Advanced Studies in Theoretical Physics
Vol. 9, 2015, no. 13, 609 - 616
HIKARI Ltd, www.m-hikari.com
http://dx.doi.org/10.12988/astp.2015.5772

Time-Varying Gravitomagnetism in Brans-Dicke Theory of Gravity

João Ribamar Silva

Secretaria de Estado de Educação e Desporto de Roraima
CEP 69300-000, Boa Vista, RR, Brazil

A. Barros

Centro de Desenvolvimento Sustentável do Semiárido
Universidade Federal de Campina Grande, CEP 58540-000
Sumé, PB, Brazil

Abstract

In this work we consider the time-varying gravitomagnetism formalism in the context of the Brans-Dicke theory of gravity. Initially, the gravitoelectric and gravitomagnetic potentials are exhibited. Then, a linear model for the temporal variation of the angular momentum of a massive body is presented. As an application, we examine the gravitomagnetic time delay, comparing the results obtained with those predicted by General Relativity.

Subject Classification: 04.50.Kd, 04.25.Nx

Keywords: Gravitomagnetism, Brans-Dicke theory, Weak field approximation

1 Introduction

In the linear approximation of the General Relativity theory is possible to separate the gravitational effects in two parts: the first, linked with mass, and the second, connected with mass currents. In this framework, we say that mass currents generate a field called, by analogy with electromagnetism, the gravitomagnetic field [5]. Thereby, the rotation of the Earth produces a gravitomagnetic field that causes a precession in gyroscopes orbiting around the planet. This effect, named the Lense-Thirring effect, was verified by the GP-B experiment with an accuracy of 19% [7]. However, other aspects of gravitomagnetism have been studied [9] and there is an investigation line that takes into account situations where the angular momentum of the source varies with time [10]. Indeed, rarely the rotation rate of an astronomical source is constant and, in the Earth case, the rotation rate decreases slowly due mostly to tidal friction. As a consequence, the Earth-Moon distance increases at a rate of 4 *cm* per year to ensure the conservation of angular momentum of the system [3].

On the other hand, there are alternative theories to the General Relativity, among which are highlighted the scalar-tensor theories [11]. These theories introduce an additional field, the scalar field ϕ, which together with the space-time metric $g_{\mu\nu}$ will be responsible for the description of gravitational phenomena. In general, the scalar field coupling factor is $\omega(\phi)$; the Brans-Dicke theory [4] is obtained when $\omega(\phi) = \omega = constant$. The scalar-tensor theories are popular, among other reasons [1], because they incorporate key ingredients of string theories, such as a dilaton-like gravitational scalar field and its non-minimal coupling to the curvature [8].

In this paper, we develop the time-varying gravitomagnetism formalism in the context of Brans-Dicke theory. In this way, the gravitoelectric and gravitomagnetic potentials are obtained in Section 2. Following, in Section 3, we consider a simple model in which the intrinsic angular momentum of a massive body varies linearly with time. Then, in Section 4, we show an expression for the gravitomagnetic time delay, comparing the result with General Relativity predictions. Finally, in Section 5, the conclusions are presented.

2 Gravitomagnetism

The Brans-Dicke field equations are given by [4]

$$G_{\mu\nu} = \frac{8\pi}{c^4\phi}T_{\mu\nu} + \frac{\omega}{\phi^2}(\phi_{,\mu}\phi_{,\nu} - \frac{1}{2}g_{\mu\nu}\phi_{,\alpha}\phi^{,\alpha}) + \frac{1}{\phi}(\phi_{,\mu;\nu} - g_{\mu\nu}\Box\phi), \tag{1}$$

$$\Box\phi = \frac{8\pi T}{c^4(2\omega+3)}, \tag{2}$$

where ω is the scalar field coupling constant and $T = T^{\alpha}{}_{\alpha}$.

