \tag{2.27}$$

for a convenient $C > 0$. Since v_j has compact support, one can sees that

$$\int_{B^c_{\frac{r}{\varepsilon_n}}(0)} F_2'(w_n)v_j \longrightarrow 0 \quad \text{as} \quad n \to \infty.$$

By using that $w_n \rightharpoonup w$ in X, we firstly take the limit of $n \to \infty$ and after the limit of $j \to \infty$ in the inequality (2.27) to get

$$\int_{\mathbb{R}^N} (|\nabla w|^2 + C|w|^2) + \int_{\mathbb{R}^N} F_1'(w)w \leq 0,$$

which yields $w = 0$. This contradiction proves the claim.

The preceding claim ensures that, going to a subsequence if necessary, $\varepsilon_n y_n \to y_0 \in \overline{\Lambda}$ for some y_0. Actually, we will prove that $y_0 \in \Lambda$. To this aim, note that for each $R > 0$ the sequence $\chi_n(x) := \chi_\Lambda(\varepsilon_n x + \varepsilon_n y_n)$ is a bounded sequence in $L^q(B_R(0))$, for any $q \in [2, \infty)$. Since $L^q(B_R(0))$ is a reflexive space for all $q \in [2, \infty)$, then there exists a function $\chi_R \in L^q(B_R(0))$ such that

$$\chi_n \rightharpoonup \chi_R \ \text{ in } \ L^q(B_R(0)).$$

The reader is invited to note that, given positive numbers $0 < R_1 < R_2$, the functions χ_{R_1} and χ_{R_2} obtained in the same way of χ_R satisfy

$$\chi_{R_1} \equiv \chi_{R_2}|_{B_{R_1}(0)}.$$

Therefore, there is a measurable function $\chi \in L^q_{loc}(\mathbb{R}^N)$ satisfying

$$\chi_n \rightharpoonup \chi \ \text{ in } \ L^q(B_R(0)), \tag{2.28}$$

for each $R > 0$. Note also that $0 \leq \chi \leq 1$.

In the same way of (2.26), for each $\phi \in C_0^\infty(\mathbb{R}^N)$ we have

$$\int_{\mathbb{R}^N} (\nabla w_n \nabla \phi + (V(\varepsilon_n x + \varepsilon_n y_n) + 1)w_n \phi) + \int_{\mathbb{R}^N} F_1'(w_n)\phi = \int_{\mathbb{R}^N} G_2'(\varepsilon_n x + \varepsilon_n y_n, w_n)\phi.$$

By Claim 2.2 and (2.28),

$$\int_{\mathbb{R}^N}(\nabla w\nabla\phi+(V(y_0)+1)w\phi)+\int_{\mathbb{R}^N}F_1'(w)\phi=\int_{\mathbb{R}^N}\tilde{G}_2'(x,w)\phi,$$

where

$$\tilde{G}_2'(z,t):=\chi(z)F_2'(t)+(1-\chi(z))\tilde{F}_2'(t).$$

It is easy to check that $\tilde{G}_2'$ satisfies

$$\tilde{G}_2'(z,t)\le C(|t|+|t|^{p-1}),$$

where $p\in(2,2^*)$. Moreover, the map $t\longmapsto\frac{\tilde{G}_2'(z,t)}{t}$, for $t>0$, is an nondecreasing function.

The above arguments guarantee that $\tilde{J}'(w)=0$, where $\tilde{J}:X\longrightarrow\mathbb{R}$ is the functional given by

$$\tilde{J}(u):=\frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u|^2+(V(y_0)+1)|u|^2)+\int_{\mathbb{R}^N}F_1(u)-\int_{\mathbb{R}^N}\tilde{G}_2(x,u),$$

and $\tilde{G}_2(x,u):=\displaystyle\int_0^t\tilde{G}_2'(x,s)\,ds$. Next, we set

$$J_{V(y_0)}:=\frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u|^2+(V(y_0)+1)|u|^2)+\int_{\mathbb{R}^N}F_1(u)-\int_{\mathbb{R}^N}F_2(u)\quad\forall u\in X,$$

$$\mathcal{M}_0:=\left\{u\in X-\{0\};\ J_{V(y_0)}'(u)u=0\right\}$$

and

$$c_{V(y_0)}=\inf_{u\in\mathcal{M}_0}J_{V(y_0)}(u)=\inf_{u\in X-\{0\}}\left\{\max_{t\ge0}J(tw)\right\}.$$

Define also $\Sigma_0:=\text{supp}\chi$ and $\mathcal{O}_0:=\{u\in X_\varepsilon;\ |\text{supp}(|u|)\cap\Sigma_0|>0\}$. Using the same ideas explored in the proof of Lemma 2.1, the conditions on $\tilde{G}_2$ allows us to conclude that

$$\tilde{J}(tv)\to-\infty,\ \text{ as }\ t\to\infty,$$

for each $v\in\mathcal{O}_0$. Since $w\neq0$ and $\tilde{J}'(w)=0$, we get $w\in\mathcal{O}_0$. Therefore, by standard arguments,

$$\tilde{J}(w)=\max_{t\ge0}\tilde{J}(tw)\ge\max_{t\ge0}J(tw)\ge c_{V(y_0)}.$$

In the same way of (2.8), we find by a change of variable,

$$
\begin{aligned}
c_{\varepsilon_n} &= J_{\varepsilon_n}(u_n) - \frac{1}{2}J'_{\varepsilon_n}(u_n)u_n = \\
&= \frac{1}{2}\int_{\mathbb{R}^N}(|w_n|^2 + [F_2(w_n) - \frac{1}{2}F_2'(w_n)w_n + \frac{1}{2}G_2'(\varepsilon_n x + \varepsilon_n y_n, w_n)w_n - G_2(\varepsilon_n x + \varepsilon_n y_n, w_n)]).
\end{aligned}
$$

From $(A_1) - iv)$,

$$
c_{\varepsilon_n} \geq \frac{1}{2}\int_{B_R(0)}(|w_n|^2 + [F_2(w_n) - \frac{1}{2}F_2'(w_n)w_n + \frac{1}{2}G_2'(\varepsilon_n x + \varepsilon_n y_n, w_n)w_n - G_2(\varepsilon_n x + \varepsilon_n y_n, w_n)])
$$

for each $R > 0$. Now, fix $p \in (2, 2^*)$. Since $w_n \to w$ in $L^p(B_R(0))$, the growth conditions on F_2' and $\tilde{F}_2'$ assures that, for some $q \in (p, 2^*)$, it holds

$$
\begin{cases}
F_2'(w_n)w_n \to F_2'(w)w, & \text{in } L^{\frac{q}{p}}(B_R(0)); \\
\tilde{F}_2'(w_n)w_n \to \tilde{F}_2'(w)w, & \text{in } L^{\frac{q}{p}}(B_R(0)).
\end{cases}
$$

The convergence in (2.28) implies that $\chi_n \rightharpoonup \chi$ in $L^r(B_R(0))$, where r is the conjugate exponent of q/p. Gathering these information,

$$
\chi_n F_2'(w_n) + (1-\chi_n)\tilde{F}_2'(w_n) \longrightarrow \chi F_2'(w) + (1-\chi)\tilde{F}_2'(w) \ \text{ in } \ L^1(B_R(0)).
$$

Now, employing the fact that

$$
G_2'(\varepsilon_n x + \varepsilon_n y_n, w_n) = \chi_n(x)F_2'(w_n) + (1 - \chi_n(x))\tilde{F}_2'(w_n),
$$

we conclude that

$$
G_2'(\varepsilon_n x + \varepsilon_n y_n, w_n) \to \tilde{G}_2'(x, w) \ \text{ in } \ L^1(B_R(0)).
$$

Using an analogous reasoning we also derive

$$
G_2(\varepsilon_n x + \varepsilon_n y_n, w_n) \to \tilde{G}_2(x, w) \ \text{ in } \ L^1(B_R(0)).
$$

Consequently, by Fatou's Lemma (recall the inequality in $(A_1) - iv)$) and Lemma 2.5,

$$
c_0 \geq \int_{B_R(0)}\left(\frac{1}{2}|w|^2 + [F_2(w) - \frac{1}{2}F_2'(w)w + \frac{1}{2}\tilde{G}_2'(x,w)w - \tilde{G}_2(x,w)]\right), \quad \forall R > 0.
$$

Letting $R \to \infty$, one gets

$$
\begin{aligned}
c_0 &\geq \int_{\mathbb{R}^N}\left(\frac{1}{2}|w|^2 + [F_2(w) - \frac{1}{2}F_2'(w)w + \frac{1}{2}\tilde{G}_2'(x,w)w - \tilde{G}_2(x,w)]\right) = \\
&= \tilde{J}(w) - \frac{1}{2}\tilde{J}'(w)w = \tilde{J}(w) \geq c_{V(y_0)}.
\end{aligned}
$$

By the definitions of levels c_0 and $c_{V(y_0)}$, the above inequality ensures that $V(y_0) \leq V(0) = \inf_{x\in\Lambda} V(x)$. Indeed, note that, if $\lambda_1 < \lambda_2$, then

$$\max_{t\geq 0} J_{\lambda_1}(tu) < \max_{t\geq 0} J_{\lambda_2}(tu),$$

so that $c_{\lambda_1} < c_{\lambda_2}$, where J_λ is the energy functional associated with the problem

$$(P_\lambda) \qquad \begin{cases} -\Delta u + \lambda u = u \log u^2, \ \text{ in } \ \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N) \cap L^{F_1}(\mathbb{R}^N). \end{cases}$$

and

$$c_\lambda := \inf_{u\in X\setminus\{0\}} \max_{t\geq 0} J_\lambda(tu).$$

Thus, by (V_2), we must have $V(y_0) = V(0) = V_0$ and $y_0 \in \Lambda$.

In order to finish the proof, it remains to prove that

$$w_n \longrightarrow w \ \text{ in } \ X \quad \text{as} \quad n \to +\infty.$$

Aiming this goal, we will prove the following result

Claim 2.3 $\displaystyle\lim_{n\to+\infty} \int_{(\Lambda_{\varepsilon_n} - y_n)} |w_n|^2 = \int_{\mathbb{R}^N} |w|^2.$

Note first that, since $\varepsilon_n y_n \to y_0 \in \Lambda$, there exists a number $r > 0$ such that $B_r(\varepsilon_n y_n) \subset \Lambda$, for all n large enough. Thereby,

$$B_{\frac{r}{\varepsilon_n}}(0) \subset \Lambda_{\varepsilon_n} - y_n,$$

for all n large enough, and so,

$$\chi_{(\Lambda_{\varepsilon_n} - y_n)}(x) \longrightarrow 1, \ \text{a.e.} \ x \in \mathbb{R}^N. \tag{2.29}$$

Now, note that, by using $\tilde{G}_2' \leq F_2'$ and that $\tilde{J}'(w)w = 0$ we get $J'_{V(y_0)}(w)w \leq 0$, so that $J'_0(w)w \leq 0$, because $V(y_0) = V_0$. Therefore, for some $t_0 \in (0, 1]$ it holds $t_0 w \in \mathcal{N}_0$. Then, from (2.29) and Lemma 2.5-iii),

$$\begin{aligned} c_0 \leq J_0(t_0 w) = \frac{t_0^2}{2} \int_{\mathbb{R}^N} |w|^2 \leq \frac{t_0^2}{2} \liminf_{n\to+\infty} \int_{(\Lambda_{\varepsilon_n} - y_n)} |w_n|^2 \leq \frac{t_0^2}{2} \limsup_{n\to+\infty} \int_{(\Lambda_{\varepsilon_n} - y_n)} |w_n|^2 \leq \\ \leq \frac{t_0^2}{2} \limsup_{n\to+\infty} c_{\varepsilon_n} \leq c_0, \end{aligned} \tag{2.30}$$

where we have used that

$$\frac{1}{2}\int_{(\Lambda_{\varepsilon_n}-y_n)}|w_n|^2=\frac{1}{2}\int_{\Lambda_{\varepsilon_n}}|u_n|^2\le J_{\varepsilon_n}(u_n)-\frac{1}{2}J'_{\varepsilon_n}(u_n)u_n=c_{\varepsilon_n}.$$

The above computations prove the claim.

Observe that the sentence in (2.30) also ensures that $t_0=1$, and so, $w\in\mathcal{N}_0$. Using that $J'_{\varepsilon_n}(u_n)u_n=0$, by a change of variable, we find

$$\begin{gathered}\int_{\mathbb{R}^N}(|\nabla w_n|^2+(V(\varepsilon_n x+\varepsilon_n y_n)+1)|w_n|^2)+\int_{\mathbb{R}^N}F_1'(w_n)w_n=\\ \int_{(\Lambda_{\varepsilon_n}-y_n)}F_2'(w_n)w_n+\int_{(\Lambda_{\varepsilon_n}-y_n)^c}\tilde{F}_2'(w_n)w_n.\end{gathered}\tag{2.31}$$

By applying Claim 2.3 and interpolation,

$$\chi_{(\Lambda_{\varepsilon_n}-y_n)}w_n\longrightarrow w\ \text{ in }\ L^p(\mathbb{R}^N)$$

and

$$\int_{(\Lambda_{\varepsilon_n}-y_n)}F_2'(w_n)w_n=\int_{\mathbb{R}^N}F_2'(w)w+o_n(1).$$

As $w\in\mathcal{N}_0$ and

$$(V(\varepsilon_n x+\varepsilon_n y_n)+1])|w_n|^2-\tilde{F}_2'(w_n)w_n)\ge 0\quad\text{in}\quad(\Lambda_{\varepsilon_n}-y_n)^c,$$

the equality (2.31) yields that

$$\begin{aligned}&\int_{\mathbb{R}^N}(|\nabla w|^2+(V(y_0)+1)|w|^2)+\int_{\mathbb{R}^N}F_1'(w)w\le\\ &\le\liminf\int_{\mathbb{R}^N}\left(|\nabla w_n|^2+\int_{(\Lambda_{\varepsilon_n}-y_n)}(V(\varepsilon_n x+\varepsilon_n y_n)+1)|w_n|^2)+\int_{\mathbb{R}^N}F_1'(w_n)w_n\right)\le\\ &\le\int_{\mathbb{R}^N}(|\nabla w|^2+(V_0+1)|w_n|^2)+\int_{\mathbb{R}^N}F_1'(w)w.\end{aligned}$$

Taking into account $V(y_0)=V_0$, we derive that

$$||w_n||^2_{H^1(\mathbb{R}^N)}\to||w||^2_{H^1(\mathbb{R}^N)}\ \text{ and }\ \int_{\mathbb{R}^N}F_1'(w_n)w_n\to\int_{\mathbb{R}^N}F_1'(w)w.$$

The above limit together with (C.6) ensure that $w_n\to w$ in X. Finally the boundedness of (w_n) in $L^\infty(\Omega)$ and the limit (2.24) follow as in [11, Lemma 3.10] ■

As a direct consequence of the computations made above, see the sentence (2.30), we have the following result

Corollary 2.2 *The levels c_ε satisfies $\lim\limits_{\varepsilon\to 0}c_\varepsilon=c_0$.*

Finally, we are ready to prove that (P_ε) has a positive solution for all ε small enough.

Theorem 2.3 *There exists $\varepsilon_0 > 0$ such that (S_ε) (and so (P_ε)) has a positive solution $u_\varepsilon \in X_\varepsilon$ for all $\varepsilon \in (0, \varepsilon_0)$.*

Proof. In what follows, we will prove that

$$u_\varepsilon(x) < t_1, \ \ \forall x \in \mathbb{R}^N - \Lambda_\varepsilon, \tag{2.32}$$

for $\varepsilon \in (0, \varepsilon_0)$. Indeed, consider a sequence $\varepsilon_n \to 0$ and (u_{ε_n}) such that $J_{\varepsilon_n}(u_{\varepsilon_n}) = c_{\varepsilon_n}$ and $J'_{\varepsilon_n}(u_{\varepsilon_n}) = 0$. By Lemma 2.6, going to a subsequence if necessary, there exists a sequence (y_n) in $\mathbb{R}^N$ satisfying $\varepsilon_n y_n \to y_0$, with $V(y_0) = V_0$. Thus, for some $r > 0$ it holds $B_r(\varepsilon_n y_n) \subset \Lambda$, and so, $B_{\frac{r}{\varepsilon_n}}(y_n) \subset \Lambda_{\varepsilon_n}$. The last inclusion is equivalent to

$$\mathbb{R}^N - \Lambda_{\varepsilon_n} \subset \mathbb{R}^N - B_{\frac{r}{\varepsilon_n}}(y_n).$$

On the other hand, the sequence (y_n) can be chosen such that $w_n(x) = u_{\varepsilon_n}(x + y_n)$ satisfies (2.24). Therefore, for $R > 0$ large enough,

$$w_n(x) < t_1, \ \ \forall x \in \mathbb{R}^N - B_R(0),$$

which implies

$$u_{\varepsilon_n}(x) < t_1, \ \ \forall x \in \mathbb{R}^N - B_R(y_n).$$

Since for $n \in \mathbb{N}$ large enough $r/\varepsilon_n \geq R$, we have

$$\mathbb{R}^N - \Lambda_{\varepsilon_n} \subset \mathbb{R}^N - B_{\frac{r}{\varepsilon_n}}(y_n) \subset \mathbb{R}^N - B_R(y_n),$$

for all n large enough, showing that

$$u_{\varepsilon_n}(x) < t_1, \ \ \forall x \in \mathbb{R}^N - \Lambda_{\varepsilon_n}.$$

Since $\varepsilon_n \to 0$ is arbitrary, the proof is over. ■

Remark 2.2 A natural question related with the problem (P_ε) it is about the concentration of positive solutions. Using (2.24), the same arguments employed in [11, Section 4] guarantee that the below result holds.

Corollary 2.3 (Concentration phenomena) *Let $v_\varepsilon(x) = u_\varepsilon(x/\varepsilon)$. Then, v_ε is a solution of (P_ε) for $\varepsilon \in (0, \varepsilon_0)$. Moreover, if $z_\varepsilon \in \mathbb{R}^N$ is a global maximum point of v_ε, we have*

$$\lim_{\varepsilon \to 0^+} V(z_\varepsilon) = V_0.$$

2.4 Multiplicity of solution for (P_ε)

In this section we will show the existence of multiple solution for (P_ε) by using the Lusternik-Schnirelmann category theory. More precisely, setting

$$M := \{x \in \Lambda;\, V(x) = V_0\} \text{ and } M_\delta := \{x \in \mathbb{R}^N;\, d(x, M) \leq \delta\}, \tag{2.33}$$

where $\delta > 0$ is small enough of such way that $M_\delta \subset \Lambda$, our arguments will prove that (S_ε) has at least $\text{cat}_{M_\delta}(M)$ solutions. To begin with, we start by recalling some notions related with the Lusternik-Schnirelmann category theory, for further details see [83, Chapter 5, and references therein].

Definition 2.2 *Let Y be a closed subset of a topological space Z. We say that the (Lusternik-Schnirelmann) category of Y in Z is n, $cat_Z(Y) = n$ for short, if n is the least number of closed and contractible sets in Z which cover Y.*

Suppose that W is a Banach space and V is a C^1- manifold of the form $V = \Psi^{-1}(\{0\})$, where $\Psi \in C^1(W, \mathbb{R})$ and 0 is a regular value of Ψ. For a functional $I : W \longrightarrow \mathbb{R}$ denote

$$I^d := \{u \in V;\, I(u) \leq d\}.$$

The following result can be found in [83, Chapter 5] and it is our main abstract tool to get the existence of multiple solution for (P_ε).

Theorem 2.4 *Let $I \in C^1(W, \mathbb{R})$ be such that $I|_V$ is bounded from below. Suppose that I satisfies the $(PS)_c$ condition for $c \in [\inf I|_V, d]$, then $I|_V$ has at least $cat_{I^d}(I^d)$ critical points in I^d.*

In the sequel, let us introduce some notations that will be used later on. Hereafter, we denote by u_0 a positive ground state solution of (P_0). Furthermore, for each $\delta > 0$, we fix $\phi \in C^\infty([0, \infty)$ such that $0 \leq \phi \leq 1$ and

$$\phi(t) = \begin{cases} 1, & 0 \leq t \leq \dfrac{\delta}{2}; \\ 0, & t \geq \delta. \end{cases}$$

Using the above notation, for each $y \in M$ we also set

$$w_{\varepsilon,y}(x) := \phi(|\varepsilon x - y|) u_0 \left(\frac{\varepsilon x - y}{\varepsilon} \right)$$

and let $t_{\varepsilon,y} > 0$ be such that $t_{\varepsilon,y} w_{\varepsilon,y} \in \mathcal{N}_\varepsilon$. Note that $|\mathrm{supp}(w_{\varepsilon,y}) \cap \Lambda_\varepsilon| > 0$, then we know that $t_{\varepsilon,y}$ verifies $J_\varepsilon(t_{\varepsilon,y} w_{\varepsilon,y}) = \max_{t\geq 0} J_\varepsilon(t w_{\varepsilon,y})$.

