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GRAVITOMAGNETISM IN BRANS-DICKE THEORY OF GRAVITY

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We discuss the gravitomagnetism in the context of Brans-Dicke theory of gravity. We obtain the equation of motion of a particle in terms of gravitoelectric and gravitomagnetic fields. Comparing the result obtained with that predicted by general relativity we show that the difference between the two theories lies in the gravitomagnetic force.

Keywords: Gravitomagnetism; Brans-Dicke; weak field approximation.

1. Introduction

Lense and Thirring[1] realized that as a consequence of the Einstein field equations an effect called the Lense-Thirring precession (or "frame dragging") should occur near the spinning massive objects like the Earth. Frame dragging can be viewed as a Machian effect,[2] which is not predicted by Newtonian gravity. Effects of general relativity associated with the rotation of massive bodies may be better understood by using a formal analogy with electromagnetism. The idea is that mass currents generate a field called the gravitomagnetic field.[3]

There are indirect evidences of the existence of gravitomagnetism in an astrophysical context and in the weak field and slow motion approximation valid throughout the Solar System.[4,5] On the other hand, the first accurate measurement of the Lense-Thirring effect, with an error estimate of 10%, was performed using the current technology of laser ranged satellites (LAGEOS and LAGEOS II).[6] On april 2004 has been launched the Gravity Probe B experiment,[7] an ongoing space mission using orbiting gyroscopes, which plans to measure the Lense-Thirring effect with an error of about 1%. Certainly, these experimental programs will open new possibilities of testing general relativity against other metric theories of gravity,[8] in particular the scalar-tensor theory. Our aim in this paper is to obtain a "Lorentz force law" with the gravitoelectric and gravitomagnetic fields in the context of Brans-Dicke theory, and then compare the results with those predicted by general relativity. To get this result we will use the fact that, in the weak field approximation, solutions of Brans-Dicke equations are simply related to the solutions of general relativity equations for the same matter distribution.[9]

This paper is organized as follows. In Sec. 2, we introduce the basic ideas of gravitomagnetism. Then, in Sec. 3, we show how general relativity and Brans-Dicke theory of gravity are related in the weak field approximation. The gravitoelectric and gravitomagnetic fields in Brans-Dicke theory are defined in Sec. 4 and the force law is obtained. Section 5 is devoted to some remarks.

2. Gravitomagnetism in General Relativity

Considering the weak field approximation of general relativity we assume the metric tensor $g_{\mu\nu}$ as being $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\upsilon}$, where $\eta_{\mu\nu} = diag(-1,1,1,1)$ denotes Minkowski metric tensor and $h_{\mu\upsilon}$ is a small perturbation term. Then, by keeping only first-order terms in $h_{\mu\upsilon}$, the Einstein equations become

$$\Box\overline{h}_{\mu\nu} = -\frac{16\pi G}{c^4}T_{\mu\nu}, \tag{1}$$

where $\overline{h}^{\mu}_{\nu} = h^{\mu}_{\upsilon} - \frac{1}{2}\delta^{\mu}_{\nu}h$ and we are adopting the usual harmonic coordinate gauge $\left(h^{\mu}_{\upsilon} - \frac{1}{2}\delta^{\mu}_{\nu}h\right)_{,\mu} = 0$.

If we assume a non-relativistic matter distribution with a mass density ρ and velocity field $\overrightarrow{v}$, then (1) yields

$$\Box\overline{h}_{00} = -\frac{16\pi G}{c^2}\rho, \tag{2}$$

$$\Box\overline{h}_{0i} = \frac{16\pi G}{c^3}\rho v_i, \tag{3}$$

where v_i denotes the velocity components, and terms such as p and $v_i v_j/c^4$ have been neglected. Let us now specialise to the case of a stationary gravitational field of a slowly rotating body. Then (2) and (3) reduce to

$$\nabla^2\left(\frac{c^2\overline{h}_{00}}{4}\right) \equiv \nabla^2(\Phi) = -4\pi G\rho, \tag{4}$$

$$\nabla^2\overline{h}_{0i} = \frac{16\pi G}{c^3}\rho v_i, \tag{5}$$

where Φ is the gravitoelectric scalar potential. Far from the source we will have

