Universidade Federal da Paraíba
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Controle hierárquico via estratégia de Stackelberg–Nash para controlabilidade de sistemas parabólicos e hiperbólicos

por

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João Pessoa - PB

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por

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sob orientação do

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Tese apresentada ao Corpo Docente do Programa Associado de Pós-Graduação em Matemática - UFPB/UFCG, como requisito parcial para obtenção do título de Doutor em Matemática.

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Resumo

Nesta tese apresentamos resultados sobre controlabilidade exata de Equações Diferenciais Parciais (EDPs) dos tipos parabólico e hiperbólico, no contexto de controle hierárquico, usando a estratégia de Stackelberg-Nash. Em todos os problemas consideramos um controle principal (líder) e dois controles secundários (seguidores). Para cada líder obtemos um equilíbrio de Nash correspondente, associado a um problema de controle ótimo bi-objetivo; então buscamos o líder de custo que resolve o problema de controlabilidade. Para os problemas parabólicos temos controles distribuídos e na fronteira, já nos hiperbólico todos os controles são distribuídos. Consideramos casos lineares e semilineares, os quais resolvemos usando desigualdade de observabilidade obtida combinando desigualdades de Carleman adequadas. Também usamos um método de ponto fixo.

Palavras-chave: Controlabilidade; Controle hierárquico; Estratégia de Stackelberg-Nash; Desigualdade de observabilidade; Desigualdade de Carleman; Equação do calor; Equação da onda; Equação de Burgers.

Abstract

In this thesis we presents results on the exact controllability of the partial Differential Equations (PDEs) of the parabolic and hyperbolic type, in the context of hierarchic control, using the Stackelberg-Nash strategy. In every problems we consider a main control (leader) and two secondary controls (followers). To each leader we obtain a correnponding Nash equilibrium, associated to a bi-objective optimal control problem; then we look for a leader of minimal cost that solves the exact controllability problem. For the parabolic problems we have distributed and boundary controls, now in the hyperbolics every controls are distributed. We consider linear and semilinear cases, which we solve using observability inequality obtained combining right Carleman inequalities. Also we use a fixed point method.

Keywords: Controllability; Hierarchic control; Stackelberg-Nash strategy; Observability inequality; Carleman inequality; Heat equation; Wave equation; Burgers' equation.

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À minha mãe Lucia e à minha esposa Joseane.

Sumário

Introdução

Nesta tese apresentamos alguns resultados sobre controlabilidade de sistemas associados a equações diferenciais parciais parabólicas e hiperbólicas. Vamos resolver os problemas de controlabilidade no contexto do que Lions [50] chamou de controle hierárquico e aplicaremos a estratégia de Stackelberg–Nash.

A grosso modo, controlar um sistema de equações diferenciais ordinárias ou parciais, como o próprio nome sugere, é atuar sobre ele para obter um comportamento desejado. Um problema de controle para sistemas governados por uma equação parabólica ou hiperbólica pode ser formulado como segue: Dados um intervalo de tempo $[0, T]$, um estado inicial e um estado final, buscamos por um controle que atua na trajetória do sistema (o lado direito da equação diferencial ou a condição de fronteira), de modo que, a solução do sistema associado a tal controle seja igual ao estado inicial no tempo $t = 0$ e ao estado final no tempo $t = T$.

A Teoria de Controle é uma parte da Matemática muito rica em aplicações, pois problemas de diversas áreas, como engenharia, biologia, medicina, economia entre outras, quando formulados matematicamente dão origem a problemas de controle. Por exemplo, questões de meio ambiente como despoluir um lago ou rio podem ser formuladas como problemas de controles de certas equações parabólicas, como equação do calor ou de Navier–Stokes. Pensando em controlar estruturas flexíveis surgem problemas de controle que envolvem equações hiperbólicas, já para controle de fluxo de tráfego aparecem problemas de controle, por exemplo, para equação de Burgers.