For each $\varepsilon > 0$, we define the map

$$\begin{aligned} \Phi_\varepsilon : M &\longrightarrow \mathcal{N}_\varepsilon \\ y &\longmapsto \Phi_{\varepsilon,y} \equiv t_{\varepsilon,y} w_{\varepsilon,y}. \end{aligned}$$

Now, fix $\rho > 0$ such that $M_\delta \subset B_\rho(0)$ and $\zeta : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ given by

$$\zeta(x) = \begin{cases} x, & |x| \leq \rho; \\ \rho \dfrac{x}{|x|}, & |x| \geq \rho. \end{cases}$$

Finally, we set $\beta : \mathcal{N}_\varepsilon \longrightarrow \mathbb{R}^N$ given by

$$\beta(u) := \frac{\displaystyle\int_{\mathbb{R}^N} \zeta(\varepsilon x)|u(x)|^p}{||u||_p^p}.$$

Lemma 2.7 *The following limit holds*

$$\lim_{\varepsilon\to 0} J_\varepsilon(\Phi_{\varepsilon,y}) = c_0, \qquad \textit{uniformly in} \quad y \in M.$$

Proof. Arguing by contradiction, we get sequences (ε_n) and (y_n), with $\varepsilon_n \to 0$ and $(y_n) \subset M$, such that

$$|J_{\varepsilon_n}(\Phi_{\varepsilon_n,y_n}) - c_0| \geq \delta_0, \tag{2.34}$$

for some $\delta_0 > 0$. Setting $t_n = t_{\varepsilon_n,y_n}$ and using that $\Phi_{\varepsilon_n,y_n} \in \mathcal{N}_{\varepsilon_n}$, we find

$$\begin{aligned} J_{\varepsilon_n}(\Phi_{\varepsilon_n,y_n}) = \frac{t_n^2}{2} \int_{\mathbb{R}^N} (|\nabla \phi(\varepsilon_n z) u_0(z)|^2 + (V(\varepsilon_n z + y_n) + 1)|\phi(\varepsilon_n z) u_0(z)|^2) + \\ + \int_{\mathbb{R}^N} F_1(t_n \phi(\varepsilon_n z) u_0(z)) - \int_{\mathbb{R}^N} G_2(\varepsilon_n z + y_n, t_n \phi(\varepsilon_n z) u_0(z)) \end{aligned} \tag{2.35}$$

and

$$\begin{aligned} t_n^2 \int_{\mathbb{R}^N} (|\nabla \phi(\varepsilon_n z) u_0(z)|^2 + (V(\varepsilon_n x + \varepsilon_n y_n) + 1)|\phi(\varepsilon_n z) u_0(z)|^2) = \\ = \int_{\mathbb{R}^N} G_2'(\varepsilon_n z + y_n, t_n \phi(\varepsilon_n z) u_0(z)) t_n \phi(\varepsilon_n z) u_0(z) - \int_{\mathbb{R}^N} F_1'(t_n \phi(\varepsilon_n z) u_0(z)) t_n \phi(\varepsilon_n z) u_0(z). \end{aligned} \tag{2.36}$$

Note that, if $z \in B_{\frac{\delta}{\varepsilon_n}}(0)$, then $\varepsilon_n z + y_n \in B_\delta(y_n) \subset M_\delta$. By (2.33), we derive that $\varepsilon_n z + y_n \in \Lambda$. Hence, for $z \in B_{\frac{\delta}{\varepsilon_n}}(0)$ one has $G_2' \equiv F_2'$. This information together with

(2.36) yields

$$\int_{\mathbb{R}^N}(|\nabla\phi(\varepsilon_n z)u_0(z)|^2+(V(\varepsilon_n x+y_n)+1)|\phi(\varepsilon_n z)u_0(z)|^2)=$$
$$=\int_{\mathbb{R}^N}|\phi(\varepsilon_n z)u_0(z)|^2\log(|t_n\phi(\varepsilon_n z)u_0(z)|^2)=$$
$$=\int_{\mathbb{R}^N}|\phi(\varepsilon_n z)u_0(z)|^2\log(|\phi(\varepsilon_n z)u_0(z)|^2)+\log(|t_n|^2)\int_{\mathbb{R}^N}|\phi(\varepsilon_n z)u_0(z)|^2.$$

Our next step is proving that, going to a subsequence, $t_n\to 1$. Since $y_n\in M$, we can assume $y_n\to y_0\in M$. In this way, the above equality ensures that (t_n) is a bounded sequence. Otherwise, going to a subsequence if necessary, we would have $t_n\to\infty$ and thus $\log(|t_n|^2)\to\infty$. Gathering this information with the Lebesgue Dominated Convergence Theorem in the above equality we arrive at a contradiction.

We may assume that $t_n\to t_0\geq 0$. Using the same ideas of preceding paragraph, one can see that $t_0>0$. Finally, by combining the Lebesgue's Theorem with the last equality we find

$$t_0^2\int_{\mathbb{R}^N}(|\nabla u_0|^2+V_0|u_0|^2)=\int_{\mathbb{R}^N}|t_0u_0|^2\log(t_0|u_0|^2),$$

which shows that $t_0=1$, because u_0 is a ground state solution of (P_0). As $t_n\to 1$, the sentence in (2.35) implies that $J_{\varepsilon_n}(\Phi_{\varepsilon_n,y_n})\to J_0(u_0)=c_0$, contradicting (2.34). The proof is now complete. ■

Let us introduce the following set

$$\tilde{\mathcal{N}}_\varepsilon:\{u\in\mathcal{N}_\varepsilon;\ J_\varepsilon(u)\leq c_0+o_1(\varepsilon)\}.$$

Note that the last lemma assures that $\Phi_{\varepsilon,y}\in\tilde{\mathcal{N}}_\varepsilon$.

Lemma 2.8 *The map β satisfies*

$$\lim_{\varepsilon\to 0}\beta(\Phi_{\varepsilon,y})=y,\quad \textit{uniformly in}\quad y\in M.$$

Proof. The idea is the same found in [14, Lemma 4.2]. If the result is false, there are sequences $\varepsilon_n\to 0$ and $(y_n)\subset M$ such that

$$|\beta(\Phi_{\varepsilon_n,y_n})-y_n|\geq\delta_1,$$

for some $\delta_1>0$. By using the definition of β and setting $z=\frac{\varepsilon_n x-y}{\varepsilon_n}$, we find

$$\beta(\Phi_{\varepsilon_n,y_n})=y_n+\frac{\displaystyle\int_{\mathbb{R}^N}(\zeta(\varepsilon z+y_n)-y_n)|\phi(|\varepsilon_n z|)u_0(z)|^p}{\displaystyle\int_{\mathbb{R}^N}|\phi(|\varepsilon_n z|)u_0(z)|^p}.$$

Without loss of generality, we may assume that $y_n \to y_0 \in M \subset B_\rho(0)$. Thus, the definition of ζ together with the Lebesgue Dominated Convergence Theorem implies that

$$|\beta(\Phi_{\varepsilon_n,y_n}) - y_n| = o_n(1),$$

which is absurd. ■

In the next lemma we prove a version of result of Cingolani-Lazzo in [43, Claim 4.2]. In that paper the authors have considered a homogenous type nonlinearity while in our case we are working with a logarithmic nonlinearity.

Lemma 2.9 *Let $u_n \in \mathcal{N}_{\varepsilon_n}$. Suppose that $J_{\varepsilon_n}(u_n) \to c_0$, where $\varepsilon_n \to 0$. Then, there exists a sequence (y_n) in $\mathbb{R}^N$ such that $w_n(x) := u_n(x+y_n)$ has a convergent subsequence in X. Furthermore,*

$$\lim_{n\to+\infty}(\varepsilon_n y_n) = y_0,$$

for some $y_0 \in M$.

Proof. As made in the proof of Lemma 2.6, we have that $\sup_{n\in\mathbb{N}}||u_n||_{\varepsilon_n} < \infty$, and so, (u_n) is a bounded sequence in X. By Lemmas 2.5-ii) and 2.34, we know that $c_{\varepsilon_n} = \inf_{u\in\mathcal{N}_{\varepsilon_n}} J_{\varepsilon_n}(u)$ and $J_{\varepsilon_n}(u_n) = c_{\varepsilon_n} + o_n(1)$. Therefore, by a slight variant of Ekeland's Variational Principle, there is $v_n \in \mathcal{N}_{\varepsilon_n}$ such that

i) $J_{\varepsilon_n}(v_n) = c_{\varepsilon_n} + o_n(1)$;

ii) $||v_n - u_n||_{\varepsilon_n} \leq o_n(1)$;

iii) $||J'_{\varepsilon_n}(v_n)||_* = o_n(1)$.

The reasoning employed in the proof of the Proposition 2.5 shows that $||J'_{\varepsilon_n}(v_n)||_{X'_{\varepsilon_n}} \to 0$, where X'_{ε_n} designates the topological dual space of X_{ε_n}. From the condition ii) above,

$$J'_{\varepsilon_n}(v_n)v_n = o_n(1).$$

Now, by following the steps in the proof of Lemma 2.6, we get a sequence $(y_n) \subset \mathbb{R}^N$ such that

$$\lim_{n\to+\infty}(\varepsilon_n y_n) = y_0,$$

for some $y_0 \in M$. Moreover, the sequence $\tilde{w}_n = v_n(\cdot + y_n)$ has a convergent subsequence in X and thus, using ii) above, $w_n := u_n(\cdot + y_n)$ has a convergent subsequence in X. This finishes the proof. ■

The below result relates the number of solutions of $(\tilde{S}_\varepsilon)$ with $\text{cat}_{M_\delta}(M)$.

Proposition 2.7 *Assume that $(V_1)-(V_2)$ hold and that δ is small enough. Then, problem $(\tilde{S}_\varepsilon)$ has at least $cat_{M_\delta}(M)$ solutions, with $\varepsilon \in (0, \varepsilon_1)$, for some $\varepsilon_1 > 0$.*

Proof. In this proof we will employ the Theorem 2.4 with $I = J_\varepsilon$, $V = \mathcal{N}_\varepsilon$ and $d = c_o + o_1(\varepsilon)$. In this case, we have $J_\varepsilon^d = \tilde{\mathcal{N}}_\varepsilon$. On account of Proposition 2.5, the functional $J_\varepsilon|_{\mathcal{N}_\varepsilon}$ verifies the (PS) condition, and so, the Theorem 2.4 guarantees that $J_\varepsilon|_{\mathcal{N}_\varepsilon}$ has at least $\text{cat}_{\tilde{\mathcal{N}}_\varepsilon}(\tilde{\mathcal{N}}_\varepsilon)$ critical points in $\tilde{\mathcal{N}}_\varepsilon = J_\varepsilon^d$. Thereby, by Proposition 2.4, J_ε has $\text{cat}_{\tilde{\mathcal{N}}_\varepsilon}(\tilde{\mathcal{N}}_\varepsilon)$ critical points, from where it follows that $(\tilde{P}_\varepsilon)$ has at least $\text{cat}_{\tilde{\mathcal{N}}_\varepsilon}(\tilde{\mathcal{N}}_\varepsilon)$ solutions.

In order to finish the proof, we will prove

$$\text{cat}_{\tilde{\mathcal{N}}_\varepsilon}(\tilde{\mathcal{N}}_\varepsilon) \geq \text{cat}_{M_\delta}(M).$$

Our argument follows the ideas of [43, Section 6]. It suffices to consider the case $\text{cat}_{\tilde{\mathcal{N}}_\varepsilon}(\tilde{\mathcal{N}}_\varepsilon) < \infty$. Let $n = \text{cat}_{\tilde{\mathcal{N}}_\varepsilon}(\tilde{\mathcal{N}}_\varepsilon)$ and take $A_1, ...A_n$ closed and contractible sets in $\tilde{\mathcal{N}}_\varepsilon$ satisfying $\tilde{\mathcal{N}}_\varepsilon = \bigcup_{i=1}^{n} A_i$. In this way, it is possible to find $h_i \in C([0,1] \times A_i, \tilde{\mathcal{N}}_\varepsilon)$, with $h_i(0,u) = u$ and $h_i(1,u) = h_i(1, v_0^i)$, for some fixed $v_0^i \in A_i$, $i \in \{1, ..., n\}$. Note that, by Lemma 2.7, we have $\Phi_\varepsilon(M) \subset \tilde{\mathcal{N}}_\varepsilon$ for $\varepsilon \approx 0^+$. Also, the map

$$\beta \circ \Phi_\varepsilon : M \longrightarrow M_\delta$$

is well defined for $\varepsilon \approx 0^+$. Set

$$\begin{aligned} \eta : [0,1] \times M &\longrightarrow M_\delta \\ (t,y) &\longmapsto \eta(t,y) = t\beta(\Phi_{\varepsilon,y}) + (1-t)y. \end{aligned}$$

By using the properties related with β, one can see that η is well defined and $\beta \circ \Phi_\varepsilon$ is homotopic to inclusion map $i : M \longrightarrow M_\delta$. Since Φ_ε is a continuous map, the sets $B_i := \Phi_\varepsilon^{-1}(A_i)$ are closed subsets of M. In addition,

$$M = \bigcup_{i=1}^{n} B_i. \tag{2.37}$$

Now we are able to show that $n \geq \text{cat}_{M_\delta}(M)$. Indeed, it remains to prove that, for each $i \in \{1, ..., n\}$, the set B_i is contractible in M_δ. To this aim, let

$$H_i : [0,1] \times B_i \longrightarrow M_\delta$$

be given by

$$H_i(t,u)=\begin{cases}\eta(2t,u), & 0\le t\le \dfrac{1}{2};\\ g_i(2t-1), & \dfrac{1}{2}\le t\le 1,\end{cases}$$

with $g_i(t,u):=\beta(h_i(t,\Phi_{\varepsilon,y}))$. The above conditions on η and h_i ensure that H_i is well defined. Furthermore,

$$H_i(0,y)=\eta(0,y)=y \quad\text{and}\quad H_i(1,y)=\beta(h_i(1,v_0^i)),\ \forall y\in B_i,$$

which shows that B_i is contractible in M_δ. From (2.37) we get the desired inequality.

■

The result below points out an important property of the solutions of $(\tilde{S}_\varepsilon)$ obtained in the last theorem.

Proposition 2.8 (Positive solutions counting) *There exists $\varepsilon_2>0$ such that, for $\varepsilon\in(0,\varepsilon_2)$, it holds*

i) $(\tilde{S}_\varepsilon)$ has at least $\frac{cat_{M_\delta}(M)}{2}$ positive solutions, if $cat_{M_\delta}(M)$ is an even number;

ii) $(\tilde{S}_\varepsilon)$ has at least $\frac{cat_{M_\delta}(M)+1}{2}$ positive solutions, if $cat_{M_\delta}(M)$ is an odd number.

Proof. Take $\varepsilon_2\approx 0^+$ and fix $\varepsilon\in(0,\varepsilon_2)$. If v_ε is a critical point of $J_\varepsilon(v_\varepsilon)\le c_0+o_\varepsilon(1)$, we must have $v_\varepsilon^+=0$ or $v_\varepsilon^-=0$. Otherwise, we would have $v_\varepsilon^+,\ v_\varepsilon^-\in\mathcal{N}_\varepsilon$, and so,

$$2c_\varepsilon\le J_\varepsilon(v_\varepsilon^+)+J_\varepsilon(v_\varepsilon^-)=J_\varepsilon(v_\varepsilon)\le c_0+o_\varepsilon(1),$$

which is a contradiction for $\varepsilon_2\approx 0^+$. Therefore, using the same arguments of Lemma 2.6, we deduce that either $v_\varepsilon>0$ or $v_\varepsilon<0$.

Now, suppose that $k:=\mathrm{cat}_{M_\delta}(M)$ is an even number and let $v_1,\dots,v_k$ be the solutions of $(\tilde{P}_\varepsilon)$ given in the preceding proposition. If at least $\frac{k}{2}$ of the solutions $v_1,\dots,v_k$ are positive solutions, the item $i)$ is proved. Otherwise, we know that at least $\frac{k}{2}$ of the solutions $v_1,\dots,v_k$ are negative. Denote by $w_1,\dots,w_{\frac{k}{2}}$ such negative solutions. Since $g_2(x,\cdot)-F_1'$ is an odd function, the functions $-w_1,\dots,-w_{\frac{k}{2}}$ are positive solutions of the problem

$$(\tilde{S}_\varepsilon)\qquad \begin{cases}-\Delta u+(V(\varepsilon x)+1)u=g_2(\varepsilon x,u)-F_1'(u), \ \text{ in } \mathbb{R}^N,\\ u\in H^1(\mathbb{R}^N)\cap L^{F_1}(\mathbb{R}^N).\end{cases}$$

and thus $i)$ is proved. The proof of $ii)$ follows by a similar reasoning. ■

2.4.1 Proof of Theorem 2.1

The proof is as follows.

Proof of Theorem 2.1. Let v_ε be a critical point of $J_\varepsilon(v_\varepsilon) \leq c_0 + o_\varepsilon(1)$. It suffices to show that there exists $\varepsilon_3 \approx 0^+$ such that, for $\varepsilon \in (0, \varepsilon_3)$,

$$0 < v_\varepsilon(x) < t_1, \quad \forall x \in \mathbb{R}^N - \Lambda_\varepsilon, \tag{2.38}$$

for each solution v_ε of $(\tilde{S}_\varepsilon)$ given in the items $i) - ii)$ of the last proposition. Arguing by contradiction, we get a sequence (v_{ε_n}) of solutions of $(\tilde{S}_{\varepsilon_n})$ where $\varepsilon_n \to 0$ and $v_n := v_{\varepsilon_n}$ does not satisfy (2.38). Note that the obtained sequence (v_n) satisfies the hypothesis of Lemma 2.9 and that the sequence (w_n) given in the lemma must satisfy (2.24). Thus, a contradiction is obtained by following closely the same ideas used in the proof of Theorem 2.3. This argument ensures that (S_ε) verifies $i) - ii)$ in the statement of the Theorem 2.1. Now, the result follows by a change of variable. ■

We finish this chapter by pointing out an important question related with the number of positive solutions obtained in our previous results.

Remark 2.3 In [14, 43] the result of multiplicity of solution involving the Lusternik-Schnirelmann category assures the existence of at least $\mathrm{cat}_{M_\delta}(M)$ positive solutions. In [14], for example, the key point is the fact that the nonlinearity f was assumed such that $f(t) = 0$, $t \leq 0$. In our case, this framework lead us to consider $f(t) = |t^+|^2 \log |t^+|^2$, as well as,

$$J_\varepsilon(u) := \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + (V(\varepsilon x) + 1)|u|^2) + \int_{\mathbb{R}^N} F_1(u^+) - \int_{\mathbb{R}^N} G_2(\varepsilon x, u^+), \ \ \forall u \in X_\varepsilon.$$

However, we were not able to reproduce some estimates made throughout this work by considering J_ε given as above. For example, in the Lemma 2.2, we were not able to show the boundedness of the (PS) sequences when J_ε is chosen in this way. In fact, since the norm on X_ε involves the norm $||\cdot||_{F_1}$ of Orlicz space $L^{F_1}(\mathbb{R}^N)$, we need of the information of term $\displaystyle\int_{\mathbb{R}^N} F_1(u)$ in our computations. This justifies because our number of positive solutions by using the Lusternik-Schnirelmann category is a little bit different from that given in [14, 43].

CHAPTER 3

Existence of positive solution for a class of Schrödinger logarithmic equations on exterior domains

In the study developed in Chapter 2, the new function space introduced in the Section 2.1 allowed us to apply C^1-variational methods to find solutions for a class of elliptical problems with logarithmic nonlinearity. Inspired in such ideas, in the present chapter we intent to treat on the existence of positive solution for the following class of logarithmic equations.

$$\begin{cases} -\Delta u + u = Q(x) u \log u^2, & \text{in } \Omega, \\ \mathcal{B}u = 0 \text{ on } \partial\Omega, \end{cases}$$

with $\Omega \subset \mathbb{R}^N$, $N \geq 3$, an *exterior domain* (i.e., $\Omega^c = \mathbb{R}^N \setminus \Omega$ is a bounded smooth domain) and $\mathcal{B}u = u$ or $\mathcal{B}u = \frac{\partial u}{\partial \nu}$.