$$\Phi = \frac{GM}{r}, \tag{6}$$

$$\overrightarrow{\overline{h}} = -\frac{2G(\overrightarrow{J}\times\overrightarrow{r})}{c^3r^3} \equiv -\frac{2\overrightarrow{A}}{c^2}, \tag{7}$$

where $\overrightarrow{A}$ is the gravitomagnetic vector potential vector, $\overline{h}_{0i}$ are the components of the vector $\overrightarrow{\overline{h}}$, M and $\overrightarrow{J}$ are the total mass and angular momentum of the source. In close analogy with electrodynamics we define the gravitoelectric field to be $\overrightarrow{E} = -\nabla\Phi$ and the gravitomagnetic field to be

$$\overrightarrow{B} = \overrightarrow{\nabla} \times \overrightarrow{A} = \frac{G}{c}\left[\frac{3\widehat{r}(\widehat{r}\cdot\overrightarrow{J}) - \overrightarrow{J}}{r^3}\right]. \tag{8}$$

With these conditions, the spacetime metric has the form

$$ds^2 = -c^2\left(1 - 2\frac{\Phi}{c^2}\right)dt^2 - \frac{4}{c}\left(\overrightarrow{A}\cdot d\overrightarrow{x}\right)dt + \left(1 + 2\frac{\Phi}{c^2}\right)\delta_{ij}dx^i dx^j. \tag{9}$$

The Lagrangian for the motion of a test particle of mass m is $L = -mcds/dt$. To first order in Φ and $\overrightarrow{A}$ it becomes

$$L = -mc^2\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}} + m\gamma\left(1 + \frac{v^2}{c^2}\right)\Phi - \frac{2m}{c}\gamma\overrightarrow{v}\cdot\overrightarrow{A}, \tag{10}$$

where $\gamma = 1/\sqrt{1 - v^2/c^2}$. Indeed, in the weak gravitational field, we assume that the particle has a small velocity.[10] Then, we obtain from (10) that

$$L = -mc^2\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}} + m\Phi - \frac{2m}{c}\overrightarrow{v}\cdot\overrightarrow{A}. \tag{11}$$

This equation is analogous to the electromagnetic case.[11] Thus, the equation of motion, $\overrightarrow{F} = d\overrightarrow{p}/dt$, with $\overrightarrow{p} = \gamma m\overrightarrow{v}$, takes a Lorentz force law form

$$\overrightarrow{F} = -m\overrightarrow{E} - 2m\frac{\overrightarrow{v}}{c}\times\overrightarrow{B}. \tag{12}$$

3. The Weak Field Approximation of Brans-Dicke Theory

In Brans-Dicke theory of gravity the field equations are given by[12]

$$G_{\mu\upsilon} = \frac{8\pi}{c^4\phi}T_{\mu\upsilon} + \frac{\omega}{\phi^2}(\phi,_{\mu}\phi,_{\upsilon} - \frac{1}{2}g_{\mu\upsilon}\phi,_{\alpha}\phi^{,\alpha}) + \frac{1}{\phi}(\phi,_{\mu;\upsilon} - g_{\mu\upsilon}\Box\phi). \tag{13}$$

One can linearize Brans-Dicke field equations by assuming that the metric $g_{\mu\nu}$ and the scalar field ϕ can be written as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\upsilon}$ and $\phi = \phi_0 + \epsilon$, where ϕ_0 is a constant and $\epsilon = \epsilon(x)$ is a first-order term (it is assumed that both $|h_{\mu\upsilon}|$ and $\left|\epsilon\phi_0^{-1}\right|$ are $\ll 1$). In this procedure, we have used the Brans-Dicke gauge[12] $\left(h^{\mu}_{\upsilon} - \frac{1}{2}\delta^{\mu}_{\nu}h\right),_{\mu} = \epsilon,_{\upsilon}\phi_0^{-1}$.

The problem of finding solutions of Brans-Dicke equations of gravity in the weak field approximation may be reduced to solving the linearized Einstein field equations for the same energy-momentum tensor.[9] Indeed, if $g^*_{\mu\nu}(G,x)$ is a known solution of the Einstein equations in the weak field approximation for a given $T_{\mu\upsilon}$, then the Brans-Dicke solution corresponding to the same $T_{\mu\upsilon}$ will be given in the weak field approximation by

$$g_{\mu\upsilon}(x) = [1 - \epsilon G_0]g^*_{\mu\nu}(G_0, x), \tag{14}$$

where G is the gravitational constant and $G_0 = \phi_0^{-1} = \left(\frac{2\omega+3}{2\omega+4}\right) G$ and the function $\epsilon(x)$ is a solution of the scalar field equation

$$\Box\epsilon = \frac{8\pi T}{c^4(2\omega+3)}, \tag{15}$$

with T denoting the trace of $T_{\mu\upsilon}$.