Quando falamos em equações parabólicas e hiperbólicas merecem destaque as equações do calor e da onda, pois as técnicas usadas para análise dessas equações, em particular nos problemas de controlabilidade, geralmente podem ser adaptadas para outras do mesmo tipo. Merece destaque também a equação de Burgers (que é parabólica), muito importante entre as equações que descrevem fluxo de fluidos, que tem sido usada como uma versão unidimensional da equação de Navier–Stokes, que por sua vez tem sido intensivamente estudada nos últimos anos.

Para definir os tipos de problemas controlabilidade vamos considerar um problema linear abstrato:

$$\begin{cases} y_t + Ay = Bu & \text{em} \quad (0,T), \\ y(\cdot,0) = y^0 & \text{em} \quad \Omega, \end{cases} \tag{1}$$

onde A e B são operadores lineares, $u : [0,T] \to U$ é o controle e $y : [0,T] \to H$ é o estado. Por H e U denotamos dois espaços de funções adequados. Então fixado $T > 0$, podemos definir vários tipos de problemas de controlabilidade no tempo T .

Controlabilidade exata: Dados y^0 e y^1 estados possíveis do sistema, obter um controle u tal que a correspondente solução y de (1) satisfaz $y(T) = y^1$.

Controlabilidade aproximada: Dados y^0 e y^1 estados possíveis do sistema e um número $\varepsilon > 0$, obter um controle u tal que a correspondente solução y de (1) satisfaz $||y(T) - y^1||_H < \varepsilon$.

Controlabilidade nula: Dado y^0, provar que existe um controle u tal que a correspondente solução y de (1) satisfaz $y(T) = 0$.

Controlabilidade exata por trajetórias: Dado y^0 e uma trajetória arbitrária $\bar{y}$ do sistema (1) (solução associada a um controle $\bar{u}$), obter um controle u tal que a correspondente solução y de (1) satisfaz $y(T) = \bar{y}(T)$.

A controlabilidade exata é um conceito mais forte do que o de controlabilidade aproximada, nula e exata por trajetórias. No caso de sistemas lineares como (1) controlabilidade nula é equivalente a controlabilidade exata por trajetórias. Estes dois últimos tipos de controlabilidade são frequentemente analisados quando trabalhamos com sistemas não reversíveis e com efeito regularizante, por exemplo sistemas parabólicos, já que neste caso não podemos esperar a controlabilidade exata, veja [13], [22], [25], [54].

O controle de equações diferenciais parciais tem sido intensivamente estudado nas últimas décadas. Um dos trabalhos que podemos destacar é o de D.L.Russel de 1978, [60], onde o autor desenvolveu importantes ferramentas para resolver problemas de controlabilidade fazendo relação com outros métodos de equações diferenciais parciais: multiplicadores, problemas de momento, etc. Outros que destacamos são os trabalhos de J-L. Lions de 1988, [46], [47], [48], onde o autor introduziu o Método da Unicidade de Hilbert (em inglês Hilbert Uniqueness Method–H.U.M.), que logo se tornou uma ferramenta muito importante para o desenvolvimento da Teoria de Controle.

A ferramenta usada para resolver problemas de controlabilidade de sistemas lineares é a desigualdade de observabilidade para o sistema adjunto. Considerando, por exemplo o sistema linear abstrato, a controlabilidade no tempo $T > 0$ é equivalente a analisar sobrejetividade de um certo operador linear $\mathcal{F}_T : L^2(0,T;U) \to H$ que associa cada $u \in L^2(0,T;U)$ a um elemento $\mathcal{F}_T(u) := y(T,\cdot)$, onde $y \in C^0([0,T];H)$ é a solução de (1) correspondente a u com $y^0 := 0$. Por sua vez, devido a resultados clássicos de Análise Funcional, essa sobrejetividade é equivalente a existência de uma constante $C > 0$ tal que

$$||\mathcal{F}_T^*(z^T)||_{L^2(0,T;U)} \geq C||z^T||_H \quad z^T \in H, \tag{2}$$

onde $\mathcal{F}_T^* : H \to L^2(0,T;U)$ é o adjunto de $\mathcal{F}_T$. A desigualdade (2) é chamada desigualdade de observabilidade. Para os detalhes sobre a construção desses operadores pode-se consultar [13].