As in the problem (P_ε) in Chapter 2, if one tries to apply variational methods to the above problem, it is required to deal with the lack of smoothness of the natural candidate to energy functional associated to the problem.

In order to overcome such difficulty, we borrow the ideas of the preceding chapter and we consider a decomposition of the nonlinearity $f(t) = t \log t^2$, as well as a function space on which we will can to use the classical variational methods.

Our study is divided into two cases.

Case 1. *Dirichlet case*: In this case we will assume $Q \equiv 1$ and $\mathcal{B}u = u$. These conditions lead us to consider the problem:

$$(P_0) \qquad \begin{cases} -\Delta u + u = u \log u^2, & \text{in } \Omega, \\ \quad u \in H_0^1(\Omega). \end{cases}$$

The main result associated with (P_0) to be proved in this chapter is the following:

Theorem 3.1 *There exists $\rho_0 \approx 0^+$ such that, if $\Omega^c \subset B_\rho(0)$, then the problem (P_0) has a positive solution for each $\rho \in (0, \rho_0)$.*

Case 2. *Neumann case*: this case corresponds to the choosing $\mathcal{B}u := \dfrac{\partial u}{\partial \eta}$. On the function Q, we will assume in this case that

(Q_1) $\lim\limits_{|x|\to\infty} Q(x) = Q_0$ and $q_0 := \inf\limits_{x\in\mathbb{R}^N} Q(x) > 0$ for all $x \in \mathbb{R}^N$;

(Q_2) $Q_0 \geq Q(x) \geq Q_0 - Ce^{-M_0|x|^2}$, for $x \geq R_0$,

with $Q_0, C, M_0,\ R_0 > 0$.

In Case 2 our problem takes the following form:

$$(S_0) \qquad \begin{cases} -\Delta u + u = Q(x) u \log u^2, & \text{in } \Omega \\ \dfrac{\partial u}{\partial \eta} = 0, & \text{on } \partial\Omega, \end{cases}$$

The main result on the problem (S_0) is the following.

Theorem 3.2 *If the conditions $(Q_1) - (Q_2)$ hold, then for some M_0 large enough, the problem (S_0) has a positive ground state solution.*

It is important to mention that the conditions $(Q_1) - (Q_2)$ are inspired in the works [4, 33].

The new approach introduced in Chapter 2 and used in this chapter plays a crucial role in order to study the problems (P_0) and (S_0), because it permits to adapt several arguments explored in the literature about problems in exterior domains related with C^1-functionals to the problems (P_0) and (S_0); here, we have adapted and modified a lot of arguments present in the papers [3, 4, 9, 18, 27, 33, 54].

We would like to emphasize the results in the sequel can be found in the work due to Alves and da Silva in [6].

3.1 The variational framework

This section is devoted to show some technical results that will be used later on. We start by recalling an important result involving the uniqueness of positive solution for the logarithmic equation on the whole $\mathbb{R}^N$. After that, we recall some notions studied in Chapter 2 and we introduce the convenient function space that allows us to apply the C^1-variational methods in order to get solutions for our problem. Next, a result of nonexistence of ground state solution for (P_0) is also established. Finally, we prove a compactness lemma analogous to the result of Benci and Cerami in [27, Lemma 3.1] that plays a crucial role in our study.

Our first result in this section can be found in [44, Section 1] (see also [30]) and it concerns with the uniqueness of solution for the following class of problems

$$\begin{cases} -\Delta u + \kappa u = u \log u^2, & \text{in } \mathbb{R}^N, \\ \quad u \in H^1(\mathbb{R}^N), \end{cases} \tag{3.1}$$

where $\kappa > 0$.

Theorem 3.3 *The problem (3.1) has a unique positive solution $u \in C^2(\mathbb{R}^N, \mathbb{R})$, up to translations, such that $u(x) \to 0$ as $|x| \to \infty$. More precisely, the solution u is given by*

$$u(x) = C_{\kappa,N}\, e^{\frac{-|x|^2}{2}}.$$

The theorem above ensures that any positive solution of (3.1) has an exponential decaying.

3.1.1 The energy functional

In the same way of Chapter 2 (see also [6, 10, 11, 62]), we will explore a suitable decomposition of the function

$$F(s) = \int_0^s t \log t^2 \, dt = \frac{1}{2} s^2 \log s^2 - \frac{s^2}{2}, \quad s \in \mathbb{R},$$

which allows us to introduce an energy functional associated with (P_0). For each $\delta > 0$ sufficiently small, let F_1, $F_2 \in C^1(\mathbb{R})$ be given as in the Section 2.1.1 verifying

$$F_2(s) - F_1(s) = \frac{1}{2} s^2 \log s^2, \quad \forall s \in \mathbb{R}. \tag{3.2}$$

Recall that F_1 and F_2 satisfy the properties $(P_1) - (P_2)$ below:

(P_1) F_1 is an even function with $F_1'(s)s \geq 0$ and $F_1(s) \geq 0$ for all $s \in \mathbb{R}$. Moreover $F_1 \in C^1(\mathbb{R}, \mathbb{R})$ and it is also convex if $\delta \approx 0^+$;

(P_2) $F_2 \in C^1(\mathbb{R}, \mathbb{R})$ and for each $p \in (2, 2^*)$, there exists $C = C_p > 0$ such that

$$|F_2'(s)| \leq C|s|^{p-1} \quad \forall s \in \mathbb{R}.$$

As in Subsection 2.1.1, it will be explored the fact that F_1 is a N-function verifying the (Δ_2) condition (see the Appendix C for the proof). This fact ensures that the Orlicz space

$$L^{F_1}(\Omega) = \left\{ u \in L^1_{loc}(\Omega) \; ; \; \int_\Omega F_1(|u|)\, dx < +\infty \right\}$$

with the norm

$$||u||_{F_1} = \inf \left\{ \lambda > 0 \; ; \; \int_\Omega F_1\left(\frac{|u|}{\lambda}\right) \leq 1 \right\}$$

is a reflexive and separable Banach space.

From now on, we will set $X := H_0^1(\Omega) \cap L^{F_1}(\Omega)$ endowed with the norm

$$|| \cdot ||_X := || \cdot ||_{H_0^1(\Omega)} + || \cdot ||_{F_1}.$$

Here, $L^{F_1}(\Omega)$ designates the Orlicz space associated with F_1 and $|| \cdot ||_{F_1}$ denotes the usual norm associated with $L^{F_1}(\Omega)$. In view of the last proposition, the space X is a separable and reflexive Banach space. Furthermore, the embeddings $X \hookrightarrow H^1(\Omega)$ and $X \hookrightarrow L^{F_1}(\Omega)$ are continuous.

The natural candidate for the energy functional associated with (P_0) is given by

$$I(u) := \frac{1}{2}\int_\Omega (|\nabla u|^2 + 2|u|^2) + \int_\Omega F_1(u) - \int_\Omega F_2(u), \quad \forall u \in X.$$

It will be convenient to take the norm of $H_0^1(\Omega)$ as being

$$||u||_{H_0^1(\Omega)} := \left(\int_\Omega (|\nabla u|^2 + 2|u|^2) \right)^{\frac{1}{2}},$$

which is equivalent to the usual norm of $H_0^1(\Omega)$. Moreover, it is associated with the inner product

$$\langle u, v \rangle_{H_0^1(\Omega)} := \int_\Omega (\nabla u \nabla v + 2uv), \quad \forall u, v \in H_0^1(\Omega).$$

Similarly, we will consider

$$||u||_{H^1(\mathbb{R}^N)} := \left(\int_{\mathbb{R}^N} (|\nabla u|^2 + 2|u|^2) \right)^{\frac{1}{2}}, \quad \forall u \in H^1(\mathbb{R}^N),$$

as the norm in $H^1(\mathbb{R}^N)$.

From $(P_1) - (P_2)$, $I \in C^1(X, \mathbb{R})$ and

$$I'(u)v = \int_\Omega (\nabla u \nabla v + 2uv) + \int_\Omega F_1'(u)v - \int_\Omega F_2'(u)v, \quad \forall v \in X.$$

In our approach, we will use some properties of the limit problem below

$$(P_\infty) \qquad \begin{cases} -\Delta u + u = u \log u^2, & \text{in } \mathbb{R}^N, \\ \quad u \in H^1(\mathbb{R}^N). \end{cases}$$

Associated with (P_∞), we have the functional

$$I_\infty(u) := \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + 2u^2) + \int_{\mathbb{R}^N} F_1(u) - \int_{\mathbb{R}^N} F_2(u), \quad \forall u \in Y,$$

where $Y := (H^1(\mathbb{R}^N) \cap L^{F_1}(\mathbb{R}^N), ||\cdot||_Y)$ and $||\cdot||_Y := ||\cdot||_{H^1(\mathbb{R}^N)} + ||\cdot||_{L^{F_1}(\mathbb{R}^N)}$. Related to the functionals I and I_∞, we also have the Nehari sets

$$\mathcal{N} := \{u \in X - \{0\};\ I'(u)u = 0\}$$

and

$$\mathcal{N}_\infty := \{u \in Y - \{0\};\ I'_\infty(u)u = 0\},$$

which can be characterized by

$$\mathcal{N} := \Psi_0^{-1}(0) \text{ and } \mathcal{N}_\infty := \Psi_\infty^{-1}(0),$$

with

$$\Psi_0(u) = I(u) - \frac{1}{2} \int_\Omega |u|^2 \text{ and } \Psi_\infty(u) = I_\infty(u) - \frac{1}{2} \int_{\mathbb{R}^N} |u|^2. \tag{3.3}$$

A direct computation shows that $\Psi_0 \in C^1(X, \mathbb{R})$ and $\Psi_\infty \in C^1(Y, \mathbb{R})$. Furthermore, associated with $\mathcal{N}$ and $\mathcal{N}_\infty$, we consider the levels d_0 and d_∞ given by

$$d_0 := \inf_{u \in \mathcal{N}} I(u) \text{ and } d_\infty := \inf_{u \in \mathcal{N}_\infty} I_\infty(u).$$

The next result presents an important property of the sets $\mathcal{N}$ and $\mathcal{N}_\infty$ that is crucial in our approach

Proposition 3.1 *The sets $\mathcal{N}$ and $\mathcal{N}_\infty$ are C^1-manifolds with the topology of $(X, ||\cdot||_X)$ and $(Y, ||\cdot||_Y)$ respectively,. Furthermore, the critical points of $I|_{\mathcal{N}}$ and $I_\infty|_{\mathcal{N}_\infty}$ are critical points of I and I_∞ respectively*

Proof. For the first part, from (3.3), it is sufficient to show that 0 is a regular value for Ψ_0 and Ψ_∞. Indeed, if $u \in \Psi_0^{-1}(\{0\})$, then

$$\Psi_0'(u)u = I'(u)u - \int_\Omega |u|^2 = -\int_\Omega |u|^2 < 0,$$

since $u \neq 0$. Consequently, $\Psi_0'(u) \neq 0$ and 0 is a regular value of Ψ_0. A similar reasoning shows that 0 is also a regular value of Ψ_∞.

Now, note that if $u \in \mathcal{N}$ is a critical point of $I|_{\mathcal{N}}$, then it holds

$$I'(u) = \lambda \Psi_0'(u),$$

for some $\lambda \in \mathbb{R}$. So, one can see that $0 = \lambda \Psi_0'(u)u$, which implies that $\lambda = 0$ and $I'(u) = 0$, because $\Psi_0'(u)u < 0$ for $u \in \mathcal{N}$. In a similar way, the result follows for $I_\infty|_{\mathcal{N}_\infty}$. ■

The last proposition yields that a critical point of $I|_{\mathcal{N}}$ is a point $u \in X$ such that

$$||I'(u)||_* := \min_{\lambda \in \mathbb{R}} ||I'(u) - \lambda \Psi_0'(u)|| = 0. \quad \text{(See [83, Section 5.3])}$$

Analogously, we define a critical point of $I_\infty|_{\mathcal{N}_\infty}$.

Remark 3.1 Note that in the preceding proposition, it is crucial the fact that in our approach, in view of the topology induced by the spaces X and Y, the energy functionals I and I_∞ are of C^1 class. This fact is not verified if we consider, for example, I and I_∞ with the usual topology of $H_0^1(\Omega)$ and $H^1(\mathbb{R}^N)$.

In the next result, we point out an important property related with the sets $\mathcal{N}$ and $\mathcal{N}_\infty$ that will be explored later on.

Proposition 3.2 *There exist ρ_1, $\rho_2 > 0$ such that*

$$\rho_1 \leq ||u||_X, \ \ \forall \, u \in \mathcal{N}$$

and

$$\rho_2 \leq ||u||_Y, \ \ \forall \, u \in \mathcal{N}_\infty.$$

Proof. In fact, for $u \in \mathcal{N}$ it holds

$$0 < ||u||^2_{H^1_0(\Omega)} \leq ||u||^2_{H^1_0(\Omega)} + \int_\Omega F_1'(u)u = \int_\Omega F_2'(u)u \leq ||u||^p_{H^1_0(\Omega)},$$

with $p \in (2, 2^*]$. Using the embedding $X \hookrightarrow H^1_0(\Omega)$, one gets

$$0 < 1 \leq ||u||^{p-2}_{H^1_0(\Omega)} \leq C||u||^{p-2}_X,$$

for a convenient $C = C(p) > 0$. Thus, the first part of the result follows by setting $\rho_1 := (C^{-1})^{\frac{1}{p-2}}$. The second part of the lemma is proved with a similar argument. ■

From now on, let us designate by u_∞ a positive ground state solution of (P_∞) that can be assumed radial, that is,

$$I_\infty(u_\infty) = d_\infty > 0 \quad \text{and} \quad I'_\infty(u_\infty) = 0. (\text{ See Theorem 3.3 })$$

The next result relates the levels d_0 and d_∞.

Lemma 3.1 *It holds $d_0 = d_\infty$.*

Proof. Fix $\rho > 0$ the smallest positive number such that $\mathbb{R}^N \setminus \Omega \subset B_\rho(0)$. Now, let $\phi \in C^\infty(\mathbb{R}^N)$ satisfying

$$\begin{cases} \phi(x) = 0, & x \in B_\rho(0) \\ \phi(x) = 1, & x \in B_{2\rho}(0)^c, \end{cases}$$

with $0 \leq \phi \leq 1$. Take $(y_n) \subset \mathbb{R}^N$ with $|y_n| \to \infty$ and set

$$\phi_n(x) := \phi(x)u_\infty(x - y_n).$$

For each $n \in \mathbb{N}$, fix $t_n > 0$ of a such way that $t_n\phi_n \in \mathcal{N}$. Thereby,

$$d_0 \leq I(t_n\phi_n) = I_\infty(t_n\phi_n), \quad \forall n \in \mathbb{N}. \tag{3.4}$$

Note that, from the Lebesgue's Dominated Convergence Theorem,

$$\phi(\cdot + y_n)u_\infty \longrightarrow u_\infty. \tag{3.5}$$

Our next step is proving that $t_n \to 1$. To see why, firstly we recall that $t_n\phi_n \in \mathcal{N}$ leads to

$$\int_{\mathbb{R}^N} (|\nabla(t_n\phi_n)|^2 + |(t_n\phi_n)|^2) = \int_{\mathbb{R}^N} (t_n\phi_n)^2 \log(|t_n\phi_n|^2). \tag{3.6}$$

This combined with (3.2) gives

$$\int_{\mathbb{R}^N} (|\nabla(\phi_n)|^2 + |(\phi_n)|^2) = 2\int_{\mathbb{R}^N} (F_2(\phi_n) - F_1(\phi_n)) + \log t_n^2 \int_{\mathbb{R}^N} \phi_n^2. \quad (3.7)$$

Using (3.5) and the invariance by translation of $\mathbb{R}^N$, one finds

$$\int_{\mathbb{R}^N} F_i(\phi_n) \longrightarrow \int_{\mathbb{R}^N} F_i(u_\infty) \text{ for } i \in \{1,2\} \quad \text{and} \quad \int_{\mathbb{R}^N} |\phi_n|^2 \longrightarrow \int_{\mathbb{R}^N} |u_\infty|^2.$$

Gathering the limits above with (3.5), one sees that (t_n) is a bounded. So, we may assume that $t_n \to t_0 \geq 0$. If $t_0 = 0$, the equality (3.7) gives a contradiction. Therefore, it holds $t_0 > 0$ and, from the Lebesgue's Theorem,

$$\int_{\mathbb{R}^N} (|\nabla(t_0 u_\infty)|^2 + |(t_0 u_\infty)|^2) = \int_{\mathbb{R}^N} |t_0 u_\infty|^2 \log(t_0 u_\infty|^2),$$

showing that $t_0 = 1$, that is, $t_n \to 1$ as $n \to +\infty$. Using this limit together (3.4), we arrive at

$$d_0 \leq \lim I_\infty(t_n\phi_n) = I_\infty(u_\infty) = d_\infty.$$

As $X \subset Y$, the reverse inequality follows directly of the definition of I_∞, by noting that the condition $I'(u)u = 0$ also implies $I'_\infty(u)u = 0$. ■

Next, we establish the nonexistence of *ground state solution* for (P_0), i.e., we are going to prove that it does not exist a positive solution u_0 of (P_0) such that $I(u_0) = d_0$.

Theorem 3.4 *The problem (P_0) has no ground state solution.*

Proof. Seeking for a contradiction, assume that (P_0) has a positive ground state solution $w \in X$. Then,

$$I'(w) = 0 \quad \text{and} \quad I(w) = d_0.$$

Let v be the null extension of w, i.e., $v(x) = w(x)$ for $x \in \Omega$ and $v(x) = 0$ otherwise. It follows that $I'_\infty(v)v = I'(w)w = 0$, and by Lemma 3.1, $I_\infty(v) = I(w) = d_0 = d_\infty$. Therefore, $v \in \mathcal{N}_\infty$ is a critical point for $I_\infty|_{\mathcal{N}_\infty}$, and so, v is a critical point of I_∞. As made in [44, Section 3.1], by using a suitable version of the maximum principle found in [82], one deduces that $v > 0$ in whole $\mathbb{R}^N$, which is absurd because $v = 0$ in $\mathbb{R}^N \setminus \Omega$, finishing the proof. ■

In order to prove our next proposition, we recall the inequality in (1.61):

$$\int_{\mathbb{R}^N} |u|^2 \log|u|^2 \, dx \leq \frac{b^2}{\pi}||\nabla u||_2^2 + (\log||u||_2^2 - N(1+\log b))||u||_2^2, \ \ \forall u \in H^1(\mathbb{R}^N), \quad (3.8)$$

where $b > 0$ is a fixed positive constant.

Let us recall that a $(PS)_c$ sequence for $I|_{\mathcal{N}}$ is a sequence $(u_n) \subset \mathcal{N}$ such that

$$||I'(u_n)||_* \to 0 \quad \text{and} \quad I(u_n) \to c.$$

Lemma 3.2 *If (u_n) is a $(PS)_c$ sequence for $I|_{\mathcal{N}}$, then (u_n) is bounded in X.*

Proof. Let (u_n) be a $(PS)_c$ sequence for $I|_{\mathcal{N}}$. Since $I'(u_n)u_n = 0$, one has

$$c + o_n(1) = I(u_n) - \frac{1}{2}I'(u_n)u_n = \frac{1}{2}\int_\Omega |u_n|^2, \tag{3.9}$$

and so,

$$\int_\Omega |u_n|^2 \leq C, \quad \forall n \in \mathbb{N},$$

for a convenient $C > 0$. Applying the logarithmic inequality for some $b \approx 0^+$, we derive that

$$\int_{\mathbb{R}^N} |v|^2 \log|v|^2 \leq \frac{1}{2}||\nabla v||_2^2 + C(\log||v||_2^2 + 1)||v||_2^2, \quad v \in H^1(\mathbb{R}^N),$$

which leads to

$$\int_\Omega |u_n|^2 \log|u_n|^2 \leq \frac{1}{2}||\nabla u_n||_2^2 + C,$$

for some $C > 0$ independent of n. Therefore, by (3.8), there are C_1, $C_2 > 0$ such that

$$C_1 \geq \frac{1}{2}||u_n||^2_{H_0^1(\Omega)} - \frac{1}{2}\int_\Omega |u_n|^2 \log|u_n|^2 \geq C_2||u_n||^2_{H_0^1(\Omega)},$$

showing that

$$\sup_{n\in\mathbb{N}} ||u_n||^2_{H_0^1(\Omega)} < \infty. \tag{3.10}$$

The definition of I gives

$$\int_\Omega F_1(u_n) = I(u_n) - ||u_n||^2_{H_0^1(\Omega)} + \int_\Omega F_2(u_n).$$

Hence, by (3.9) and (3.10),

$$\sup_{n\in\mathbb{N}} \int_\Omega F_1(u_n) < \infty. \tag{3.11}$$

The sentences (3.10) and (3.11) guarantee that (u_n) is a bounded sequence in X. ■

By using the definition of the functions F_1 and F_2 and a Brezis-Lieb type result (Proposition C.1), it is possible to prove the lemma below whose the idea for the proof can be found in [80, Lemma 3.1].