4. Gravitomagnetism in Brans-Dicke Theory

Let us consider the metric of the spacetime in the context of Brans-Dicke theory. From (14) and (9), we have

$$ds^2_{BD} = (1-\epsilon G_0)\left[-c^2\left(1-2\frac{\Phi(G_0)}{c^2}\right)dt^2 - \frac{4}{c}\left(\overrightarrow{A}(G_0)\cdot d\overrightarrow{x}\right)dt + \left(1+2\frac{\Phi(G_0)}{c^2}\right)\delta_{ij}dx^i dx^j\right]. \tag{16}$$

This line element can be written as

$$ds^2_{BD} = -c^2\left(1-2\frac{\Phi(G_0)}{c^2} - \epsilon G_0\right)dt^2 - \frac{4}{c}\left(\overrightarrow{A}(G_0)\cdot d\overrightarrow{x}\right)dt + \left(1+2\frac{\Phi(G_0)}{c^2} - \epsilon G_0\right)\delta_{ij}dx^i dx^j. \tag{17}$$

Now, if we define

$$2\frac{\Phi_1}{c^2} = 2\frac{\Phi(G_0)}{c^2} + \epsilon G_0, \tag{18}$$

$$2\frac{\Phi_2}{c^2} = 2\frac{\Phi(G_0)}{c^2} - \epsilon G_0, \tag{19}$$

the metric will be given by

$$ds^2_{BD} = -c^2\left(1-2\frac{\Phi_1}{c^2}\right)dt^2 - \frac{4}{c}\left(\overrightarrow{A}(G_0)\cdot d\overrightarrow{x}\right)dt + \left(1+2\frac{\Phi_2}{c^2}\right)\delta_{ij}dx^i dx^j. \tag{20}$$

In close analogy to the general relativity approach, we will have the Lagrangian of a particle of mass m

$$L_{BD} = -mc^2\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}} + m\gamma\Phi_1 + m\gamma\frac{v^2}{c^2}\Phi_2 - \frac{2m}{c}\gamma\overrightarrow{v}\cdot\overrightarrow{A}(G_0). \tag{21}$$

However, since that $\frac{v}{c} \ll 1$, the Lagrangian can be simplified

$$L_{BD} = -mc^2\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}} + m\Phi_1 - \frac{2m}{c}\gamma\overrightarrow{v}\cdot\overrightarrow{A}(G_0). \tag{22}$$

Again, we immediately arrive at the equation of motion

$$\overrightarrow{F}_{BD} = -m\overrightarrow{E} - 2m\frac{\overrightarrow{v}}{c}\times\overrightarrow{B}(G_0), \tag{23}$$

where

$$\overrightarrow{E} = -\nabla\Phi_1, \tag{24}$$

and

$$\overrightarrow{B}(G_0) = \overrightarrow{\nabla} \times \overrightarrow{A}(G_0) = \frac{G_0}{c}\left[\frac{3\widehat{r}(\widehat{r}\cdot\overrightarrow{J}) - \overrightarrow{J}}{r^3}\right] = \left(\frac{2\omega+3}{2\omega+4}\right)\overrightarrow{B}. \tag{25}$$

It is interesting to note that the gravitoelectric field is exactly the same of the general relativity case. Really, the scalar field produced for a stationary mass point of mass M is[12] $\epsilon = 2M/c^2r(2\omega+3)$. Therefore, from (18), it follows that $\Phi_1 = GM/r$. On the other hand, the difference between the two theories, in this approximation, lies in the gravitomagnetic field due to the factor $\frac{2\omega+3}{2\omega+4}$.

5. Final Remarks

We have examined the equation of motion of a particle in gravitoelectric and gravitomagnetic fields in Brans-Dicke theory of gravity. It has been verified that the gravitomagnetic force predicted by Brans-Dicke theory differs of the corresponding force in general relativity by the correction factor $\frac{2\omega+3}{2\omega+4}$. As a consequence, the Lense-Thirring effect will be quantitatively different in the two theories.[13]

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