Let us consider the weak field approximation, in which $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $\eta_{\mu\nu} = diag(-1, 1, 1, 1)$ denotes Minkowski metric tensor and $h_{\mu\nu}$ is a small perturbation term, so that we keep only first-order terms in $h_{\mu\nu}$. Besides, we take $\phi = \phi_0 + \varepsilon$, where ϕ_0 is constant and $\varepsilon = \varepsilon(x)$ is also a small perturbation term with $|\varepsilon/\phi_0| \ll 1$. Thus, the field equations, using the Brans-Dicke gauge $(h^{\mu}{}_{\nu} - \frac{1}{2}\delta^{\mu}{}_{\nu}h)_{,\mu} = \varepsilon_{,\nu}\phi_0^{-1}$, reduce to [2]

$$\Box h_{\mu\nu} = -\frac{16\pi}{c^4\phi_0}\left[T_{\mu\nu} - \frac{\omega+1}{2\omega+3}\eta_{\mu\nu}T\right], \tag{3}$$

$$\Box\varepsilon = \frac{8\pi T}{c^4(2\omega+3)}, \tag{4}$$

with $h = h^{\alpha}{}_{\alpha}$. And more, with the definition

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h - \varepsilon\phi_0^{-1}\eta_{\mu\nu}, \tag{5}$$

the equation (3) becomes

$$\Box\bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4}\left(\frac{2\omega+3}{2\omega+4}\right)T_{\mu\nu}, \tag{6}$$

being $\phi_0^{-1} = \left(\frac{2\omega+3}{2\omega+4}\right)G$, where G is the Newton's gravitational constant [4].

The general solution of the equation (6) is a superposition of the particular retarded solution plus the general solution of the homogeneous wave equation. Here, we are interested in the particular solution

$$\overline{h}_{\mu\nu} = \frac{4G}{c^4}\left(\frac{2\omega+3}{2\omega+4}\right)\int\frac{T_{\mu\nu}(ct - |\vec{r}-\vec{r}'|, \vec{r}')}{|\vec{r}-\vec{r}'|}d^3x'. \tag{7}$$

Assuming a localized matter distribution with density ρ and velocity field $\overrightarrow{v}$, with the condition $|\overrightarrow{v}| << c$, the components of the energy-momentum tensor will be given by:

$$\begin{aligned} T_{00} &= \rho c^2, \\ T_{0i} &= -cj_i, \\ T_{ij} &= \rho v_i v_j + p\delta_{ij}, \end{aligned} \tag{8}$$

where $\vec{j} = \rho\vec{v}$ is the mass current and p is the pressure. Considering the expression of T_{ij} and equation (7), it follows that

$$\overline{h}_{ij} = O(c^{-4}). \tag{9}$$

The terms of this order will be neglected. Now, we define [6]:

$$\overline{h}_{00} = \frac{4\Phi(t,\vec{r})}{c^2}, \tag{10}$$

$$\overline{h}_{0i} = -\frac{2A_i(t,\vec{r})}{c^2}, \tag{11}$$

being $\Phi(t,\vec{r})$ the gravitoelectric potential and $\vec{A}(t,\vec{r})$ the gravitomagnetic vector potential. Then, with these definitions and the expressions to T_{00} and T_{0i} given in (8), we find from (6) that

$$\Box\Phi(t,\vec{r}) = -4\pi G\left(\frac{2\omega+3}{2\omega+4}\right)\rho, \tag{12}$$

$$\Box A_i(t,\vec{r}) = -\frac{8\pi G}{c}\left(\frac{2\omega+3}{2\omega+4}\right)j_i. \tag{13}$$

In the stationary case, (12) and (13) reduce to

$$\nabla^2\Phi(\vec{r}) = -4\pi G\left(\frac{2\omega+3}{2\omega+4}\right)\rho, \tag{14}$$

$$\nabla^2 A_i(\vec{r}) = -\frac{8\pi G}{c}\left(\frac{2\omega+3}{2\omega+4}\right)j_i. \tag{15}$$

If the matter distribution is confined around the origin of spatial coordinates, so far from the source we will have the solutions

$$\Phi(\vec{r}) = \left(\frac{2\omega+3}{2\omega+4}\right)\frac{GM}{r}, \tag{16}$$

$$\vec{A}(\vec{r}) = \left(\frac{2\omega+3}{2\omega+4}\right)\frac{G(\vec{J}\times\vec{r})}{cr^3}, \tag{17}$$

where $r = |\vec{r}|$, M and $\vec{J}$ are the mass and angular momentum of the source.