Lemma 3.3 *Let (u_n) be a bounded sequence in X such that $u_n \to u$ a.e. in Ω. Then,*

$$\int_\Omega |u_n - u|^2 \log |u_n - u|^2 = \int_\Omega u_n^2 \log u_n^2 - \int_\Omega u^2 \log u^2 + o_n(1).$$

Proof. The proof could be made following the reasoning in [80, Lemma 3.1]. However, for the reader's comfort, we will present the idea of the proof. The argument consists in a suitable application of a Brezis-Lieb type result: By (3.2), one gets

$$2(F_2(u_n - u) - F_1(u_n - u)) = |u_n - u|^2 \log |u_n - u|^2,$$

from where we derive that

$$\int_\Omega |u_n - u|^2 \log |u_n - u|^2 = 2\int_\Omega (F_2(u_n - u) - F_1(u_n - u)).$$

Now, the proof follows by noting that, since F_2 has subcritical growth, the Lemma 3.1 in [4] assures that

$$\int_\Omega F_2(u_n - u) = \int_\Omega F_2(u_n) - \int_\Omega F_2(u) + o_n(1).$$

In a similar way,

$$\int_\Omega F_1(u_n - u) = \int_\Omega F_1(u_n) - \int_\Omega F_1(u) + o_n(1),$$

by the Brezis-Lieb type result valid for N-functions in Proposition C.1. ■

Our next result is an important compactness lemma that describes the behavior of $(PS)_c$ sequences for $I|_{\mathcal{N}}$.

Lemma 3.4 *Let (u_n) be a $(PS)_c$ sequence for $I|_{\mathcal{N}}$ with $u_n \rightharpoonup u_0$. Then, going to a subsequence if necessary, either*

i) $u_n \to u_0$ in X, or

ii) There exist $k \in \mathbb{N}$ and k sequences $(u_n^j)_{n\in\mathbb{N}}$, $u_n^j \in Y$, with

$$u_n^j \rightharpoonup u_j$$

and u_j nontrivial solutions of (P_∞), $j \in \{1, ..., k\}$. Furthermore, it holds

$$||u_n||^2_{H_0^1(\Omega)} \to ||u_0||^2_{H_0^1(\Omega)} + \sum_{j=1}^k ||u_j||^2_{H^1(\mathbb{R}^N)} \quad \text{and} \quad I(u_n) \to I(u_0) + \sum_{j=1}^k I_\infty(u_j).$$

Proof. Initially, for a convenient sequence of real numbers (λ_n), we must have

$$I'(u_n) = \lambda_n \Psi_0'(u_n) + o_n(1). \tag{3.12}$$

As $I'(u_n)u_n = 0$, one gets

$$\lambda_n \Psi_0'(u_n)u_n = o_n(1).$$

From this information, we claim that $\lambda_n = o_n(1)$. Indeed, notice that $|\Psi_0'(u_n)u_n| \nrightarrow 0$, otherwise we would have

$$\Psi_0'(u_n)u_n = \int_\Omega |u_n|^2 = o_n(1),$$

and so, since (u_n) is a bounded sequence in X, by interpolation, it follows that

$$||u_n||_p = o_n(1), \quad \forall p \in (2, 2^*).$$

This combines with (P_2) to give

$$\int_\Omega F_2'(u_n)u_n = o_n(1).$$

Now, the limit above together with the fact that $I'(u_n)u_n = 0$ leads to

$$\int_\Omega (|\nabla u_n|^2 + 2|u_n|^2) + \int_\Omega F_1'(u_n)u_n = o_n(1).$$

Since F_1 is convex with $F_1(0) = 0$, we know that $F_1'(s)s \geq F_1(s)$ for all $s \in \mathbb{R}$. Then, we can infer that

$$\int_{\mathbb{R}^N} (|\nabla u_n|^2 + 2|u_n|^2) + \int_\Omega F_1(u_n) = o_n(1).$$

Using the fact that $F_1 \in (\Delta_2)$, the last limit yields $u_n \to 0$ in X, which contradicts the fact that $u_n \in \mathcal{N}$ in view of the Proposition 3.2. So, it follows that $|\Psi_0'(u_n)u_n| \nrightarrow 0$ and $\lambda_n = o_n(1)$. By (3.12), since (u_n) is a bounded sequence, it holds $I'(u_n) \to 0$, that is, the sequence (u_n) is a $(PS)_c$ sequence for I. In addition, accounting that $u_n \rightharpoonup u_0$ and the growth conditions on F_1 and F_2, we deduce that $I'(u_0)v = 0$, for any $v \in X$, implying that u_0 is a solution of (P_0).

From now on, inspired in the ideas of [27], we set

$$\psi_n^1(x) := \begin{cases} u_n - u_0, & x \in \Omega \\ 0, & x \in \mathbb{R}^N \setminus \Omega. \end{cases}$$

A direct verification shows that $\psi_n^1 \rightharpoonup 0$ in X. In [4, 27], it was proved that $(\psi_n^1|_\Omega)$ is a (PS) sequence for $I_\infty|_{H_0^1(\Omega)}$ with

$$I_\infty(\psi_n^1) = I(u_n) - I(u_0) + o_n(1). \tag{3.13}$$

However, since we are working with a logarithmic nonlinearity, we are not able to show that $(\psi_n^1|_\Omega)$ is also a (PS) sequence. In our case we will prove that a weaker condition occurs. More precisely, the following properties hold:

i) $I_\infty(\psi_n^1) = I(u_n) - I(u_0) + o_n(1)$;

ii) Let $\phi \in C_0^\infty(\Omega)$ with $||\phi||_Y \leq 1$ and, for each $y \in \mathbb{R}^N$, define $\phi^{(y)}(x) = \phi(x+y)$ for all $x \in \mathbb{R}^N$. Then,

$$\sup_{y\in\mathbb{R}^N} \|I'_\infty(\psi_n^1)\|||\phi^{(y)}||_Y = o_n(1).$$

Verification of $i)$ By simplicity, in what follows ψ_n^1 also denotes $\psi_n^1|_\Omega$. The definition of ψ_n^1 gives $I_\infty(\psi_n^1) = I(\psi_n^1)$, then by a simple computation, the Lemma 3.3 guarantees that $i)$ holds.

Verification of $ii)$ First of all, note that

$$I'_\infty(\psi_n^1)\phi^{(y)} = \int_\Omega (\nabla\psi_n^1\nabla\phi^{(y)} + 2\psi_n^1\phi^{(y)}) + \int_\Omega F_1'(\psi_n^1)\phi^{(y)} - \int_\Omega F_2'(\psi_n^1)\phi^{(y)}. \tag{3.14}$$

In order to prove the item $ii)$, we will need to show the following claim

Claim 3.1

$$\sup_{y\in\mathbb{R}^N}\int_\Omega |F_i'(u_n-u_0) - (F_i'(u_n) - F_i'(u_0))||\phi^{(y)}| = o_n(1), \quad \textit{for} \quad i\in\{1,2\}.$$

In the proof of the claim above, we adapt some ideas presented in [12, Proof of (3.39)]. In what follows, we will only show that the claim for function F_1, because the proof for F_2 follows by using similar arguments (see also [4, Lemma 3.1]).

Given $\varepsilon > 0$ and $r \in (1,2)$, the definition of F_1 guarantees that there is $t_0 > 0$ small enough such that

$$|F_1'(t)| \leq \varepsilon|t|^{r-1}, \ \ |t| \leq 2t_0. \tag{3.15}$$

On the other hand, note that it is possible to get $t_1 > t_0$ large enough such that

$$|F_1'(t)| \leq \varepsilon|t|^{2^*-1}, \ \ |t| \geq t_1 - 1, \tag{3.16}$$

as well as

$$|F_1'(t) - F_1'(s)| \leq \varepsilon |t_0|^{r-1}, \;\; |t-s| \leq s_0, \;\text{ and }\; |t|, |s| \leq t_1 + 1, \tag{3.17}$$

for some $s_0 > 0$ small enough. Therefore,

$$|F_1'(t)| \leq C_\varepsilon |t|^{r-1} + \varepsilon |t|^{2^*-1}, \quad t \in \mathbb{R}, \tag{3.18}$$

for some $C_\varepsilon > 0$. Now, fixing $R > 0$ of such way $B_R^c(0) \subset \Omega$ and using then fact that F_1 has a subcritical growth, it is easy to prove that

$$\int_{B_R(0)\cap\Omega} \mid F_1'(u_n - u_0) - (F_1'(u_n) - F_1'(u_0)) \mid\mid \phi^{(y)} \mid = o_n(1), \quad \text{uniformly in} \quad y \in \mathbb{R}^N.$$

Our next step is to estimate the integral below

$$\int_{B_R^c(0)\cap\Omega} \mid F_1'(u_n - u_0) - (F_1'(u_n) - F_1'(u_0)) \mid\mid \phi^{(y)} \mid .$$

Fix $\varepsilon > 0$. From (3.18), since $R > 0$ can be chosen large enough, one has

$$\begin{aligned}\int_{B_R^c(0)\cap\Omega} \mid F_1'(u_0) \mid\mid \phi^{(y)} \mid &\leq C_\varepsilon \int_{B_R^c(0)\cap\Omega} \mid u_0 \mid^{r-1} \mid \phi^{(y)} \mid + \varepsilon \int_{B_R^c(0)\cap\Omega} \mid u_0 \mid^{2^*-1} \mid \phi^{(y)} \mid \\ &\leq C(||u_0||_2^{r-1} ||\phi^{(y)}||_{\frac{2}{3-r}} + ||u_0||_{2^*}^{2^*-1} ||\phi^{(y)}||_{2^*}) \\ &\leq \varepsilon C ||\phi^{(y)}||_Y,\end{aligned} \tag{3.19}$$

where C does not depend on $y \in \mathbb{R}^N$. Setting

$$A_n := \{x \in B_R^c(0);\, |u_n(x)| \leq t_0\}$$

and

$$B_n := \{x \in B_R^c(0);\, |u_n(x)| \geq t_1\},$$

we have by (3.15),

$$\begin{aligned}&\int_{A_n \cap [|u_0| \leq \delta]} \mid F_1'(u_n - u_0) - F_1'(u_n) \mid\mid \phi^{(y)} \mid \leq \\ &\leq \varepsilon \int_{A_n \cap [|u_0| \leq \delta]} (\mid u_n - u_0 \mid^{r-1} \mid \phi^{(y)} \mid + \mid u_n \mid^{r-1} \mid \phi^{(y)} \mid \leq \\ &\leq \varepsilon\, C ||\phi||_Y,\end{aligned} \tag{3.20}$$

where C does not depend on $y \in \mathbb{R}^N$. Here, we have explored the fact that $|\mathrm{supp}\, \phi^{(y)}| = |\mathrm{supp}\, \phi|$ for any $y \in \mathbb{R}^N$. In a similar way, by using (3.16),

$$\int_{B_n \cap [|u_0| \leq \delta]} \mid F_1'(u_n - u_0) - F_1'(u_n) \mid\mid \phi^{(y)} \mid \leq \varepsilon C ||\phi||_Y. \tag{3.21}$$

Next, let us consider $C_n := \{x \in B_R^c(0);\, t_0 \leq |u_n(x)| \leq t_1\}$. Accounting that (u_n) is a bounded sequence in X, we derive that

$$M := \sup_{n\in\mathbb{N}} |C_n| < \infty.$$

Thereby, by (3.17),

$$\int_{C_n\cap[|u_0|\leq\delta]} | F_1'(u_n-u_0) - F_1'(u_n) \,||\, \phi^{(y)} | \leq t_0^{r-1}\varepsilon |C_n|^{1/2} ||\phi^{(y)}||_2 \leq \varepsilon C ||\phi||_Y, \quad (3.22)$$

for a convenient C independent of ε and $y \in \mathbb{R}^N$. From (3.20), (3.21) and (3.22),

$$\int_{B_R^c(0)\cap[|u_0|\leq\delta]} | F_1'(u_n-u_0) - F_1'(u_n) \,||\, \phi^{(y)} | \leq \varepsilon C ||\phi||_Y. \quad (3.23)$$

Now, we are going to analyze the case that $|u_0| > \delta$. The boundedness of (u_n) in X together with the inequality (3.18) give

$$\int_{B_R^c(0)\cap[|u_0|>\delta]} | F_1'(u_n-u_0) - F_1'(u_n) \,||\, \phi^{(y)} | \leq$$
$$\leq C_\varepsilon \int_{B_R^c(0)\cap[|u_0|>\delta]} (| \, u_n - u_0 \,|^{r-1} | \, \phi^{(y)} \,| + | \, u_n \,|^{r-1} | \, \phi^{(y)} \,| + \varepsilon C ||\phi||_Y,$$

where C is independent of ε and y. Since $u_0 \in X \subset H_0^1(\Omega)$, one has

$$|B_R^c(0) \cap [|u_0| > \delta]| \longrightarrow 0, \quad \text{as} \quad R \to +\infty.$$

Thereby,

$$C_\varepsilon \int_{B_R^c(0)\cap[|u_0|>\delta]} (| \, u_n - u_0 \,|^{r-1} | \, \phi^{(y)} \,| + | \, u_n \,|^{r-1} | \, \phi^{(y)} \,| \leq$$
$$\leq C_\varepsilon (||(u_n - u_0||_{2^*}^{r-1} + ||u_n||_{2^*}^{r-1}) ||\phi||_{2^*} |B_R(0)^c \cap [|u_0| > \delta]|^{(2^*-r)/2^*} \leq$$
$$\leq \varepsilon C ||\phi||_Y,$$

for $R > 0$ large enough and C independent of ε and y. Using the last information together with (3.19) and (3.23), one finds

$$\sup_{y\in\mathbb{R}^N} \int_{B_R^c(0)\cap\Omega} | F_1'(u_n-u_0) - (F_1'(u_n) - F_1'(u_0)) \,||\, \phi^{(y)} | \leq \varepsilon C ||\phi||.$$

Since ε is an arbitrary positive number, the last inequality with $||\phi||_Y \leq 1$ ensures that the Claim 3.1 is valid for the function F_1 and this finishes the proof of the claim.

Now, we are ready to show the item $ii)$. In fact, fix $\phi \in C_0^\infty(\Omega)$. So, by (3.14),

$$[I_\infty'(\psi_n^1) - (I'(u_n) - I'(u_0))](\phi^{(y)}) = \int_\Omega [F_1'(u_n-u_0) - (F_1'(u_n) - F_1'(u_0))]\phi^{(y)} +$$
$$+ \int_\Omega [F_2'(u_n-u_0) - (F_2'(u_n) - F_2'(u_0))]\phi^{(y)}.$$

Hence, by Claim 3.1,

$$\sup_{y\in\mathbb{R}^N} |I'_\infty(\psi_n^1) - (I'(u_n) - I'(u_0))|||\phi^{(y)}||_Y = o_n(1),$$

from where it follows that

$$\sup_{y\in\mathbb{R}^N} \|I'_\infty(\psi_n^1)\| \, ||\phi^{(y)}||_Y = o_n(1),$$

and the item $ii)$ is proved. If $\psi_n^1 \to 0$, then the proof would be finished. Thereby, in order to get the desired result, let us consider that

$$\psi_n^1 \nrightarrow 0 \;\text{ in }\; Y. \tag{3.24}$$

In this way, we can prove that the following claim holds

Claim 3.2 *There exist* $\lambda_0 > 0$ *and* $n_0 \in \mathbb{N}$ *such that*

$$I_\infty(\psi_n^1) \geq \lambda_0, \;\; \forall n \geq n_0.$$

Otherwise, considering a subsequence of (ψ_n^1) if necessary, we would have

$$I_\infty(\psi_n^1) \leq o_n(1).$$

Now, recalling that

$$F_2'(t)t - F_1'(t)t = t^2 \log t^2 + t^2, \;\; t \in \mathbb{R}$$

the same arguments explored in the proof of item $i)$ ensure that

$$I'_\infty(\psi_n^1)\psi_n^1 = I'(u_n)u_n - I'(u_0)u_0 = o_n(1),$$

and so,

$$I_\infty(\psi_n^1) = I_\infty(\psi_n^1) - \frac{1}{2}I'_\infty(\psi_n^1)\psi_n^1 + o_n(1) = \frac{1}{2}\int_{\mathbb{R}^N} |\psi_n^1|^2 + o_n(1).$$

Consequently, one finds $\displaystyle\int_{\mathbb{R}^N} |\psi_n^1|^2 = o_n(1)$, and by interpolation, $\displaystyle\int_{\mathbb{R}^N} |\psi_n^1|^p = o_n(1)$. So, the growth condition on F_2 allows us to conclude that

$$\int_{\mathbb{R}^N} F_2'(\psi_n^1)\psi_n^1 = o_n(1).$$

From the computations above, one has

$$||\psi_n^1||^2_{H_1(\mathbb{R}^N)} + \int_{\mathbb{R}^N} F_1'(\psi_n^1)\psi_n^1 = o_n(1),$$

which contradicts (3.24). Then, the claim is proved.

Now, lets us consider a decomposition of $\mathbb{R}^N$ into unit hypercubes B_i with vertices having integer coordinates and set

$$d_n := \max_{i\in\mathbb{N}} ||\psi_n^1||_{L^p(B_i)},$$

for a fixed $p \in (2, 2^*)$.

Claim 3.3 *There exist $\lambda_1 > 0$ and $n_1 \in \mathbb{N}$ such that*

$$d_n \geq \lambda_1, \quad \forall n \geq n_1.$$

Arguing as in the last claim,

$$I'_\infty(\psi_n^1)\psi_n^1 = I'(u_n)u_n - I'(u_0)u_0 = o_n(1),$$

and so

$$||\psi_n^1||^2_{H^1(\mathbb{R}^N)} + \int_{\mathbb{R}^N} F'_1(\psi_n^1)\psi_n^1 = \int_{\mathbb{R}^N} F'_2(\psi_n^1)\psi_n^1 + o_n(1).$$

By (2.4),

$$C\left(||\psi_n^1||^2_{H^1(\mathbb{R}^N)} + \int_{\mathbb{R}^N} F_1(\psi_n^1)\right) \leq \int_{\mathbb{R}^N} F'_2(\psi_n^1)\psi_n^1 + o_n(1),$$

for some constant $C > 0$. Combining this inequality with (P_2), one finds

$$\begin{aligned} I_\infty(\psi_n^1) &= \frac{1}{2}||\psi_n^1||^2_{H^1(\mathbb{R}^N)} + \int_{\mathbb{R}^N} F_1(\psi_n^1) - \int_{\mathbb{R}^N} F_2(\psi_n^1) \leq \\ &\leq C\int_{\mathbb{R}^N} |\psi_n^1|^p + o_n(1) = C\sum_{i\in\mathbb{N}} ||\psi_n^1||^p_{L^p(B_i)} + o_n(1). \end{aligned}$$

Since each B_i is a unit hypercube of $\mathbb{R}^N$, there is a constant $\tilde{C} > 0$ independent of i such that

$$||\psi_n^1||_{L^p(B_i)} \leq \tilde{C}||\psi_n^1||_{H^1(B_i)}, \quad \forall i \in \mathbb{N}. \tag{3.25}$$

Hence, modifying $\tilde{C} > 0$ if necessary, it holds

$$I_\infty(\psi_n^1) + o_n(1) \leq Cd_n^{p-2}\sum_{i\in\mathbb{N}} ||\psi_n^1||^2_{H^1(B_i)} \leq Md_n^{p-2},$$

for some $M > 0$. Now, we apply Claim 3.2 to get the desired result.

