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Refractive Index Associated with Global Monopole Spacetime in the Brans-Dicke Theory of Gravity

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Abstract

In this work, we take the Brans-Dicke theory as the fundamental theory of gravity, considering that the spacetime is static and spherically symmetric, making no assumptions about the isotropy of internal pressures of the source generating the gravitational field. We employ the weak field approximation in order to derive formulae for the refractive index associated with the adopted spacetime. Then, we calculate the expression of the spacetime refractive index for the global monopole.

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1 Introduction

Einstein was the first to suggest the idea of the analogy between the gravitational field and a refractive medium [6]. Indeed, considering the General Relativity theory, the gravitational field, in relation to the propagation of light, can be interpreted as a medium with refractive index n [8, 9].

On the other hand, there are alternative theories of gravity [11]. One of the most popular is the Brans-Dicke scalar-tensor theory [4], which is the more simplest generalization of Einstein's theory of gravity, with the gravitational effects described by the spacetime metric $g_{\alpha\beta}$ and also by a scalar field ϕ. We must note that the interest in the alternative theories of gravity occurs in many cases because the high-energy theories, seeking the quantization of gravity or its unification with the other interactions, generally make predictions that diverge from General Relativity; the scalar-tensor theories, for instance, naturally incorporate elements of string theory, such as a dilaton-like gravitational scalar field [7].

In this work, let us obtain the expression of a refractive index n in the context of the Brans-Dicke theory; this index simulates gravitational effects of a static spherically symmetric spacetime possessing a source with non-zero pressures. Then, as an application, we calculate n for the global monopole spacetime [1, 5]. Such monopoles present Goldstone fields whose energy density decreases with r^{-2}, so that the energy of the monopole varies linearly with r. This suggests that global monopoles can produce appreciable gravitational effects. A striking feature is that the curved spacetime generated by the monopoles has a solid angle deficit in the hypersurfaces $t = constant$, being the area of a sphere of radius r in this space different from $4\pi r^2$.

The paper is organized as follows: in Section 2, we get the solution of Brans-Dicke equations for a static metric with spherical symmetry considering the weak field approximation, given that the source of gravitational field has radial and transverse internal pressures. In Section 3, we show the expression of the spacetime refractive index. Then, in Section 4, we calculate the refractive index associated with the global monopole spacetime. Finally, Section 5 is devoted to our conclusions.

2 Static Spherically Symmetric Solution

The Brans-Dicke field equations are given by

$$G_{\alpha\beta} = \frac{8\pi}{\phi} T_{\alpha\beta} + \frac{\omega}{\phi^2}(\phi_{,\alpha}\phi_{,\beta} - \frac{1}{2} g_{\alpha\beta}\phi_{,\mu}\phi^{,\mu}) + \frac{1}{\phi}(\phi_{,\alpha;\beta} - g_{\alpha\beta}\Box\phi), \tag{1}$$

$$\Box\phi = \frac{8\pi T}{2\omega + 3}, \tag{2}$$

where ω is the scalar field coupling constant, $T = T^{\alpha}{}_{\alpha}$ and we use units in which $c = 1$.

Let us consider the weak field approximation, in which $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$, being $\eta_{\alpha\beta}$ the flat spacetime metric and $h_{\alpha\beta}$ a small perturbation term, so that we keep only first-order terms in $h_{\alpha\beta}$. Besides, we take $\phi = \phi_0 + \varepsilon$, where ϕ_0 is constant and $\varepsilon = \varepsilon(x)$ is also a small perturbation term with $\mid \varepsilon/\phi_0 \mid \ll 1$.

Thus, the field equations, using the Brans-Dicke gauge $(h^{\alpha}{}_{\beta} - \frac{1}{2}\delta^{\alpha}{}_{\beta}h)_{;\alpha} = \varepsilon_{,\beta}\phi_0^{-1}$ and considering the static case, reduce to [2]

$$\nabla^2 h_{\alpha\beta} = -16\pi\phi_0^{-1}\left[T_{\alpha\beta} - \frac{\omega+1}{2\omega+3}\eta_{\alpha\beta}T\right], \tag{3}$$

$$\nabla^2\varepsilon = \frac{8\pi T}{2\omega + 3}, \tag{4}$$

being $h = h^{\alpha}{}_{\alpha}$. And more, with the definition

$$\bar{h}_{\alpha\beta} = h_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}h - \varepsilon\phi_0^{-1}\eta_{\alpha\beta}, \tag{5}$$

the equation (3) becomes

$$\nabla^2\bar{h}_{\alpha\beta} = -16\pi G_0 T_{\alpha\beta}, \tag{6}$$

with $G_0 = \left(\frac{2\omega+3}{2\omega+4}\right)G = \phi_0^{-1}$, where G is the Newton's gravitational constant [4].