The Brans-Dicke gauge can be written as $\overline{h}^{\mu\nu}{}_{,\mu} = 0$. Thus, using equations (10) and (11), we obtain

$$\frac{2}{c}\frac{\partial\Phi(t,\vec{r})}{\partial t} + \nabla\cdot\vec{A}(t,\vec{r}) = 0. \tag{18}$$

3 Linear Model for Time-Varying Gravitomagnetism

A simple model for time-varying gravitomagnetism can be developed if we admit that the angular momentum of the source varies linearly with time [10]. Indeed, in many cases of astrophysical interest, the temporal variation of the angular momentum of the source cannot be effectively linear, but a linear model corresponds to a first approximation of the problem. Also, in our discussion, only nonradiative situations are considered, in which the angular momentum slowly varies with time.

To get the desired model, let us define the following time-dependent gravitational potentials:

$$\Phi(t,\vec{r}) = \sigma(t)\Phi(\vec{r}), \tag{19}$$

$$\vec{A}(t,\vec{r}) = f(t)\vec{A}(\vec{r}), \tag{20}$$

where $\sigma(t)$ and $f(t)$ are functions to be determined. Hence, in according to (18), we have

$$\frac{2}{c}\Phi(\vec{r})\frac{d\sigma}{dt} + f(t)\nabla\cdot\vec{A}(\vec{r}) = 0. \tag{21}$$

However, for the stationary potentials $\Phi(\vec{r})$ and $\vec{A}(\vec{r})$, the equation (18) implies that $\nabla\cdot\vec{A}(\vec{r}) = 0$. Thus, it follows from (21) that

$$\frac{d\sigma}{dt} = 0. \tag{22}$$

Then, for simplicity, we make $\sigma = 1$. Now, using (11) and (20), we obtain

$$\bar{h}_{0i}(t,\vec{r}) = -\frac{2}{c^2}f(t)A_i(\vec{r}) = f(t)\bar{h}_{0i}(\vec{r}). \tag{23}$$

Substituting this result in (6), we find that

$$\frac{d^2 f(t)}{dt^2} = 0, \tag{24}$$

since $T_{0i}(t,\vec{r}) = f(t)T_{0i}(\vec{r})$. Therefore, it is concluded that $f(t)$ is a linear function of time; furthermore, $f(t)$ must be such that the relation $\frac{2\,|A_i(t,\overrightarrow{r})|}{c^2} \ll 1$ is satisfied in the interval of time considered, maintaining the weak field approximation valid.

Taking into account the previous results, the gravitational potentials, far from the source, will be given by (16) and

$$\vec{A} = f(t)\vec{A}(\vec{r}) = f(t)\left(\frac{2\omega+3}{2\omega+4}\right)\frac{G(\vec{J}\times\vec{r})}{cr^3} = \left(\frac{2\omega+3}{2\omega+4}\right)\frac{GJ(t)(\hat{J}\times\vec{r})}{cr^3}, \tag{25}$$

being $f(t)\vec{J} \equiv J(t)\hat{J}$, where $\hat{J}$ is a fixed unit angular momentum vector of the source.

We can obtain an expression for $J(t)$ considering that the source undergoes a linear change in its angular momentum, that goes since J_1 at initial time t_1 until J_2 at t_2. Then, we get

$$J(t) = J_1 + \left(\frac{J_2 - J_1}{t_2 - t_1}\right)(t - t_1). \tag{26}$$

4 Gravitomagnetic Time Delay

Let us calculate the gravitational time delay when a ray of electromagnetic radiation propagate from a point P_1 to a point P_2, with the angular momentum of source linearly varying since J_1 until J_2. The effect can be separated into two parts: the Shapiro time delay [11] and the gravitomagnetic time delay, the latter due to rotation of source.