Hereafter, for our goals, let us consider y_n^1 the center of B_i in such way that

$$d_n = ||\psi_n^1||_{L^p(B_i)}.$$

In this way, one can see that, by taking a subsequence, $|y_n^1| \to \infty$. Otherwise, for some $R > 0$ large enough we must have

$$\int_{B_R(0)} |\psi_n^1|^p \geq \int_{B_i} |\psi_n^1|^p = d_n^p \geq \lambda_1^p > 0,$$

which is a contradiction, because the weak limit $\psi_n^1 \rightharpoonup 0$ in Y implies that

$$\int_{B_R(0)} |\psi_n^1|^p \longrightarrow 0.$$

Thereby, we may assume that $|y_n^1| \to \infty$.

Notice that, by the invariance of translations of $\mathbb{R}^N$, we conclude that $(\psi_n^1(\cdot + y_n^1))$ is bounded in Y. Then, for some $u_1 \in Y$,

$$\psi_n^1(\cdot + y_n^1) \rightharpoonup u_1 \text{ in } Y. \tag{3.26}$$

Our next step is to prove that u_1 is a nontrivial solution of (P_∞).

Claim 3.4 *The function u_1 is a nontrivial solution of (P_∞).*

Initially, let us prove that $u_1 \neq 0$. To see why, let us denote by B_0 the unit hypercube of $\mathbb{R}^N$ centered at the origin. Then, by the Claim 3.3,

$$\int_{B_0} |\psi_n^1(\cdot + y_n^1)|^p = \int_{B_i} |\psi_n^1|^p = d_n^p \geq \lambda_1^p > 0.$$

Observe that, by (3.26), $\psi_n^1(\cdot + y_n^1) \to u_1$ in $L^p(B_0)$. Hence,

$$\int_{B_0} |u_1|^p \geq \lambda_1^p > 0,$$

showing that $u_1 \neq 0$.

Set

$$\Omega_n := \{x \in \mathbb{R}^N;\, x + y_n^1 \in \Omega\}.$$

Note that, for each $v \in C_0^\infty(\mathbb{R}^N)$, we have that $\operatorname{suppt} v \subset \Omega_n$ for n large enough. Setting $v^{(n)}(x) := v(x - y_n^1)$, it follows that

$$\operatorname{suppt} v^{(n)} \subset \Omega \quad \text{and} \quad v^{(n)} \in H_0^1(\Omega).$$

Taking $v \in C_0^\infty(\mathbb{R}^N)$ with $||v||_Y \leq 1$, we see that $||v^{(n)}||_X = 1$ and

$$I'_\infty(\psi_n^1(\cdot + y_n^1))v = I'_\infty(\psi_n^1)v^{(n)}.$$

Thus, by item *ii*), $I'_\infty(\psi_n^1(\cdot + y_n^1))v = o_n(1)$. On the other hand, standard arguments involving the weak convergence of $(\psi_n^1(\cdot + y_n^1))$ yield

$$I'_\infty(\psi_n^1(\cdot + y_n^1))v = I'_\infty(u_1)v.$$

By gathering these information, we derive that $I'_\infty(u_1)v = 0$, then u_1 is a nontrivial critical point of I_∞, and so, u_1 is a solution of (P_∞).

Define $\psi_n^2 := (\psi_n^1(\cdot + y_n^1) - u_1)$. If $\psi_2^n \to 0$, then the proof is finished. Otherwise, we use the fact that $\psi_n^2 \rightharpoonup 0$ and the ideas explored above to find a unbounded sequence (y_n^2) of $\mathbb{R}^N$ and to produce $u_2 \in Y$ a nontrivial solution of (P_∞). Continuing with this procedure, for each $j \geq 2$ it is possible to define

$$\psi_n^j := \psi_n^{j-1}(\cdot + y_n^{j-1}) - u_{j-1},$$

with

$$\begin{cases} y_n^{j-1} \to \infty \\ \psi_n^{j-1} \rightharpoonup u_{j-1}, \end{cases}$$

and u_{j-1} a nontrivial solution of (P_∞). By exploring the same type of argument used in the prove of item *i*), one can prove that

iii): $||\psi_n^j||^2_{H^1(\mathbb{R}^N)} = ||u_n||^2_{H_0^1(\Omega)} - ||u_0||^2_{H_0^1(\Omega)} - \sum_{i=1}^{j-1} ||u_i||^2_{H^1(\mathbb{R}^N)} + o_n(1);$

iv): $I_\infty(\psi_n^j) = I(u_n) - I(u_0) - \sum_{i=1}^{j-1} I_\infty(u_i) + o_n(1).$

v): $\liminf_{n\to\infty} I_\infty(\psi_n^j) > 0$ for each $j \in \mathbb{N}$.

We finish the proof by proving that the following claim holds.

Claim 3.5 *There is a number $k \in \mathbb{N}$ such that $\psi_n^k \to 0$ in Y.*

In fact, otherwise it would be possible to get by the preceding procedure a nontrivial solution u_j of (P_∞) for each $j \in \mathbb{N}$, and so,

$$I_\infty(u_j) \geq d_\infty = \inf_{u \in \mathcal{N}_\infty} I_\infty(u) > 0, \ \ \forall j \in \mathbb{N}.$$

Thus, from *iv*),

$$I_\infty(\psi_n^j) \leq I(u_n) - I(u_0) - (j-1)d_\infty + o_n(1).$$

As $(I(u_n))$ is a bounded sequence, for j large enough the last inequality implies that $\liminf_{n\to\infty} I_\infty(\psi_n^j) < 0$, which contradicts *v*). From this, the Claim 3.5 is proved and the proof is over. ■

3.2 Technical Results

In this section we prove some technical results that are crucial in the proof of Theorem 3.1. The main goal is to prove that $I|_{\mathcal{N}}$ satisfies the $(PS)_c$ condition for all $c \in (d_\infty + \varepsilon, 2d_\infty - \varepsilon)$, for some $\varepsilon > 0$ small enough.

In the sequel,

$$\chi(t) := \begin{cases} 1, & 0 \leq t \leq R; \\ \dfrac{R}{t}, & R \leq t, \end{cases}$$

where $R > 0$ is such that $\Omega^c \subset B_R(0)$. Next, let $\tau : Y \longrightarrow \mathbb{R}^N$ be given by

$$\tau(u) := \int_{\mathbb{R}^N} |u|^2 \chi(|x|)x$$

and set

$$P := \{u \in X;\, u \geq 0\} \quad \text{and} \quad T_0 := \{u \in \mathcal{N} \cap P;\, \tau(u) = 0\}.$$

Employing the above notations, let us define the level

$$c_0 := \inf_{u \in T_0} I(u),$$

which satisfies

$$d_\infty = d_0 \leq c_0. \tag{3.27}$$

Our first result is the following

Lemma 3.5 *The number c_0 satisfies $d_\infty < c_0$.*

Proof. Arguing by contradiction, in view of (3.27), if the lemma does not hold, then it occurs

$$d_\infty = d_0 = c_0.$$

Thus, it is possible to take a sequence (v_n) in $\mathcal{N} \cap P$ such that

$$\tau(v_n) = 0 \text{ and } I(v_n) \longrightarrow d_0 = \inf_{u \in \mathcal{N}} I(u).$$

By applying the Ekeland's Variational Principle, there is a sequence (u_n) in $\mathcal{N}$ satisfying $I(u_n) \leq I(v_n)$, $||u_n - v_n||_X = o_n(1)$ and (u_n) is also a $(PS)_{d_0}$ sequence for $I|_{\mathcal{N}}$ (see e.g. [83, Theorem 8.5]). Thanks to Lemma 3.4, there are $k \in \mathbb{N}$ and nontrivial solutions $u_1, ..., u_k$ of (P_∞) with

$$||u_n||^2_{H^1_0(\Omega)} \longrightarrow ||u_0||^2_{H^1_0(\Omega)} + \sum_{j=1}^{k} ||u_j||^2_{H^1(\mathbb{R}^N)} \tag{3.28}$$

and

$$I(u_n) \longrightarrow I(u_0) + \sum_{j=1}^{k} I_\infty(u_j), \tag{3.29}$$

where u_0 has been chosen in a such way that $u_n \rightharpoonup u_0$ and u_0 is a solution of (P_0). Using the fact that $d_\infty = d_0$, it holds

$$I(u_0) + \sum_{j=1}^{k} I_\infty(u_j) \geq I(u_0) + kd_0.$$

Since $I(u_n) \to d_0$ and $I(u_0) \geq 0$, from (3.29) one has $k = 0$ or $k = 1$. If $k = 0$, accounting (3.28), we find

$$u_n \longrightarrow u_0 \text{ in } \quad H_0^1(\Omega).$$

Now, as (u_n) is a $(PS)_{d_0}$ sequence for $I|_{\mathcal{N}}$ (and also for I) and u_0 is a solution of (P_0), one gets

$$\begin{aligned} ||u_n||^2_{H_0^1(\Omega)} + \int_\Omega F_1'(u_n)u_n = \int_\Omega F_2'(u_n)u_n = \\ = \int_\Omega F_2'(u_0)u_0 + o_n(1) = \\ = ||u_0||^2_{H_0^1(\Omega)} + \int_\Omega F_1'(u_0)u_0 + o_n(1), \end{aligned}$$

that is,

$$||u_n||^2_{H_0^1(\Omega)} + \int_\Omega F_1'(u_n)u_n \longrightarrow ||u_0||^2_{H_0^1(\Omega)} + \int_\Omega F_1'(u_0)u_0.$$

In particular, one has

$$||u_n||^2_{H_0^1(\Omega)} \longrightarrow ||u_0||^2_{H_0^1(\Omega)} \text{ and } \int_\Omega F_1'(u_n)u_n \longrightarrow \int_\Omega F_1'(u_0)u_0,$$

which yields that $u_n \to u_0$ in $H_0^1(\Omega)$ and $u_n \to u_0$ in $L^{F_1}(\Omega)$, since $F_1 \in (\Delta_2)$. From this, $u_n \to u_0$ in X, and so,

$$I(u_n) \longrightarrow I(u_0) = d_0,$$

showing that u_0 is a ground state solution for (P_0), which contradicts Theorem 3.4. So, $k = 1$ and $u_0 = 0$. Otherwise, if $u_0 \neq 0$, the function u_0 would be a nonzero solution of (P_0), and so,

$$d_0 = \lim I(u_n) \geq 2d_0,$$

giving a new contradiction. By following the notation in the proof of Lemma 3.4, one finds

$$\begin{cases} u_n(x + y_n^1) = \psi_n^1(x + y_n^1) \rightharpoonup u_1; \\ y_n^1 \to \infty. \end{cases}$$

Note also that $||u_n||^2_{H^1_0(\Omega)} \to ||u_1||^2_{H^1_0(\Omega)}$ and $I_\infty(u_1) = d_\infty$. Thus, u_1 is a ground state solution of (P_∞).

Now, on accounting of Theorem 3.3 one can gets a contradiction by following the same ideas in [27, Lemma 4.3]. For the sake of completeness, we recall some steps made in [27, Lemma 4.3]. Denote, by simplicity, $y_n := y_n^1$,

$$(\mathbb{R}^N)^+_n := \{x \in \mathbb{R}^N;\ \langle x, y_n\rangle_{\mathbb{R}^N} > 0\},$$

$$(\mathbb{R}^N)^-_n := \mathbb{R}^N - (\mathbb{R}^N)^+_n,$$

and

$$w_n(x) := u_n(x) - u_1(x - y_n).$$

The above information gives $w_n \to 0$ in $H^1(\mathbb{R}^N)$.

By Theorem 3.3, without loss of generality we may assume that u_1 is a radially symmetric solution of (P_∞). In the same way as [27, Lemma 4.3] (see also [2, Lemma 4.3]), we derive that

$$\begin{cases} u_1(x - y_n) \geq \dfrac{1}{2}u_1(0) > 0, \ \ x \in B_r(y_n); \\ u_1(x - y_n) \to 0, \ \ \text{a.e } x \in (\mathbb{R}^N)^-_n \ \ \text{and} \ \ \displaystyle\int_{(\mathbb{R}^N)^-_n} |u_1(x - y_n)|^2 \chi(|x|)|x| = o_n(1), \end{cases}$$

for some $r > 0$, as well as

$$\langle \tau(u_1(x - y_n)), y_n/|y_n|\rangle_{\mathbb{R}^N} \geq C > 0, n \geq n_0, \tag{3.30}$$

for some $C > 0$. On the other hand, taking into accounting that $\tau(u_1(\cdot - y_n)) = \tau(u_n - w_n)$, and that $|\tau(u_n)|$, $|\tau(w_n)| = o_n(1)$, we derive that

$$|\tau(u_1(x - y_n))| = o_n(1). \tag{3.31}$$

From (3.30)-(3.31), we find a contradiction, finishing the proof. ■

Hereafter we will fix $\rho > 0$ as the smallest positive number such that $\Omega^c \subset B_\rho(0)$. Let $\phi(x) := \varphi(\frac{|x|}{\rho})$, where $\varphi \in C_0^\infty([0, \infty))$ is an increasing function such that $\varphi(t) = 0$, $0 \leq t \leq 1$, and $\varphi(t) = 1$, $t \geq 2$. Now, for each $y \in \mathbb{R}^N$, we set

$$\psi_{y,\rho}(x) := \phi(x)u_\infty(x - y),$$

where $u_\infty \in \mathcal{N}_\infty$ is a ground state solution of (P_∞), which is assumed to be a decreasing and radially symmetric at the origin. Finally, fix $t_{y,\rho} > 0$ satisfying

$$\phi_\rho(y) := t_{y,\rho}\psi_{y,\rho} \in \mathcal{N}_\infty.$$

Next, we prove an important property related to the mappings $\phi_\rho(y)$.

Lemma 3.6 *The family of mappings $(\phi_\rho(y))$ satisfies the following limits:*

$i)$: $\lim\limits_{\rho\to 0} I_\infty(\phi_\rho(y)) = d_\infty$, uniformly in $y \in \mathbb{R}^N$;

$ii)$: For each fixed $\rho > 0$, it holds $\lim\limits_{|y|\to\infty} I_\infty(\phi_\rho(|y|)) = d_\infty$.

Proof. Verification of $i)$: From the definition of $\psi_{y,\rho}$ and the properties of u_∞ (see Theorem 3.3 above), for each fixed $p \in [2, 2^*]$, one has

$$\begin{aligned}||\psi_{y,\rho} - u_\infty(\cdot - y)||_p^p &\leq C\int_{B_{2\rho}(0)} |u_\infty(\cdot - y)|^p \\ &\leq C\int_{B_{2\rho}(0)} |u_\infty(0)|^p \\ &\leq \tilde{C}\rho^N = o_\rho(1), \quad \forall y \in \mathbb{R}^N.\end{aligned}$$

Similarly, since $N \geq 3$,

$$\begin{aligned}||\nabla(\psi_{y,\rho} - u_\infty(\cdot - y))||_2^2 &\leq C\int_{B_{2\rho}(0)} |\nabla\phi|^2|u_\infty(\cdot - y)|^2 + C\int_{B_{2\rho}(0)} |\phi(x) - 1|^2|\nabla u_\infty(\cdot - y)|^2 \\ &\leq C_1\rho^N + C_2\rho^{N-2}, \quad \forall y \in \mathbb{R}^N.\end{aligned}$$

Hence,

$$||\psi_{y,\rho}||_p \longrightarrow ||u_\infty(\cdot - y)||_p \ \text{ as } \ \rho \to 0,$$

as well as

$$||\psi_{y,\rho}||_{H^1(\mathbb{R}^N)} \longrightarrow ||u_\infty(\cdot - y)||_{H^1(\mathbb{R}^N)}, \ \text{ as } \ \rho \to 0,$$

uniformly in $y \in \mathbb{R}^N$. From this,

$$\int_{\mathbb{R}^N} F_2(\psi_{y,\rho}) \longrightarrow \int_{\mathbb{R}^N} F_2(u_\infty), \ \text{ as } \ \rho \to 0,$$

uniformly in $y \in \mathbb{R}^N$. Now, using the definition of $\psi_{y,\rho}$, one gets

$$\int_{\mathbb{R}^N} |F_1(\psi_{y,\rho}) - F_1(u_\infty(\cdot - y))| = \int_{B_\rho(0)} |F_1(\psi_{y,\rho}) - F_1(u_\infty(\cdot - y))|. \tag{3.32}$$

By the mean value theorem,

$$\int_{B_\rho(0)} |F_1(\psi_{y,\rho}) - F_1(u_\infty(\cdot - y))| = \int_{B_\rho(0)} |F_1'(\theta_{y,\rho})||\phi(x) - 1||u_\infty(\cdot - y))|, \qquad (3.33)$$

where $|\theta_{y,\rho}| \leq |\psi_{y,\rho}| + |u_\infty(\cdot - y)|$. Then, since $(\theta_{y,\rho}) \subset \mathbb{R}$ is a bounded and $F_1 \in C^1(\mathbb{R})$, we derive that

$$\int_{B_\rho(0)} |F_1'(\theta_{y,\rho})||\phi(x) - 1||u_\infty(\cdot - y))| \leq C \int_{B_\rho(0)} |\phi(x) - 1||u_\infty(0)| = o_\rho(1).$$

From (3.32)-(3.33),

$$\int_{\mathbb{R}^N} F_1(\psi_{y,\rho}) \longrightarrow \int_{\mathbb{R}^N} F_1(u_\infty(\cdot - y)), \ \ \forall y \in \mathbb{R}^N.$$

Adapting the ideas used in the proof of Lemma 3.1, we can show that $t_{y,\rho} \to 1$ as $\rho \to 0$, and so,

$$||\phi_\rho(y)||_{H^1(\mathbb{R}^N)} = ||t_{y,\rho}\psi_{y,\rho}||_{H^1(\mathbb{R}^N)} \longrightarrow ||u_\infty||_{H^1(\mathbb{R}^N)} \ \text{as} \ \rho \to 0,$$

and

$$\int_{\mathbb{R}^N} F_i(\phi_\rho(y)) \longrightarrow \int_{\mathbb{R}^N} F_i(u_\infty), \ \ i \in \{1, 2\}.$$

The last convergences yield that

$$\lim_{\rho \to 0} I_\infty(\phi_\rho(y)) \longrightarrow I_\infty(u_\infty) = d_\infty,$$

uniformly in $y \in \mathbb{R}^N$, proving the part $i)$ of lemma.

Verification of $ii)$: The proof follows as in the proof Lemma 3.1 and it will be omitted.

■

A byproduct of the last lemma is the following corollary.

Corollary 3.1 *Given $\varepsilon \approx 0^+$, there exists $\rho_0 > 0$ such that*

$$\sup_{y \in \mathbb{R}^N} I_\infty(\phi_\rho(y)) < 2d_\infty - \varepsilon, \quad \forall \rho \in (0, \rho_0).$$

Next, we establish more two important properties of the mappings $\phi_\rho(y)$.

Lemma 3.7 *Fixed $\rho > 0$, there exists $R_0 > \rho$ such that*

$i)$: $d_\infty < I(\phi_\rho(y)) < \frac{c_0 + d_\infty}{2}$, $|y| \geq R_0$;

$ii)$: $\langle \tau(\phi_\rho(y)), y \rangle$, $|y| = R_0$.

Proof. Verification of i**):** By the definition of $\phi_\rho(y)$,

$$d_\infty \leq I_\infty(\phi_\rho(y)) = I(\phi_\rho(y)).$$

On the other hand, as $d_\infty = d_0$ (see Lemma 3.1) and (P_0) has no ground state solution, it follows that

$$d_\infty < I(\phi_\rho(y)), \quad \text{for any} \quad \rho > 0 \quad \text{and} \quad y \in \mathbb{R}^N.$$

Finally, note that, by part ii) of Lemma 3.6,

$$I(\phi_\rho(y)) < \frac{c_0 + d_\infty}{2}, \;\; |y| \geq R_0,$$

for some $R_0 > 0$ large enough, because $c_0 > d_\infty$. This completes the proof of item i).
Verification of ii**):** The proof follows as in [27, Lemma 4.3 (b)]. ■

We finish this section by showing that $I|_{\mathcal{N}}$ satisfies the $(PS)_c$ for some levels $c \in \mathbb{R}$.