Now, we consider a spherically symmetric weak field for which the energy-momentum tensor is given by [3]

$$T_{\alpha\beta} = \begin{pmatrix} \rho(r) & 0 & 0 & 0 \\ 0 & p_r(r) & 0 & 0 \\ 0 & 0 & p_t(r)r^2 & 0 \\ 0 & 0 & 0 & p_t(r)r^2\sin^2\theta \end{pmatrix}, \tag{7}$$

where ρ is the energy density, p_r and p_t are the radial and transverse pressures, respectively; at the origin, we have $p_r(0) = p_t(0)$. Therefore, with the conservation condition

$$T^{\alpha\beta}{}_{;\beta} = 0, \tag{8}$$

we obtain

$$T^{1\beta}{}_{;\beta} = \partial_r p_r(r) + \frac{2\left[p_r(r) - p_t(r)\right]}{r} = 0, \tag{9}$$

or

$$p_t(r) = p_r(r) + \frac{1}{2} r \partial_r p_r(r), \tag{10}$$

because $\eta_{\alpha\beta} = diag(-1, 1, r^2, r^2 \sin^2\theta)$ in spherical polar coordinates.

Also, one can define

$$\bar{h}_{\alpha\beta} = \begin{pmatrix} F_0(r) & 0 & 0 & 0 \\ 0 & F_r(r) & 0 & 0 \\ 0 & 0 & F_t(r)r^2 & 0 \\ 0 & 0 & 0 & F_t(r)r^2\sin^2\theta \end{pmatrix}, \tag{11}$$

with $F_r(0) = F_t(0)$. Now, the Brans-Dicke gauge will be written as $\bar{h}^{\alpha\beta}{}_{;\beta} = 0$. Thus, for similarity with the equations (7) and (8), we have immediately

$$F_t(r) = F_r(r) + \frac{1}{2} r \partial_r F_r(r). \tag{12}$$

To solve the field equations (6), we must have

$$\nabla^2 \bar{h}_{\alpha\beta} = \bar{h}_{\alpha\beta;\gamma}{}^{;\gamma} = \begin{pmatrix} L_0(r) & 0 & 0 & 0 \\ 0 & L_r(r) & 0 & 0 \\ 0 & 0 & L_t(r)r^2 & 0 \\ 0 & 0 & 0 & L_t(r)r^2\sin^2\theta \end{pmatrix}, \tag{13}$$

where

$$L_0(r) = \frac{1}{r^2}\partial_r(r^2\partial_r F_0), \tag{14}$$

$$L_r(r) = \frac{1}{r^2}\partial_r(r^2\partial_r F_r) - 4\left(\frac{F_r - F_t}{r^2}\right) = \frac{1}{r^4}\partial_r(r^4\partial_r F_r), \tag{15}$$

$$L_t(r) = \frac{1}{r^2}\partial_r(r^2\partial_r F_t) + 2\left(\frac{F_r - F_t}{r^2}\right). \tag{16}$$

Then, using (6) and (7), we find

$$F_0 = -16\pi\left(\frac{2\omega+3}{2\omega+4}\right) G \int \left[\int \rho r^2 dr\right] r^{-2} dr, \tag{17}$$

$$F_r = -16\pi\left(\frac{2\omega+3}{2\omega+4}\right) G \int \left[\int p_r r^4 dr\right] r^{-4} dr. \tag{18}$$

The function F_t can be easily obtained from (12) and (18). On the other hand, ε can be obtained from (4). So, in agreement with (7), we have

$$\frac{1}{r^2}\frac{d}{dr}\left[r^2\frac{d\varepsilon}{dr}\right] = \frac{8\pi}{2\omega+3}(-\rho + p_r + 2p_t),$$

and the solution

$$\varepsilon = \frac{8\pi}{2\omega+3}\int\left[\int(-\rho+p_r+2p_t)r^2dr\right]r^{-2}dr. \tag{19}$$

We obtain the formal solution of the Brans–Dicke field equations for a static metric with spherical symmetry. Let us check the consistency of the solution by getting the gravitational field produced by a point of mass M, for which

$$\rho = M\delta(\overrightarrow{r}), \quad p_r = p_t = 0. \tag{20}$$

In this case, according to (17), (18) and (12), we get

$$F_0 = -16\pi\left(\frac{2\omega+3}{2\omega+4}\right)GM\int\left[\int r^2\delta(\overrightarrow{r})dr\right]r^{-2}dr, \qquad F_r = F_t = 0. \tag{21}$$

Since $\delta(\overrightarrow{r}) = \dfrac{\delta(r)}{4\pi r^2}$, it follows that

$$F_0 = \frac{4GM}{r}\left(\frac{2\omega+3}{2\omega+4}\right). \tag{22}$$

And also, from (19), the scalar field is

$$\varepsilon = \frac{2M}{(2\omega+3)r}. \tag{23}$$

Then, considering the equations (5) and (21)-(23), we find

$$h_{00} = \frac{2GM}{r}, \tag{24}$$

$$h_{11} = h_{00}\left(\frac{\omega+1}{\omega+2}\right), \quad h_{22} = h_{11}r^2, \quad h_{33} = h_{11}r^2\sin^2\theta. \tag{25}$$

This is the solution for a central body in the context of the Brans-Dicke theory [4].