If the ray propagate from a point $P_1 : (ct_1, \vec{r}_1)$ to a point $P_2 : (ct_2, \vec{r}_2)$ in the spacetime with $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, then we will have [6]

$$\int_{t_1}^{t_2} dt = \frac{1}{c}\int_{P_1}^{P_2} \mid d\vec{r} \mid + \frac{1}{2c}\int_{P_1}^{P_2} h_{\mu\nu}k^\mu k^\nu dl, \tag{27}$$

where $k^\mu = (1, \hat{k})$, $\hat{k}$ is the constant unit propagation vector of the signal and $dl = \mid d\vec{r} \mid = (\delta_{ij}dx^i dx^j)^{1/2}$ denotes the Euclidean length element along the straight line that joins P_1 to P_2. Therefore, the gravitational time delay is defined as

$$\Delta = \frac{1}{2c}\int_{P_1}^{P_2} h_{\mu\nu}k^\mu k^\nu dl. \tag{28}$$

Moreover, using (5) and also that $\eta_{\mu\nu}k^\mu k^\nu = k^\mu k_\mu = -1 + \hat{k}^i\hat{k}_i = -1 + 1 = 0$, we can write

$$\Delta = \frac{1}{2c}\int_{P_1}^{P_2} \bar{h}_{\mu\nu}k^\mu k^\nu dl. \tag{29}$$

Thus, from (10) and (11), one obtains

$$\Delta = \frac{2}{c^3}\int_{P_1}^{P_2} \Phi(t, \vec{r})dl - \frac{2}{c^3}\int_{P_1}^{P_2} \vec{A}(t, \vec{r}) \cdot \hat{k}dl, \tag{30}$$

where the Shapiro time delay is given by

$$\Delta_S = \frac{2}{c^3}\int_{P_1}^{P_2} \Phi(t, \vec{r})dl \tag{31}$$

and the gravitomagnetic time delay is expressed by

$$\Delta_{GM} = -\frac{2}{c^3}\int_{P_1}^{P_2} \vec{A}(t, \vec{r}) \cdot \hat{k}dl. \tag{32}$$

Substituting the potentials (16) and (25) in (31) and (32), we find that

$$\Delta_S = \frac{2GM}{c^3}\left(\frac{2\omega + 3}{2\omega + 4}\right)\int_{P_1}^{P_2} \frac{dl}{r}, \tag{33}$$

$$\Delta_{GM} = -\frac{2G}{c^4}\left(\frac{2\omega+3}{2\omega+4}\right)\int_{P_1}^{P_2}\frac{J(t)(\hat{J}\times\vec{r})}{r^3}\cdot\hat{k}dl. \quad (34)$$

The equation (33) is the Brans-Dicke Shapiro time delay [2]. On the other hand, from (26), it follows that

$$J(t) = J(l) = J_1 + \frac{J_2 - J_1}{L}l, \quad (35)$$

where we take $t = l/c$, $t_1 = 0$ and $t_2 = L/c$, being L the length of the straight line that joins P_1 to P_2. Thereby, (34) reads

$$\Delta_{GM} = -\frac{2G}{c^4}\left(\frac{2\omega+3}{2\omega+4}\right)\hat{J}\cdot\left(J_1\int_0^L\frac{\vec{r}\times\hat{k}}{r^3}dl + \frac{J_2-J_1}{L}\int_0^L\frac{\vec{r}\times\hat{k}}{r^3}ldl\right). \quad (36)$$