Proposition 3.3 *For each fixed* $\varepsilon \approx 0^+$*, the functional* $I|_{\mathcal{N}}$ *satisfies the* $(PS)_c$ *condition for* $c \in (d_\infty + \varepsilon, 2d_\infty - \varepsilon)$*.*

Proof. Let (u_n) be a $(PS)_c$ sequence for $I|_{\mathcal{N}}$. By Lemma 3.2, we know that (u_n) is a bounded sequence in X. Since X is a reflexive space, we may assume that

$$u_n \rightharpoonup u_0 \;\text{ in }\; X.$$

If $u_n \nrightarrow u_0$, by Lemma 3.4 there are $u_1, ..., u_k$ solutions of (P_∞) such that

$$||u_n||^2_{H^1_0(\Omega)} \longrightarrow ||u_0||^2_{H^1_0(\Omega)} + \sum_{j=1}^{k} ||u_j||^2_{H^1(\mathbb{R}^N)}$$

and

$$I(u_n) \longrightarrow I(u_0) + \sum_{j=1}^{k} I_\infty(u_j).$$

Supposing that $u_0 \neq 0$, we arrive at

$$I(u_n) \geq (k+1)d_\infty + o_n(1).$$

Since $k \geq 1$, it follows that

$$c \geq (k+1)d_\infty \geq 2d_\infty,$$

which is absurd, because $c < 2d_\infty$. This contradiction allows us to infer that $u_0 = 0$. Moreover, we must have $k = 1$, because if $k > 1$, then

$$I(u_n) \geq kd_\infty \geq 2d_\infty,$$

obtaining again a contradiction. From this, the unique possibility is $u_0 = 0$ and $u_1 > 0$, and so,

$$c + o_n(1) = I(u_n) = I_\infty(u_1) + o_n(1) = d_\infty + o_n(1).$$

The last equality implies that $c = d_\infty$, which is absurd. This reasoning shows that $u_n \to u_0$ and the proof is finished. ■

3.3 Existence of positive solution for (P_0) (Dirichlet case)

Along this section we show how the technical results of the preceding section imply in the existence of positive solution for (P_0). The key point is to show that the functional I possesses a $(PS)_c$ sequence in a suitable level $c \in (d_\infty + \varepsilon, 2d_\infty - \varepsilon)$, $\varepsilon \approx 0^+$. Bearing this in mind, set

$$G := \{\phi_\rho(y);\ |y| \leq R_0\}$$

and

$$H := \left\{\eta \in C(\mathcal{N} \cap P, \mathcal{N} \cap P);\ \eta(u) = u, \text{ if } I(u) < \frac{c_0 + d_\infty}{2}\right\}.$$

Hereafter, we are using the same notations introduced in Section 4. Now, fix

$$\Gamma := \{\eta(G);\ \eta \in H\}$$

and

$$c := \inf_{A \in \Gamma} \sup_{u \in A} I(u).$$

In view of Lemma 3.7-*ii*), as made in [9, 27], we can prove the lemma below.

Lemma 3.8 *It holds*

$$A \cap T_0 \neq \emptyset, \quad \forall A \in \Gamma.$$

Our second result in this section ensures that, for some convenient $\varepsilon > 0$, we must have

$c \in (d_\infty + \varepsilon, 2d_\infty - \varepsilon)$, which is a key step to show the $(PS)_c$ condition of I restricted to $\mathcal{N}$.

Lemma 3.9 *There exists $\varepsilon > 0$ such that $c \in (d_\infty + \varepsilon, 2d_\infty - \varepsilon)$.*

Proof. Using the preceding lemma, for each $A \in \Gamma$ there exists $u_0 \in A \cap T_0$. Therefore,

$$c_0 = \inf_{u \in T_0} I(u) \leq I(u_0) \leq \sup_{u \in A} I(u),$$

and so,

$$c_0 \leq c.$$

Take $\varepsilon \in (0, \frac{d_\infty}{2})$, $\varepsilon \approx 0^+$, such that

$$d_\infty + \varepsilon < c_0 \leq c, \tag{3.34}$$

which is possible in view of Lemma 3.5. On the other hand, since

$$c \leq \sup_{u \in A} I(u), \quad \forall A \in \Gamma,$$

we know that,

$$c \leq \sup_{\phi_\rho(y) \in G} I(\eta(\phi_\rho(y))), \quad \forall \eta \in H.$$

Choosing $\eta := Id_{(\mathcal{N} \cap P)}$ and applying the Corollary 3.1, one finds

$$c < 2d_\infty - \varepsilon,$$

for ε and ρ small enough. This combines with (3.34) to give

$$c \in (d_\infty + \varepsilon, 2d_\infty - \varepsilon).$$

■

Now we are able to prove that the problem (P_0) has a positive solution.

Proof of Theorem 3.1: Combining the preceding lemma with the Proposition 3.3, it suffices to show that $I|_{\mathcal{N}}$ has a $(PS)_c$ sequence in P. More precisely, we will prove that the following condition holds:

(D): For each $\lambda \in (0, c - \frac{c_0 + d_\infty}{2})$, there exists $u_\lambda \in I^{-1}([c - \lambda, c + \lambda])$ with $u_\lambda \in \mathcal{N} \cap P$ and

$$||I'(u_\lambda)||_* < \lambda.$$

Arguing by contradiction, we find $\lambda_0 \in (0, c - \frac{c_0+d_\infty}{2})$ such that

$$||I'(u_\lambda)||_* \geq \frac{\lambda_0}{2}, \quad \forall u \in I([c-\lambda_0, c+\lambda_0]) \cap (\mathcal{N} \cap P).$$

By applying the version of quantitative deformation lemma in [83], we get $\eta \in C([0,1] \times \mathcal{N} \cap P, \mathcal{N} \cap P)$ satisfying

$i)$: $\eta(t,u) = u, \ \ \forall u \in I^{-1}([c-\lambda_0, c+\lambda_0])$;

$ii)$: $\eta(1, I^{c+\frac{\lambda_0}{2}}) \subset I^{c-\frac{\lambda_0}{2}}$, with $I^d := \{u \in \mathcal{N} \cap P; \, I(u) \leq d\}$.

By the definition of c, it holds

$$\sup_{u \in A_0} I(u) \leq c + \frac{\lambda_0}{2},$$

for some $A_0 \in \Gamma$, that is,

$$A_0 \in I^{c+\frac{\lambda_0}{2}}.$$

Then, by item $ii)$,

$$\eta(1, A_0) \in I^{c-\frac{\lambda_0}{2}}. \tag{3.35}$$

Note that $A_0 = \eta_0(G)$ for some $\eta_0 \in H$. Setting $\gamma_1 := \eta(1,\cdot) \circ \eta_0$ we derive that $\gamma_1 \in C(\mathcal{N} \cap P, \mathcal{N} \cap P)$ and, if $I(u) < \frac{c_0+d_\infty}{2}$,

$$\gamma_1(u) = \eta(1, \eta_0(u)) = u$$

(Note that $c - \lambda_0 > \frac{c_0+d_\infty}{2}$). Thus, $\gamma_1 \in H$ and

$$\eta_(1, A_0) = \eta(1, \eta_0(G)) = \gamma_1(G) \in \Gamma.$$

Consequently, by (3.35),

$$c \leq \sup_{u \in \eta(1,A_0)} I(u) \leq c - \lambda_0.$$

This contradiction completes the proof. □

3.4 Existence of positive solution for (S_0) (Neumann case)

In this section, we study the existence of solution for the following class of problems

$$(S_0) \qquad \begin{cases} -\Delta u + u = Q(x) u \log u^2, & \text{in } \Omega \\ \dfrac{\partial u}{\partial \eta} = 0, & \text{in } \partial\Omega, \end{cases}$$

where Ω is an exterior domain as in the problem (P_0), and $Q : \mathbb{R}^N \longrightarrow \mathbb{R}$ is a continuous function satisfying the following conditions:

(Q_1) $\lim_{|x|\to\infty} Q(x) = Q_0$ and $q_0 := \inf_{x\in\mathbb{R}^N} Q(x) > 0$ for all $x \in \mathbb{R}^N$;

(Q_2) $Q_0 \geq Q(x) \geq Q_0 - Ce^{-M|x|^2}$, for $x \geq R_0$, $M \geq M_0$,

with $Q_0, C, M_0,\ R_0 > 0$.

The reader will see in this section that different of the Dirichlet case, we will prove that if $M_0 > 0$ is large enough, then the Problem (S_0) has a ground state solution.

Let $(E, ||\cdot||_E)$ be a Banach space and $d \in \mathbb{R}$. We recall that a *Cerami sequence* for a functional $J \in C^1(E, \mathbb{R})$ at level d (shortly $(C)_d$-*sequence*), is a sequence $(u_n) \subset E$ satisfying

$$J(u_n) \longrightarrow d \ \text{ and } \ (1 + ||u_n||_E)||J'(u_n)||_{E'} \longrightarrow 0.$$

We say that J verifies the Cerami condition at level d, or $(C)_d$-condition for short, if each $(C)_d$-sequence for J admits a convergent subsequence. Note that a $(C)_d$-sequence for J is also a $(PS)_d$-sequence. Therefore, if $u_n \to u_0$ and (u_n) is a $(C)_d$-sequence, then u_0 is a critical point of J. See [35] for further details.

Hereafter, we will need of the auxiliary problem below

$$(S_\infty) \qquad \begin{cases} -\Delta u + u = Q_0 u \log u^2, & \text{in } \mathbb{R}^N \\ u \in H^1(\mathbb{R}^N). \end{cases}$$

Note that, in view of the condition (Q_1), the problem (S_∞) is the limit problem of (S_0).

Applying the Theorem 3.3, by a change of variable, we get the uniqueness of positive solution for (S_∞). In fact, if u_1 is a solution for (3.1), by defining $v_1(x) := u_1(\sqrt{k^{-1}}x)$, by a direct computation, we find

$$-\Delta v_1 = -v_1 + \frac{1}{k} v_1 \log v_1^2 \ \text{ in } \mathbb{R}^N.$$

So, we get the existence and uniqueness of positive solution for (S_∞) by choosing $k = Q_0^{-1}$.

From now on, we may assume that, up to translations, the problem (S_∞) has a unique positive solution of the form

$$v_\infty(x) = C_1 e^{-C_2|x|^2}, \quad \forall x \in \mathbb{R}^N, \tag{3.36}$$

for convenient C_1, $C_2 > 0$.

Related with the problems (S_0) and (S_∞) we have the energy functionals

$$J(u) := \frac{1}{2}\int_\Omega (|\nabla u|^2 + (1+Q(x))|u|^2) + \int_\Omega Q(x)F_1(u) - \int_\Omega Q(x)F_2(u), \quad \forall u \in Z,$$

and

$$J_\infty(u) := \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + (1+Q_0)|u|^2) + \int_{\mathbb{R}^N} Q_0F_1(u) - \int_{\mathbb{R}^N} Q_0F_2(u), \quad \forall u \in Y,$$

with $Z := (H^1(\Omega) \cap L^{F_1}(\Omega), ||\cdot||_Z)$, $||\cdot||_Z := ||\cdot||_{H^1(\Omega)} + ||\cdot||_{L^{F_1}(\Omega)}$, and Y is chosen as in the previous sections. Thus, $J \in C^1(Z, \mathbb{R})$, $J_\infty \in C^1(Y, \mathbb{R})$ and critical points of J and J_∞ correspond respectively to solutions of (S) and (S_∞).

The Nehari sets associated with the functionals J and J_∞ respectively are given by

$$\mathcal{M} := \{u \in Z - \{0\};\ J'(u)u = 0\}$$

and

$$\mathcal{M}_\infty := \{u \in Y - \{0\};\ J'_\infty(u)u = 0\}.$$

Arguing as in the proof of Proposition 3.1, we also derive that the sets $\mathcal{M}$ and $\mathcal{M}_\infty$ are C^1-manifolds. Indeed, it suffices to replace Ψ_0 and Ψ_∞ in the proof of Proposition 3.1 by

$$\tilde{\Psi}_0(u) = J(u) - \frac{1}{2}\int_\Omega Q(x)|u|^2 \text{ and } \tilde{\Psi}_\infty(u) = J_\infty(u) - \frac{1}{2}\int_{\mathbb{R}^N} Q_0|u|^2,$$

respectively. From now on, we will denote by l_0 and l_∞ the levels

$$l_0 := \inf_{u \in \mathcal{M}} J(u) \text{ and } l_\infty := \inf_{u \in \mathcal{M}_\infty} J_\infty(u).$$

It is not difficulty to prove that the function v_∞ given in (3.36) satisfies

$$J_\infty(v_\infty) = l_\infty. \tag{3.37}$$

The next result is a version of Lemma 3.4 for the $(C)_d$-sequences of the functional J.

Lemma 3.10 *Let (v_n) be a $(C)_d$-sequence for J. Assume that $v_n \rightharpoonup v_0$. Then, going to a subsequence if necessary, either*

i) $v_n \to v_0$ in Z, or

ii) There exists $k \in \mathbb{N}$ and k nontrivial solutions v_j of (S_∞), $j \in \{1, ..., k\}$, satisfying

$$\left\| v_n - v_0 - \sum_{j=1}^{k} v_n^j \right\|_{H^1(\Omega)}^2 = o_n(1) \quad \text{and} \quad J(u_n) \to J(v_0) + \sum_{j=1}^{k} J_\infty(u_j),$$

with $v_n^j := v_j(\cdot \ - y_n^j)$, and $(y_n^j) \subset \mathbb{R}^N$ with $|y_n^j| \to \infty$ for each $j \in \{1, ..., k\}$.

Proof. The proof is a slight variant of the argument made in Lemma 3.4 (see also the ideas in [4, Lemma 3.3] and [27, Lemma 3.1]). In fact, since (v_n) is $(C)_d$-sequence for J, it holds $J'(v_n)v_n = o_n(1)$. So, it is possible to prove that (v_n) is bounded in the same way of the proof of Lemma 3.4. From this, it follows that (v_n) is a bounded $(PS)_d$ sequence for J. Accounting that $v_n \rightharpoonup v_0$, we derive that $J'(v_0) = 0$, and so, v_0 is a solution of (S_0). Following the ideas in the proof of Lemma 3.4, setting

$$\xi_n^1(x) := v_n(x) - v_0(x), \ \text{ in } \ \Omega;$$

we find that

$$\xi_n^1 \rightharpoonup 0 \ \text{ in } \ Z.$$

Then, if $\xi_n^1 \to 0$ in Z, the proof would be finished. Otherwise, if $\xi_n^1 \not\to 0$ in Z, arguing as in the proof of Lemma 3.4, see items $i) - ii)$, we find

$$J(\xi_n^1) = J(v_n) - J(v_0) + o_n(1) \tag{3.38}$$

and

$$J'(\xi_n^1)\xi_n^1 = J'(v_n)v_n - J'(v_0)v_0 + o_n(1). \tag{3.39}$$

In the same line of Lemma 3.4, let us consider $(y_n^1)_{n\in\mathbb{N}}$ in $\mathbb{R}^N$, with y_n^1 the centers of unit N-dimensional hypercubes B_i, $\mathbb{R}^N = \bigcup_{i\in\mathbb{N}} B_i$, and verify

$$||\xi_n^1||^p_{L^p(\tilde{B}_i)} = \max_{j\in\mathbb{N}} ||\xi_n^1||^p_{L^p(\tilde{B}_j)} := \delta_n,$$

where $\tilde{B}_i = (B_i \cap \Omega)$. Next, we are going to guarantee that

$$\delta_n \geq \tau_0 > 0, \ \ n \geq n_0,$$

for some $n_0 \in \mathbb{N}$, and

$$|y_n^1| \to \infty.$$

In the sequel, we set

$$\tilde{\xi}_n(x) = \xi_n^1(x + y_n^1), \quad \Omega_n^1 = \{y - y_n^1;\, y \in \Omega\}, \;\; X_n := H^1(\Omega_n^1) \cap L^{F_1}(\Omega_n^1)$$

and the functional $J_n : X_n \longrightarrow \mathbb{R}$ given by

$$J_n(u) := \frac{1}{2}\int_{\Omega_n^1} (|\nabla u|^2 + (1+Q(x+y_n^1))|u|^2) + \int_{\Omega_n^1} Q(x+y_n^1)F_1(u) - \int_{\Omega_n^1} Q(x+y_n^1)F_2(u), \quad u \in X_n.$$

The following claim holds.

Claim 3.6 *The sequence $\tilde{\xi}_n$ is such that*

$$J_n(\tilde{\xi}_n) \geq \tau_1 > 0, \tag{3.40}$$

for some $\tau_1 \in \mathbb{R}$.

It suffices to show that

$$\inf_{n\in\mathbb{N}} \left(\frac{1}{2}\int_{\Omega_n^1} (|\nabla \tilde{\xi}_n|^2 + (1 + Q(x + y_n^1))|\tilde{\xi}_n|^2) + \int_{\Omega_n^1} Q(x + y_n^1)F_1(\tilde{\xi}_n) - \int_{\Omega_n^1} Q(x + y_n^1)F_2(\tilde{\xi}_n) \right)$$

is a positive number.

Arguing as in the Claim 3.2, by considering (3.39) and the condition (Q_1), we find

$$J_n(\tilde{\xi}_n) = \int_{\Omega_n^1} Q(x + y_n^1)|\tilde{\xi}_n|^2 + o_n(1) \geq q_0 \int_{\Omega_n^1} |\tilde{\xi}_n|^2 + o_n(1).$$

Now, if for some subsequence it holds $J_n(\tilde{\xi}_n) \leq o_n(1)$, then it would have $||(\chi_{\Omega_n^1}\tilde{\xi}_n)||^2_{L^2(\mathbb{R}^N)} = o_n(1)$, and so $\displaystyle\int_{\mathbb{R}^N} |\chi_{\Omega_n^1}\tilde{\xi}_n|^p = o_n(1)$, for a fixed $p \in (2, 2^*]$, by an interpolation argument. From this, by the properties on F_2 (*vide* (P_2) above), it follows that

$$\int_{\Omega_n^1} F_2'(\tilde{\xi}_n)\tilde{\xi}_n = \int_{\mathbb{R}^N} F_2'(\chi_{\Omega_n^1}\tilde{\xi}_n)\chi_{\Omega_n^1}\tilde{\xi}_n = o_n(1).$$

Therefore,

$$\int_{\Omega_n^1} (|\nabla \tilde{\xi}_n|^2 + (1 + Q(x + y_n^1))|\tilde{\xi}_n|^2) + \int_{\Omega_n^1} Q(x + y_n^1)F_1'(\tilde{\xi}_n)\tilde{\xi}_n = o_n(1).$$

Equivalently, by a change of variable,

$$\int_\Omega (|\nabla \xi_n|^2 + (1 + Q(x))|\xi_n|^2) + \int_\Omega Q(x)F_1'(\xi_n)\xi_n = o_n(1),$$

contradicting the fact that $\xi_n \nrightarrow 0$. The proof of the claim is completed.

In the same line of Lemma 3.4, we are able to show that the next claim holds.

Claim 3.7 *There exist $\tau_0 > 0$ and $n_0 \in \mathbb{N}$ such that*

$$\delta_n \geq \tau_0, \ \ n \geq n_0.$$

Take into accounting the inequality in (3.40), the proof of the claim follows by reasoning as made in Claim 3.2. However, we would like point out an important fact related with the proof of the Claim 3.2. The inequality in (3.25) plays a crucial role in the proof of Claim 3.2. Such inequality is based in the fact that the constant associated with the embedding

$$H^1(B_i) \hookrightarrow L^p(B_i)$$

are independent of i. In the current proof a similar property also holds, more precisely

$$H^1(\tilde{B}_i) \hookrightarrow L^p(\tilde{B}_i),$$

since the sets $\tilde{B}_i = (B_i \cap \Omega)$ verify the uniform cone property (see [1]).

The preceding claim assures that

$$|y_n^1| \longrightarrow \infty.$$

In fact, otherwise, it would be possible to find $R > 0$, such that

$$\int_{(B_R(0)\cap\Omega)} |\xi_n^1|^p \geq \int_{\tilde{B}_i} |\xi_n^1|^p = \delta_n^p \geq \tau_0^p > 0.$$

This contradicts the convergence

$$\int_{(B_R(0)\cap\Omega)} |\xi_n^1|^p \longrightarrow 0,$$

which is valid in view of the weak convergence $\xi_n^1 \rightharpoonup 0$ in Z. Thus, hereafter we will assume that $|y_n^1| \to \infty$.