3 Refractive Index

For light propagation in a static spacetime, we define the $3x3$ refractive index tensor as [3]

$$n_{ij} = (1+\frac{1}{2}\bar{h}_{00})\delta_{ij} + \frac{1}{2}\bar{h}_{ij}, \tag{26}$$

so that the refractive index is given by

$$n = n_{ij}\hat{k}^i\hat{k}^j, \tag{27}$$

where $||\hat{k}|| = \sqrt{\delta_{ij}\hat{k}^i\hat{k}^j} = 1$ and the 3-vector $\hat{k}$ indicates the direction of the light propagation.

Now, written in terms of spherical polar coordinates, the condition $\delta_{ij}\hat{k}^i\hat{k}^j = 1$ takes the form

$$\hat{k}^1\hat{k}^1 + r^2\hat{k}^2\hat{k}^2 + r^2\sin^2\theta\hat{k}^3\hat{k}^3 = 1. \tag{28}$$

On the other hand, considering the equation (11), the calculation of n_{ij} leads to

$$n_{11} = 1 + \frac{1}{2}F_0 + \frac{1}{2}F_r, \tag{29}$$

$$n_{22} = \left(1 + \frac{1}{2}F_0 + \frac{1}{2}F_t\right) r^2, \tag{30}$$

$$n_{33} = \left(1 + \frac{1}{2}F_0 + \frac{1}{2}F_t\right) r^2 \sin^2\theta. \tag{31}$$

Thus, with the aid of the equations (28)-(31), one can obtain the expression for the refractive index when the metric is static and possesses spherical symmetry:

$$n = 1 + \frac{1}{2}F_0 + \frac{1}{2}\left(F_r\hat{k}^1\hat{k}^1 + F_t r^2\hat{k}^2\hat{k}^2 + F_t r^2\sin^2\theta\hat{k}^3\hat{k}^3\right). \tag{32}$$

As an application, we consider a beam of light propagating in the plane $\varphi = \varphi_0$. If the light rays go through a coordinate point r, making an angle ψ with respect to the $\widehat{r}$ direction, then $\hat{k} = \cos\psi\widehat{r} + \sin\psi\widehat{\theta}$. In this case, the refractive index reads

$$n(r,\psi) = 1 + \frac{1}{2}F_0 + \frac{1}{2}\left(F_r\cos^2\psi + F_t\sin^2\psi\right). \tag{33}$$

Using (12), we still obtain

$$n(r,\psi) = 1 + \frac{1}{2}(F_0 + F_r) + \frac{1}{4}r\partial_r F_r\sin^2\psi. \tag{34}$$

4 Refractive Index Associated with Global Monopole Spacetime

The energy-momentum tensor that expresses the global monopole configuration is:

$$T^{\mu}{}_{\nu} = diag\left(-\frac{\eta^2}{r^2}, -\frac{\eta^2}{r^2}, 0, 0\right) = diag(-\rho, p_r, p_t, p_t), \tag{35}$$

being η the energy scale of symmetry breaking. Hence, can be calculated from (17) and (18) that

$$F_0 = -16\pi\left(\frac{2\omega+3}{2\omega+4}\right)G\eta^2\ln\frac{r}{r_0}, \tag{36}$$

$$F_r = \frac{16\pi}{3}\left(\frac{2\omega+3}{2\omega+4}\right) G\eta^2 \ln\frac{r}{r_0}, \tag{37}$$

where r_0 is constant. Therefore, substituting (36) and (37) into the equation (34), one obtains

$$n(r,\psi) = 1 + \frac{16\pi}{3}\left(\frac{2\omega+3}{2\omega+4}\right) G\eta^2 \left[\frac{\sin^2\psi}{4} - \ln\frac{r}{r_0}\right]. \tag{38}$$

If the global monopoles exist, they could be detected by means of effects such as the gravitational lenses [10], so that the refractive index associated with monopole spacetime would be given by (38).

In the limit $\omega \to \infty$, the equation (38) must be reduced for the expression of the refractive index in General Relativity [11]. Thus, we have

$$n_{GR}(r,\psi) = 1 + \frac{16\pi}{3} G\eta^2 \left[\frac{\sin^2\psi}{4} - \ln\frac{r}{r_0}\right]. \tag{39}$$

The factor $\frac{2\omega+3}{2\omega+4}$ is responsible by discrepancies between the predictions of the two theories. It represents the contribution of the Brans-Dicke scalar field, since that if $\varepsilon \neq 0$ in (19) then ω is finite.

5 Conclusion

We consider static metrics in the context of the Brans-Dicke theory, obtaining the solution with spherical symmetry, including the effect of internal pressures of the gravitational sources. Then, in the development of an analogy for the gravitational field acting as an optical medium, we find an expression for the refractive index, which also incorporated the effect of the Brans-Dicke scalar field. After, as an application, it was exhibited the spacetime refractive index for the global monopole; in the limit $\omega \to \infty$, the corresponding expression for the refractive index in the General Relativity theory was obtained.

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