Then, by solving the integrals [10], we have

$$\Delta_{GM} = -\frac{2G}{c^4}\left(\frac{2\omega+3}{2\omega+4}\right)\hat{J}\cdot\left[\frac{\hat{r}_1\times\hat{r}_2}{1+\hat{r}_1\cdot\hat{r}_2}\left(\frac{J_1}{r_1}+\frac{J_2}{r_2}\right)\right], \quad (37)$$

with $\hat{r}_1$ and $\hat{r}_2$ indicating the unit vectors along $\vec{r}_1$ and $\vec{r}_2$, respectively. The factor $\frac{2\omega+3}{2\omega+4}$ represents the contribution of the scalar field, since that if $\varepsilon \neq 0$ in (4) then ω is finite. In the limit $\omega \to \infty$, the result of the gravitomagnetic time delay in General Relativity is recovered [10]. It is worth mentioning that, if the angular momentum is constant, with $J_1 = J_2 = J$, and we take the limit $\omega \to \infty$, we regain the result of General Relativity to the stationary case [6].

5 Conclusion

We study the time-varying gravitomagnetism in the Brans-Dicke theory context, exhibiting the gravitoelectric e gravitomagnetic potentials. After, was developed a model in which the angular momentum of the source is linearly dependent upon time. As an application, was obtained an expression to the gravitomagnetic time delay. In the limit $\omega \to \infty$, the result of the gravitomagnetic time delay in General Relativity was recovered.

The equation (37) may be of astrophysical interest in pulsar timing, provided that the angular momentum of a pulsar varies due to the external electromagnetic braking torques [3].

References

[1] M. Artymowski, Z. Lalak and M. Lewicki, Inflation and dark energy from the Brans-Dicke theory, *JCAP*, **2015** (2015), no. 06, 1 - 15. http://dx.doi.org/10.1088/1475-7516/2015/06/031

[2] A. Barros and C. Romero, Gravitomagnetic time delay and the Lense-Thirring effect in Brans-Dicke theory of gravity, *M*od. Phys. Lett. A, **18** (2003), 2117 - 2124. http://dx.doi.org/10.1142/S0217732303011721

[3] D. Bini, C. Cherubini, C. Chicone and B. Mashhoon, Gravitational induction, *Class. Quantum Grav.*, **25** (2008), 225014. http://dx.doi.org/10.1088/0264-9381/25/22/225014

[4] C. Brans and R. H. Dicke, Mach's principle and a relativistic theory of gravitation, *Phys. Rev.*, **124** (1961), 925 - 935. http://dx.doi.org/10.1103/physrev.124.925

[5] I. Ciufolini and J. A. Wheeler, *Gravitation and Inertia*, Princeton University Press, Princeton, 1995.

[6] I. Ciufolini, S. Kopeikin, B. Mashhoon and F. Ricci, On the gravitomagnetic time delay, *Phys. Lett. A*, **308** (2003), 101 - 109. http://dx.doi.org/10.1016/S0375-9601(02)01804-2

[7] C. W. F. Everitt, D. B. DeBra, B. W. Parkinson et al, Gravity Probe B: Final results of a space experiment to test general relativity, *Phys. Rev. Lett.*, **106** (2011). http://dx.doi.org/10.1103/PhysRevLett.106.221101

[8] V. Faraoni, Scalar field mass in generalized gravity, *Class. Quantum Grav.*, **26** (2009), 145014. http://dx.doi.org/10.1088/0264-9381/26/14/145014

[9] P. Landry and E. Poisson, Gravitomagnetic response of an irrotational body to an applied tidal field, *Phys. Rev. D*, **91** (2015). http://dx.doi.org/10.1103/physrevd.91.104026

[10] B. Mashhoon, Time-varying gravitomagnetism, *Class. Quantum Grav.*, **25** (2008), 085014. http://dx.doi.org/10.1088/0264-9381/25/8/085014

[11] C. M. Will, The confrontation between general relativity and experiment, *Living Rev. Rel.*, **17** (2014). http://dx.doi.org/10.12942/lrr-2014-4

Received: July 17, 2015; Published: September 17, 2015