Now, since $y_n^1 \to \infty$, we know that $\Omega_n^1 \to \mathbb{R}^N$, as $n \to \infty$, (in the sense of the characteristic functions $\chi_{\Omega_n^1} \to 1$ a.e. in $\mathbb{R}^N$) for each $R > 0$, there exists $m_0 \in \mathbb{N}$ such that $B_R(0) \subset \Omega_n^1$, $n \geq m_0$. Considering that (ξ_n^1) is a bounded sequence, it is possible to find $v_1 \in Y \setminus \{0\}$ satisfying

$$\tilde{\xi}_n \rightharpoonup v_1 \ \text{ in } \ H^1(B_R(0)) \cap L^{F_1}(B_R(0)),$$

for each $R > 0$ fixed. Fixed $\phi \in C_0^\infty(\Omega)$, inasmuch as $|y_n^1| \to \infty$, we know that, for some $m_1 \in \mathbb{N}$, it holds

$$\operatorname{supp} \phi(\cdot - y_n^1) \subset \Omega, \ \ n \geq m_1.$$

Hence, $\phi^{(y_n^1)} := \phi(\cdot - y_n^1) \in C_0^\infty(\Omega)$ for $n \geq m_1$.

By exploring the ideas in the proof of Lemma 3.4-ii), we derive

$$\sup_{n\in\mathbb{N}} \left(|J'(\xi_n)| \cdot ||\phi(\cdot\ - y_n^1)||_Z\right) = o_n(1).$$

By combining these information with the properties (Q_1) and (3.25) above, we derive that v_1 is a nontrivial solution of (S_∞). Set

$$\xi_n^2 := \xi_n^1 - v_1(\cdot - y_n^1), \text{ in } \Omega.$$

Hence, we can repeat the preceding steps made with ξ_n^1. Following this procedure, the reasoning made in final of Lemma 3.4 allows us to get a $k \in \mathbb{N}$ and unbounded sequences $(y_n^1), ..., (y_n^k)$ in $\mathbb{R}^N$ in such way that

$$\xi_n^j := \xi_n^{j-1}(\cdot + y_n^{j-1}) - v_{j-1} \rightharpoonup 0, \text{ in } Y,$$

with v_{j-1} a nontrivial solution of (S_∞), $\xi_n^{k+1} \to 0$, as $n \to \infty$, $j \in \{2, ..., k\}$. Setting $v_j := v_j(\cdot - y_n^j)$, these facts assure that

$$\left\| v_n - v_0 - \sum_{j=1}^k v_n^j \right\|_{H^1(\Omega)}^2 = o_n(1)$$

as well as

$$J(u_n) \longrightarrow J(v_0) + \sum_{j=1}^k J_\infty(u_j).$$

■

An immediate consequence of the preceding lemma is following corollary.

Corollary 3.2 *The functional J verifies the $(C)_d$-condition for $d \in (0, l_\infty)$.*

Proof. Let (v_n) be a $(C)_d$-sequence, with $d \in (0, l_\infty)$. In particular,

$$J'(v_n)v_n = o_n(1),$$

and so, using the same ideas explored in the begin of the proof of Lemma 3.3, we derive that (v_n) is a bounded sequence in Z and, going to a subsequence if necessary, it holds $v_n \rightharpoonup v_0$, for some $v_0 \in Z$. Since (v_n) is a $(C)_d$-sequence, we have $J'(v_0) = 0$. Now, it is sufficient to observe that the hypothesis $d \in (0, l_\infty)$ combined with the items $i) - ii)$ of the preceding lemma gives the required result. ■

We are going to show that J has a ground state solution, i.e., a positive solution v_0 satisfying $J(v_0) = l_0$. We start by showing that the functional J satisfies the mountain geometric (see e.g [83, Section 2.3]).

Lemma 3.11 *The functional J verifies the Mountain Pass geometry, i.e.,*

i) $J(0) = 0$ and there exist r, $\rho_0 > 0$ such that $J_{\partial B_r(0)} \geq \rho_0$;

ii) There exits v, $||v||_Z > r$, and $J(v) \leq J(0) = 0$.

Proof. *i*): From the conditions $(Q_1) - (Q_2)$ it follows that, for some constant $C > 0$, it holds

$$J(u) \geq C||u||^2_{H^1(\Omega)} + C\int_\Omega F_1(u) - Q_0\int_\Omega F_2(u).$$

By using (2.4) and (P_2), modifying the constant C if necessary, we can find $r \approx 0^+$ such that, for $||u||_Z = r$, is valid that

$$J(u) \geq C||u||^2_{H^1(\Omega)} + C||u||^2_{L^{F_1}(\Omega)} - C_1||u||^p_Z \geq C_2||u||^2_Z - C_1||u||^p_Z$$

with C_1, $C_2 > 0$ and $p > 2$. The property required in the item *i*) follows as a direct consequence of the last inequality.

ii): Fix $u \in Z - \{0\}$. So,

$$J(tu) = \frac{t^2}{2}\left[\int_\Omega (|\nabla u|^2 + |u|^2) - \frac{1}{2}\int_\Omega Q(x)u^2 \log u^2 - \log t\int_\Omega Q(x)u^2\right] \longrightarrow -\infty,$$

as $t \to \infty$. So, the item *ii*) holds by taking $v = tu$, for some $t \approx \infty$. ■

We are going to show that the problem (S_0) has a ground state solution. To begin with, we will show the existence of a $(C)_d$-sequence at mountain pass level. Namely, we have the following corollary.

Corollary 3.3 *The functional J has a sequence $(C)_{\tilde{l}_0}$-sequence, where $\tilde{l}_0$ is the level*

$$\tilde{l}_0 := \inf_{\gamma\in\Gamma}\sup_{t\in[0,1]} J(\gamma(t)),$$

and

$$\Gamma := \{\gamma \in C([0,1], Z);\ \gamma(0) = 0,\ \gamma(1) < 0\}.$$

Proof. The result follows by a variant of the classical Mountain Pass Theorem of Ambrosetti-Rabinowitz (see, e.g., [83, Section 2]). Note that the reasoning made in [83] can be adapted when the $(PS)_d$-sequences are replaced by $(C)_d$-sequences (see the

Proposition 1.1 in [38] for a statement of a variant Mountain Pass Theorem involving the Cerami sequences). ■

Exploring the ideas in [10, Lemma 3.3], in view of (Q_1), we can show that the level $\tilde{l}_0$ in the above corollary coincides with the level l_0, namely, it holds

$$\tilde{l}_0 = l_0 := \inf_{u \in \mathcal{M}} J(u). \tag{3.41}$$

Thereby, the last corollary assures the existence of a $(C)_{l_0}$-sequence for J. The next lemma is our main technical result in the present section, and it relates the levels l_0 and l_∞.

Lemma 3.12 *Assume the conditions $(Q_1)-(Q_2)$. Then the following inequality holds.*

$$l_0 < l_\infty.$$

Proof. Set

$$v_n(x) := v_\infty(x - x_n),$$

with $x_n := (n, 0, ..., 0) \in \mathbb{R}^N$ and v_∞ the solution of (S_∞) satisfying (3.37). By (3.41),

$$l_0 \le \max_{t \ge 0} J(tv_n) =: J(t_n v_n),$$

and $t_n \in (0, \infty)$. In this way, we derive that $t_n v_n \in \mathcal{M}$, which yields

$$t_n^2 \int_\Omega (|\nabla v_n|^2 + |v_n|^2) = \int_\Omega t_n^2 |v_n|^2 \log |t_n v_n|^2.$$

Therefore, since $|x_n| \to \infty$, the same ideas employed in the proof of Lemma 3.1 enable us to show that, going to a subsequence if necessary, it holds $t_n \to 1$.

Now, it follows that

$$\begin{aligned} l_0 \le J(t_n v_n) &= \frac{1}{2} \int_\Omega (|t_n \nabla v_n|^2 + (1 + Q(x))|t_n v_n|^2) + \int_\Omega Q(x) F_1(t_n v_n) - \int_\Omega Q(x) F_2(t_n v_n) = \\ &= J_\infty(t_n v_n) - \frac{t_n^2}{2} A_n + \int_{\Omega^c} Q_0 F_2(t_n v_n) - \int_{\Omega^c} Q_0 \left[F_1(t_n v_n) + \frac{t_n^2}{2} v_n^2 \right] + \\ &+ \int_\Omega (Q_0 - Q(x)) \left[F_2(t_n v_n) - F_1(t_n v_n) - \frac{t_n^2}{2} v_n^2 \right], \end{aligned}$$

with $A_n := \displaystyle\int_{\Omega^c} (|\nabla v_n|^2 + |v_n|^2)$. From (Q_1),

$$l_0 \le J_\infty(t_n v_n) - \frac{t_n^2}{2} A_n + \int_{\Omega^c} Q_0 F_2(t_n v_n) + \int_\Omega (Q_0 - Q(x)) F_2(t_n v_n).$$

Taking into account that $t_n \to 1$ as $|x_n| \to \infty$, the condition (Q_1) and the invariance by translations of $\mathbb{R}^N$, one finds

$$J_\infty(t_n v_n) = J_\infty(v_\infty) + o_n(1) = c_\infty + o_n(1).$$

This information together with the last inequality give

$$l_0 \le l_\infty + o_n(1) - \frac{t_n^2}{2} A_n + B_n, \tag{3.42}$$

with $B_n := \int_{\Omega^c} Q_0 F_2(t_n v_n) + \int_\Omega (Q_0 - Q(x)) F_2(t_n v_n)$.

Our next step is proving that $\frac{B_n}{A_n} \to 0$. Having this in mind, since $|\Omega^c| < \infty$, the equality in (3.36) implies

$$A_n \ge \int_{\Omega^c} |v_n|^2 \ge C e^{-2C_2 n^2}, \quad \forall n \in \mathbb{N}, \tag{3.43}$$

for a convenient $C > 0$. From the condition (P_2), for some $p \in (2, 2^*]$, it holds

$$|F_2(t)| \le C_p |t|^p, \ \ \forall t \in \mathbb{R}.$$

Therefore, using again $|\Omega^c| < \infty$, one has

$$Q_0 \int_{\Omega^c} F_2(t_n v_n) \le C e^{-pC_2 n^2}, \tag{3.44}$$

for some C. Now, take $R_n \in (0, n)$. So,

$$\int_\Omega (Q_0 - Q(x)) F_2(t_n v_n) = \int_{\Omega \cap [|x| > R_n]} (Q_0 - Q) F_2(t_n v_n) + \int_{\Omega \cap [|x| \le R_n]} F_2(t_n v_n).$$

By invoking the assumption (Q_2), it follows that

$$\int_{\Omega \cap [|x| > R_n]} (Q_0 - Q(x)) F_2(t_n v_n) \le C e^{-MR_n^2}, \tag{3.45}$$

for some $C > 0$, as well as,

$$\int_{\Omega \cap [|x| \le R_n]} (Q_0 - Q(x)) F_2(t_n v_n) \le C_N n^N e^{-pC_2 (n - R_n)^2}. \tag{3.46}$$

for some constant $C_N > 0$. The estimates in (3.43)-(3.46) combined produce, for some constant $C > 0$,

$$\frac{B_n}{A_n} \le C \left(\frac{e^{2C_2 n^2}}{e^{pC_2 n^2}} + \frac{e^{2C_2 n^2}}{e^{MR_n^2}} + \frac{C_N n^N e^{2C_2 n^2}}{e^{pC_2 (n - R_n)^2}} \right).$$

Setting $R_n := \dfrac{n}{k}$, $k \in \mathbb{N}$, we find

$$\frac{C_N n^N e^{2C_2 n^2}}{e^{pC_2(n-R_n)^2}} = \frac{C_N n^N e^{2C_2 n^2}}{e^{(\frac{k-1}{k})^2 pC_2 n^2}}.$$

Since $\left(\dfrac{k-1}{k}\right)^2$ converges to 1, as $k \to \infty$, and $p > 2$, we may fix $k_0 \approx \infty$ such that $p\left(\frac{k_0}{k_0-1}\right)^2 > 2$. Hence

$$\frac{C_N n^N e^{2C_2 n^2}}{e^{(\frac{k_0-1}{k_0})^2 pC_2 n^2}} \longrightarrow 0.$$

Then, choosing M_0 large enough in the condition (Q_2), we derive that

$$\frac{e^{2C_2 n^2}}{e^{MR_n^2}} = \frac{e^{2C_2 n^2}}{e^{(M/k_0^2)n^2}} \longrightarrow 0.$$

These convergences assure that

$$\frac{B_n}{A_n} \longrightarrow 0.$$

Recalling that $t_n \to 1$ for some $n_0 \in \mathbb{N}$,

$$-\frac{t_n^2}{2} A_n + B_n = \left(\frac{-t_n^2}{2} + \frac{B_n}{A_n}\right) A_n < 0, \quad n \geq n_0.$$

Using this information in (3.42), we derive that

$$l_0 < l_\infty,$$

proving the desired result. ■

Now we can prove our main result.

Proof of Theorem 3.2. The proof is essentially established. In fact, by combining the Corollary 3.3 with (3.41), there exists a $(C)_{l_0}$-sequence for J, which will be denotes by (v_n). Since (v_n) is bounded, it follows that

$$J(v_n) \longrightarrow l_0 \text{ and } J'(v_n) \longrightarrow 0.$$

Invoking together the Corollary 3.2 and the Lemma3.12, we may assume that

$$v_n \longrightarrow v_0 \text{ in } Z,$$

for some v_0. In this way, we derive that

$$J(v_0) = l_0 \text{ and } J'(v_0) = 0,$$

and so v_0 is a ground state solution for (S_0). Now, we would like to point out that v_0 can be chosen as a positive solution. Indeed, writing $v_0 = v_0^+ - v_0^-$, with $v_0^+ := \max\{v_0, 0\}$ and $v_0^- := \max\{-v_0, 0\}$, we find $J'(v_0^+)v_0^+ = J'(v_0^-)v_0^- = 0$ and $l_0 = J(v_0) = J(v_0^+) + J(v_0^-)$. These facts combined assure that either $v_0^+ = 0$ or $v_0^- = 0$. Hence, since $f(t) = t\log t$ is an odd function, we may assume that $v_0 \geq 0$, so that $v_0 > 0$ by a variant of maximum principle presented in [82] (see [7, 10, 11] for a similar reasoning) ■

APPENDIX A

A brief on nonsmooth critical point theory

Next, we present, in general lines, some notions of the generalized critical point theory required in our study. We subdivide the list of abstract concepts and results into two parts: firstly, we present the notions related with locally Lipschitz functionals. Secondly, we introduce the concepts referring to l.s.c. functionals. For further details and proofs, we refer Chang [36], Clarke [40, 41], Carl, Le and Motreanu [34], Motreanu and Panagiotopoulos [71, Chapters 1-2], and Szulkin [81].

A.1 The locally Lipschitz case

A real-valued functional $\varphi : X \to \mathbb{R}$ is called *locally Lipschitz continuous* (briefly $\varphi \in \mathrm{Lip}_{\mathrm{loc}}(X, \mathbb{R})$) when to every $u \in X$ there correspond a neighbourhood $V := V_u$ of u and a constant $K := K_u > 0$ such that

$$|\varphi(v) - \varphi(w)| \leq K\|v - w\|, \quad \forall v, w \in V.$$

The *generalized directional derivative* of $\varphi \in \mathrm{Lip}_{\mathrm{loc}}(X, \mathbb{R})$ at u along the direction $v \in X$ is defined by

$$\varphi^{\circ}(u; v) := \limsup_{w \to u,\, t \to 0^+} \frac{\varphi(w + tv) - \varphi(w)}{t}.$$

The *generalized gradient* of the function $\varphi \in \mathrm{Lip}_{\mathrm{loc}}(X, \mathbb{R})$ in u is the set

$$\partial\varphi(u) = \{\phi \in X^* : \varphi^{\circ}(u; v) \geq \langle \phi, v \rangle\, , \forall\, v \in X\}.$$

Proposition 2.1.2 of [41] ensures that $\partial\varphi(u)$ turns out nonempty, convex, in addition to weak* compact, and that

$$\varphi^\circ(u;v) := \max\{\langle \eta, v\rangle : \eta \in \partial\varphi(u)\}.$$

In the sequel we say that a point $u \in X$ is a *critical point* of $\varphi \in \mathrm{Lip}_{\mathrm{loc}}(X,\mathbb{R})$ if $0 \in \partial\varphi(u)$. We also recall that, when a functional $\eta : X \to \mathbb{R}$ is convex, the *subdifferential* of η at u is the set

$$\partial_s\eta(u) := \{\phi \in X^* : \eta(v) - \eta(u) \geq \langle \phi, v-u\rangle, \forall\, v \in X\}. \tag{A.1}$$

If $\eta \in \mathrm{Lip}_{\mathrm{loc}}(X,\mathbb{R})$ then $\partial_s\eta(u) = \partial\eta(u)$.

Some usual properties of the generalized directional derivative as well of the generalized gradient are listed below.

Lemma A.1 *Let* $\varphi \in \mathrm{Lip}_{\mathrm{loc}}(X,\mathbb{R})$, *then*

i) the map $(u,v) \mapsto \varphi^\circ(u,v)$ *is an upper semicontinuous functional, i.e. if* $(u_j,v_j) \to (u,v)$ *then*

$$\limsup \varphi^\circ(u_j,v_j) \leq \varphi^\circ(u,v);$$

ii) $\varphi^\circ(u,-v) = (-\varphi)^\circ(u,v)$.

Lemma A.2 *If* ψ *is continuously Fréchet differentiable in an open neighborhood of* $u \in X$, *then* $\partial\psi(u) = \{\psi'(u)\}$.

Lemma A.3 *If* φ, $\psi \in \mathrm{Lip}_{\mathrm{loc}}(X,\mathbb{R})$, *then for each* $u \in X$ *one has*

i) $\partial(\varphi+\psi)(u) \subseteq \partial\varphi(u) + \partial\psi(u)$;

ii) $\partial(\varphi+\psi)(u) = \{\varphi'(u)\} + \partial\psi(u)$, *provided that* $\varphi \in C^1(X,\mathbb{R})$.

In the next lemma we report an important property between $\varphi^\circ(u,v)$ and the Gâteaux derivatives of φ at $u \in X$ along $v \in X$, i.e.

$$\frac{\partial\varphi}{\partial v}(u) := \lim_{t\to 0^+} \frac{\varphi(u+tv)-\varphi(u)}{t}. \tag{A.2}$$

Lemma A.4 *If* $\varphi \in \mathrm{Lip}_{\mathrm{loc}}(X,\mathbb{R})$ *is convex, then* $\dfrac{\partial\varphi}{\partial v}(u)$ *exists for any* $u, v \in X$ *and*

$$\frac{\partial\varphi}{\partial v}(u) = \varphi^\circ(u,v).$$

A.2 The lower semicontinuous case

From now on, we say that a functional $I : X \to (-\infty, +\infty]$ is a *Szulkin-type functional* if

(H_0) *$I := \Phi + \Psi$, with $\Phi \in C^1(X, \mathbb{R})$ and $\Psi : X \to (-\infty, +\infty]$ is a convex lower semicontinuous functional and proper, i.e. $\Psi \not\equiv \infty$.*

The *effective domain* of I is defined by

$$D(I) := \{u \in X : I(u) < +\infty\},$$

and so, for a Szulkin-type functional I one has that $D(I) = D(\Psi)$. For each $u \in D(I)$, we say that the *subdifferential* of I at u is the set

$$\partial I(u) := \{\varphi \in X^* : \langle \Phi'(u), v - u\rangle + \Psi(v) - \Psi(u) \geq \langle \varphi, v - u\rangle, \forall v \in X\}. \tag{A.3}$$

Definition A.1 *Suppose that I is a Szulkin-type functional Then*

i) a point $u \in X$ is called a critical point of I if $0 \in \partial I(u)$, or more precisely, $u \in D(I)$ and

$$\langle \Phi'(u), v - u\rangle + \Psi(v) - \Psi(u) \geq 0, \quad \forall v \in X,$$

ii) a sequence (u_n) is called a Palais-Smale sequence (briefly (PS) *sequence) for I at level $c \in \mathbb{R}$ if $I(u_n) \to c$ and*

$$\langle \Phi'(u_n), v - u_n\rangle + \Psi(v) - \Psi(u_n) \geq -\varepsilon_n \|v - u_n\|, \quad \forall v \in X,$$

with $\varepsilon_n \to 0^+$, or equivalently (see [81, Proposition 1.2])

$$\langle \Phi'(u_n), v - u_n\rangle + \Psi(v) - \Psi(u_n) \geq \langle w_n, v - u_n\rangle, \quad \forall v \in X,$$

where $w_n \in X^*$ with $w_n \to 0$ in X^*;

iii) I satisfies the Palais-Smale condition (briefly (PS) *condition) at level $c \in \mathbb{R}$ when each* (PS) *sequence (u_n) at level c has a convergent subsequence. If I verifies the* (PS) *condition for all level c, we say simply that I satisfies the* (PS) *condition.*

For a fixed Szulkin-type functional I, denote by K and K_c respectively, the following sets

$$K := \{u \in X : u \text{ is a critical point of } I\},$$

and

$$K_c := \{u \in K : I(u) = c\}.$$

The following result holds

Proposition A.1 *Suppose that I verifies* (H_0) *and the* (PS) *condition at level* $c \in \mathbb{R}$. *Then,* K_c *is a compact set.*

APPENDIX B

Group actions on Banach spaces

This appendix is focused in discussing the main notions associated with group actions on Banach spaces. The notions described in this subsection follow closely the presentation in [83, Sections 1.6 and 3.2]; see also Bartsch [24] for additional comments and remarks. We also give a short review about the building of the Haar's integral on a compact group G; see Nachbin [72] for a abstract preview on this subject.

B.1 General settings

Let G be a topological group with neutral element e and X a Banach space. An action of G on X is a continuous function

$$\begin{aligned} \phi : G \times X &\to X \\ (g, v) &\mapsto \phi(g, v) = gv \end{aligned}$$

such that

(G_1) $ev = v, \ \forall x \in X$;

(G_2) $(gh)v = g(hv), \ \forall v \in X, \forall g, h \in G$;

(G_3) *For each* $g \in G$ *the map*

$$\begin{aligned} \phi_g : X &\to X \\ v &\mapsto \phi_g(v) = gv \end{aligned}$$

is linear.

If in addition to the above condition, the following relation holds

(G_4) $\|gv\| = \|v\|, \ \forall v \in X, \ \forall g \in G,$

then the map ϕ is said to be an isometric action. According to the above definitions, we say that G acts isometrically on X when $(G_1) - (G_4)$ hold.

The subspace of *invariant elements* of X is defined by

$$Fix(G) := \{u \in X : \ gu = u \ \ \forall g \in G\}.$$

Example B.1 **1º)** Let $Id : X \to X$ be the identity map on X and consider the usual representation $\mathbb{Z}_2 = \{Id, -Id\}$. Standard computations ensure that the group $\mathbb{Z}_2$ acts isometrically on X.
2º) Consider $G = O(N)$ the group of orthogonal maps on $\mathbb{R}^N$. We define the action of G on $H^1(\mathbb{R}^N)$ in the following way

$$gu = u \circ g^{-1}, \ \ g \in G, \, u \in H^1(\mathbb{R}^N).$$

Note that, in this case, $Fix(G) = H^1_{rad}(\mathbb{R}^N)$ and that G acts isometrically on $H^1(\mathbb{R}^N)$ (see [83, Section 1.5] for additional comments).

A subset A of X is said to be *G-invariant* if $gA = A$ for every $g \in G$, where $gA := \{gx : \ x \in A\}$. Also, when $A \subset X$ is a G-invariant set, a map $\gamma : A \to X$ is called *equivariant map* if

$$\gamma(gx) = g\gamma(x) \ \ \forall x \in A, \forall g \in G.$$

If a functional (not necessarily linear) φ defined on X satisfies $\varphi(gx) = \varphi(x)$ for any $x \in X$ and $g \in G$, we say that φ is a G-invariant functional.

Notation: $\Gamma_G(A) := \{\gamma \in C(A, X) : \ \gamma \, \text{is equivariant}\}.$

B.2 The Haar's Integral

The proofs e more detailed comments about the results and concepts in the sequel can be found in [72, Chapter II].

B.2.1 The normalized Haar measure

Suppose that G is a locally compact group and μ a positive measure on G. According to the classical literature on the subject, $\mathcal{L}(G,\mu)$ denotes here the space of the integrable functions $f : G \longrightarrow \mathbb{R}$ with respect to the measure μ, and μ is a *left invariant* measure when

$$\int_G f(g^{-1}y)d\mu = \int_G f(y)d\mu, \ \forall g \in G, \tag{B.1}$$

for every $f \in \mathcal{L}(G,\mu)$.

The next result assures the existence of a left invariant measure on a locally compact topological group G.

Theorem B.2 (Haar) *Let G be a locally compact group. Then, there exists at least one left invariant positive measure $\mu_0 \neq 0$. Moreover, the measure $\mu_0(G)$ is unique except for a strictly positive factor of proportionality, i.e. if μ_1 is a left invariant positive measure on G, there exists $c > 0$ such that $\mu_1 = c\mu_0(G)$. Finally*

$$\mu_0(G) < \infty \Leftrightarrow \ G \text{ is compact.}$$

See [72, Chapter II, Sections 4 and 5] for a detailed proof.

Corollary B.1 (Normalized Haar measure) *Let G be a compact group. Then, there exists a left invariant positive measure μ on G such that $\mu(G) = 1$.*

Proof. Take $\mu := \dfrac{1}{\mu_0(G)}\mu_0$, with μ_0 given in the Theorem B.2. ■

Remark B.1 The integral associated to μ_0 in the Theorem B.2 is the so called Haar's integral.

B.2.2 A *vector-valued* version of the Haar's integral

The Haar's integral as defined above can be extended for X-valued measurable functions, that is, for functions $f : G \longrightarrow X$. In the sequel we show how this construction can be established. The steps and arguments follow the ideas in [55, Apenddix E] and [31, Chapter 9.].

Fix G a compact group that acting isometrically in a Banach space X and let μ be the Normalized Haar's measure given in Corallary B.1. Denote by Σ a σ-algebra of G such that μ is well defined on Σ.

Definition B.1 *A function $\phi : G \longrightarrow X$ is said to be a measurable simple function if there exist $A_1, ..., A_k \in \Sigma$, $A_i \cap A_j = \emptyset$, $i \neq j$, and $v_1, ..., v_k \in X$ such that*

$$\phi = \sum_{j=1}^{k} \chi_{A_j} v_j,$$

with χ_{A_j} the characteristic function of A_j, $j \in \{1, ..., k\}$.

An arbitrary function $f : G \longrightarrow X$ is called a *measurable function* if there exists a sequence of measurable functions $(\phi_n)_{n\in\mathbb{N}}$ such that

$$\phi_n(x) \longrightarrow f(x), \quad \text{a.e. in } G.$$

By following the same ideas in the building of the Bochner's integral (see, e.g., [31, Chapter 9]) we have the following definition.

Definition B.2 *Consider a measurable simple function of the form $f = \sum_{j=1}^{k} \chi_{A_j} v_j$. We define the (vector) integral of f as follows:*

$$\int_G f \, d\mu = \int_G \left(\sum_{j=1}^{k} \chi_{A_j} v_j \right) d\mu := \sum_{j=1}^{k} \mu(A_j) v_j.$$

Given a measurable function $f : G \longrightarrow X$, we say that f is an integrable function if there exists a sequence $(f_n)_{n\in\mathbb{N}}$ of measurable simple functions satisfying

$$\lim_{n\to\infty} \int_G ||f_n - f|| \, d\mu \longrightarrow 0. \tag{B.2}$$

The convergence in (B.2) enable us to define the integral of a measurable function in the following way.

Definition B.3 *Given a measurable function $f : G \longrightarrow X$ and $B \in \Sigma$ we define the integral of f on B by the equality below:*

$$\int_B f \, d\mu := \lim_{n\to\infty} \int_G \chi_B f_n \, d\mu,$$

with $(f_n)_{n\in\mathbb{N}}$ a sequence of measurable simple functions verifying (B.2).

The following propositions, whose the proofs can be found in [31, Section 9.7], assure that last definition is well posed. In addition, some technical properties involving vector integrals are pointed out in the next results.

Proposition B.1 *Let $f : G \longrightarrow X$ be a function. The following items are valid:*

i): The function f is a measurable function if, and only if, the function $||f|| : G \longrightarrow \mathbb{R}$ is a real-valued measurable function.

ii): The function f is an integrable function if, and only if, the function $||f|| \in \mathcal{L}(G, \mu)$.

iii): If $f : G \longrightarrow X$ is an integrable function (in the sense of (B.2)), then there exists $(f_n)_{n\in\mathbb{N}}$ a sequence of measurable simple functions such that

$$f_n(x) \longrightarrow f(x), \quad a.e \text{ in } G$$

and

$$||f_n - f|| \longrightarrow 0 \text{ in } L^1(G).$$

The next result present some properties of the vector integrals which have been used in Chapter 1.

Proposition B.2 *Let $f : G \longrightarrow X$ be an integrable function and consider $B \in \Sigma$. So, it holds:*

i): $\left|\left|\int_B f\,d\mu\right|\right| \leq \int_B ||f||\,d\mu.$

ii): Let Y a Banach space and $T : X \longrightarrow Y$ a continuous linear map. Then, the function $T \circ f : G \longrightarrow Y$ is an integrable function with

$$\int_G T \circ f\,d\mu = T\left(\int_G f\,d\mu\right).$$

Next, we prove that the left invariance property of μ in (B.1) still holds for integrable X-valued functions $f : G \longrightarrow X$.

Theorem B.3 *For all integrable function $f : G \longrightarrow X$ it holds*

$$\int_G f(g^{-1}x)\,d\mu = \int_G f(x)\,d\mu.$$

Proof. Initially, consider the case that

$$f = \sum_{j=1}^{k} \chi_{A_j} v_j$$

is a measurable simple function. So, given $g \in G$, we have

$$f(g^{-1}x) = v_j,$$

for any $x \in gA_j = \{gy;\, y \in A_j\}$. Since G acts isometrically in X, it holds $gA_j \cap gA_i$, $i \neq j$, $i,\, j \in \{1, ..., k\}$, so that

$$f(g^{-1}x) = \sum_{j=1}^{k} \chi_{gA_j}(x) v_j.$$

Since μ is a left-invariant for real function $\phi \in \mathcal{L}(G, \mu)$, we get

$$\mu(gA_j) = \int_G \chi_{gA_j}(x)\, d\mu = \int_G \chi_{A_j}(g^{-1}x)\, d\mu = \int_G \chi_{A_j}(x)\, d\mu = \mu(A_j),$$

para todo $j \in \{1, ..., k\}$. Hence

$$\int_G f(g^{-1}x)\, d\mu = \sum_{j=1}^{k} \mu(gA_j) v_j = \sum_{j=1}^{k} \mu(A_j) v_j = \int_G f(x)\, d\mu,$$

showing that the result it is true for measurable simple functions.

The general case is a direct consequence of the first case. To see why, given a integrable function $f : G \longrightarrow X$, take $(f_n)_{n\in\mathbb{N}}$ a sequence of measurable simple functions verifying the Part *iii*) of Proposition B.1. Note that, for each $g \in G$, using the properties of convergence, we derive that

$$\int_G f(g^{-1}x)\, d\mu = \lim_{n\to\infty} \int_G f_n(g^{-1}x)\, d\mu.$$

The conclusion is now an application of the first part. ■

We will finish this subsection by presenting the useful example below.

Example B.4 Define $\eta : X \longrightarrow X$ as follows:

$$\eta(u) := \int_G g\beta(g^{-1}u)\, d\mu,$$

with $\beta \in C(X, X)$. From the properties of the integral, we know that $\eta \in C(X, X)$. Furthermore, given $g_0 \in G$, we get

$$\eta(g_0 u) = g_0 \int_G g_0^{-1} g\beta((g_0^{-1}g)^{-1}u)\, d\mu.$$

By the preceding theorem, we derive that

$$\eta(g_0 u) = g_0 \eta(u),$$

proving that η is an equivariant map on X, i.e., $\eta \in \Gamma_G(X)$.

APPENDIX C

A short review on Orlicz spaces

This appendix is a primer of Orlicz spaces, in which we present some notions and properties related to the Orlicz spaces needed in our work; for further details see [1, 59, 77].

C.1 On N-functions and Orlicz spaces

We start by recalling the definition of a N-function.

Definition C.1 *A continuous function $\Phi : \mathbb{R} \to [0, +\infty)$ is a N-function if:*

(i) Φ *is convex.*

(ii) $\Phi(t) = 0 \Leftrightarrow t = 0$.

(iii) $\lim_{t\to 0} \dfrac{\Phi(t)}{t} = 0$ *and* $\lim_{t\to\infty} \dfrac{\Phi(t)}{t} = +\infty$.

(iv) Φ *is an even function.*

We say that a N-function Φ verifies the Δ_2-condition, denoted by $\Phi \in (\Delta_2)$, if

$$\Phi(2t) \leq k\Phi(t) \;\; \forall\, t \geq t_0,$$

for some constants $k > 0$ and $t_0 \geq 0$.

The conjugate function $\tilde{\Phi}$ associated with Φ is given by the Legendre's transformation, more precisely,

$$\tilde{\Phi} = \max_{t\geq 0}\{st - \Phi(t)\} \;\; \text{for} \;\; s \geq 0.$$

It is possible to prove that $\tilde{\Phi}$ is also a N-function. The functions Φ and $\tilde{\Phi}$ are complementary to each other, that is, $\tilde{\tilde{\Phi}} = \Phi$.

Given an open set $A \subset \mathbb{R}^N$, we define the Orlicz space associated with the N-function Φ as

$$L^{\Phi}(A) = \left\{ u \in L^1_{loc}(A) \; ; \; \int_A \Phi\left(\frac{|u|}{\lambda}\right) < +\infty, \quad \text{for some } \lambda > 0 \right\}.$$

The space $L^{\Phi}(A)$ is a Banach space endowed with Luxemburg norm given by

$$||u||_{\Phi} = \inf \left\{ \lambda > 0 \; ; \; \int_A \Phi\left(\frac{|u|}{\lambda}\right) \leq 1 \right\}.$$

We would like to point out that in Orlicz spaces we also have a Hölder and Young type inequalities, namely

$$st \leq \Phi(t) + \tilde{\Phi}(s), \;\; \forall s, t \geq 0,$$

and

$$\left| \int_A uv \right| \leq 2||u||_{\Phi}||v||_{\tilde{\Phi}}, \;\; \forall \, u \in L^{\Phi}(A) \quad \text{and} \quad u \in L^{\tilde{\Phi}}(A).$$

Moreover, for each $\varepsilon > 0$, it holds

$$st \leq \Phi(C_{\varepsilon} t) + \varepsilon\tilde{\Phi}(s), \;\; \forall s, t \geq 0, \tag{C.1}$$

for some positive $C_{\varepsilon} > 0$. When Φ, $\tilde{\Phi} \in (\Delta_2)$, the space $L^{\Phi}(A)$ is reflexive and separable. Furthermore, the Δ_2-condition yields that

$$L^{\Phi}(A) = \left\{ u \in L^1_{loc}(A) \; ; \; \int_A \Phi\left(|u|\right) < +\infty \right\}$$

and

$$u_n \to u \;\text{ in }\; L^{\Phi}(A) \Leftrightarrow \int_A \Phi(|u_n - u|) \to 0.$$

We would like to mention an important relation involving N-functions related with the (Δ_2) condition. Let Φ be a N-function of C^1 class and $\tilde{\Phi}$ its conjugate function. Suppose that

$$1 < l \leq \frac{\Phi'(t)t}{\Phi(t)} \leq m < N, \quad t \neq 0, \tag{C.2}$$

then Φ, $\tilde{\Phi} \in (\Delta_2)$. It is very important to point out that, when Φ, $\tilde{\Phi} \in (\Delta_2)$, it holds

$$\overline{C_0^{\infty}(A)}^{||\cdot||_{\Phi}} = L^{\Phi}(A),$$

for any open set $A \subset \mathbb{R}^N$.

Finally, setting the functions

$$\xi_0(t) := \min\{t^l, t^m\} \text{ and } \xi_1(t) : \max\{t^l, t^m\}, \qquad t \geq 0,$$

it is well known that under the condition (C.2) one has

$$\xi_0(||u||_\Phi) \leq \int_A \Phi(u) \leq \xi_1(||u||_\Phi). \tag{C.3}$$

We finish this section by recalling a Brezis-Lieb type result involving N-functions found in [32, Theorem 2]

Proposition C.1 (A Brezis-Lieb type result) *Suppose Φ is a N-function with $\Phi \in (\Delta_2)$. Let (g_n) be a sequence in $L^\Phi(A)$ satisfying:*

i) (g_n) is a bounded sequence in $L^\Phi(\Omega)$;

ii) $g_n(x) \to 0$ a.e. in A.

Then, for each $w \in L^\Phi(A)$,

$$\int_A |\Phi(g_n + w) - \Phi(g_n) - \Phi(w)| = o_n(1).$$

C.2 A special example of N-function

Here we prove that the function F_1 in (2.2), used in the decomposition

$$F_2(t) - F_1(t) = \frac{1}{2} t^2 \log t^2,$$

is a N-function such that F_1, $\tilde{F}_1 \in (\Delta_2)$.

Fix a small $\delta > 0$ and recall the definition of F_1.

$$F_1(s) := \begin{cases} 0, & s = 0 \\ -\frac{1}{2} s^2 \log s^2, & 0 < |s| < \delta \\ -\frac{1}{2} s^2 (\log \delta^2 + 3) + 2\delta|s| - \frac{\delta^2}{2}, & |s| \geq \delta \end{cases} \tag{C.4}$$

The following proposition is the main result of this section.

Proposition C.2 *The function F_1 is a N-function. Furthermore, it holds that F_1, $\tilde{F}_1 \in (\Delta_2)$.*

Proof. A direct computation shows that F_1 verifies $i)-iv)$ of the Definition C.1. Now, in order to finish the proof we will show that F_1 satisfies the relation (C.2). First of all, notice that

$$F_1'(s) := \begin{cases} -(\log s^2+1)s, & 0<s<\delta, \\ -s(\log \delta^2+3)+2\delta & s\geq \delta. \end{cases}$$

Next, we will analyze separately the cases $0<s<\delta$ and $s\geq\delta$.

Case 1: $0<s<\delta\approx 0^+$.

In this case,

$$\frac{F_1'(s)s}{F_1(s)} = 2+\frac{1}{\log s},$$

which implies the existence of $l_1>1$ satisfying

$$1<l_1\leq \frac{F_1'(s)s}{F_1(s)} \leq m_1 := \sup_{0<s<\delta}\left(2+\frac{1}{\log s}\right)\leq 2, \tag{C.5}$$

for δ small enough.

Case 2: $s\geq\delta$.

In this case,

$$\frac{F_1'(s)s}{F_1(s)} = \frac{-(\log\delta^2+3)s^2+2\delta s}{-\frac{1}{2}(\log\delta^2+3)s^2+2\delta s-\frac{1}{2}\delta^2}.$$

From this,

$$\sup_{s\geq\delta}\frac{F_1'(s)s}{F_1(s)} \leq \sup_{s\geq\delta}\left(\frac{-(\log\delta^2+3)s^2+2\delta s+(2\delta s-\delta^2)}{-\frac{1}{2}(\log\delta^2+3)s^2+2\delta s-\frac{1}{2}\delta^2}\right)\leq 2.$$

Since

$$\lim_{s\to+\infty}\frac{F_1'(s)s}{F_1(s)}=2 \quad \text{and} \quad \frac{F_1'(s)s}{F_1(s)}>1, \quad \forall s>0,$$

one gets

$$1<\inf_{s>0}\frac{F_1'(s)s}{F_1(s)}.$$

The last inequalities ensure the existence of $l\in(1,2)$ such that

$$1<l\leq\frac{F_1'(s)s}{F_1(s)}\leq 2, \quad \forall s>0. \tag{C.6}$$

As F_1 is an even function, the sentence above holds for any $s\neq 0$ and the proof is finished. ■

Given an open set $\Omega\subset\mathbb{R}^N$, by the remarks in the previous section, the last proposition assures that

$$\overline{C_0^\infty(\Omega)}^{||\cdot||_{L^{F_1}(\Omega)}} = L^{F_1}(\Omega),$$

as well as that the Orlicz space $L^{F_1}(\Omega)$ is a reflexive and separable Banach space.

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