Nos problemas de controlabilidade para sistemas não lineares, primeiro resolvemos o problema de controlabilidade nula para o sistema linear associado. Então, para obter a desigualdade de observabilidade usamos as chamadas desigualdades de Carleman, que são estimativas de normas L^2 ponderadas com funções peso convenientes. O uso dessas desigualdades para resolver problemas de controlabilidade teve notória popularização após o trabalho de Fursikov e Imanuvilov [34]. Também podemos destacar: [38], [39], [40], [41] e [44].

Uma parte importante em Teoria de Controle é a Otimização. Neste caso, além de resolver o problema de controlabilidade, buscamos o controle ótimo no sentido de minimizar custos (ou maximizar os benefícios). Isto é, considere novamente o sistema (1), se denotarmos por $\mathcal{U}_{ad}$ o espaço de todos os controles admissíveis e por Y o espaço dos estados associados, assumindo que ambos sejam espaços de Banach com as normas $||\cdot||_{\mathcal{U}_{ad}}$ e $||\cdot||_Y$, respectivamente, então podemos considerar o problema de obter um controle que minimiza um funcional da forma:

$$J(u) := \frac{1}{2}||y(u) - y_d||_Y^2 \quad \forall u \in \mathcal{U}_{ad},$$

onde $y_d \in Y$ é dada. Ou ainda, mais geralmente, podemos considerar o funcional

$$J(u) := \frac{1}{2}||y(u) - y_d||_Y^2 + \frac{\mu}{2}||u||_{\mathcal{U}_{ad}} \quad \forall u \in \mathcal{U}_{ad}, \tag{3}$$

com $\mu \geq 0$. A situação a ser analisada com o funcional (3.6) é: além de atingir o estado final desejado, com o controle de menor custo, esperamos que a solução do sistema esteja próxima de uma função dada.

Para resolver um problema de controle ótimo, Lions em [50] introduziu o conceito de controle hierárquico. A motivação veio da noção de otimização introduzida e usada em Economia pelo economista H. Von Stackelberg, [61]. Neste processo tem-se pelo menos dois controles atuando, porém apenas um controle, chamado líder, é independente e os outros são considerados a partir de cada escolha do líder, estes são chamados seguidores. Em seu trabalho, Lions considerou um conjunto aberto $\Omega \subset \mathbb{R}^n$ com fronteira suave Γ e um sistema da forma

$$\begin{cases} y_t + Ay = v_1 1_{\mathcal{O}_1} + v_2 1_{\mathcal{O}_2} & \text{em} \quad Q = \Omega \times (0,T), \\ y = 0 & \text{sobre} \quad \Sigma = \Gamma \times (0,T), \\ y(\cdot, 0) = 0 & \text{em} \quad \Omega. \end{cases} \tag{4}$$

onde,

$$Ay = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial y}{\partial x_j}\right) + \sum_{i=1}^{n} a_i(x)\frac{\partial y}{\partial x_i},$$

com

$$-\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \alpha \sum_{i=1}^{n} \xi_i^2, \; \alpha > 0.$$

onde $1_{\mathcal{O}_i}$ denota a função característica do conjunto aberto $\mathcal{O}_i \subset \Omega$ e $\mathcal{O}_1 \cap \mathcal{O}_2 = \emptyset$. Os objetivos cumpridos foram: aproximar $y(\cdot, T)$, onde y é solução de (4), de um estado desejado $y^1 \in L^2(\Omega)$, e considerar a solução de (4) que não se afaste de uma função dada $y_2 \in L^2(\Omega \times (0,T))$. Para conseguir esses objetivos foram definidos os seguintes funcionais custo

$$J_1(v_1) := \frac{1}{2}\iint_{\mathcal{O}_1 \times (0,T)} |v_1|^2 dxdt, \tag{5}$$

e

$$J_2(v_1, v_2) = \frac{1}{2}\iint_{\Omega \times (0,T)} |y(v_1, v_2) - y_2|^2 dxdt + \frac{\beta}{2}\iint_{\mathcal{O}_2 \times (0,T)} |v_2|^2, \; \beta > 0. \tag{6}$$

Então foi usada a estratégia de Stackelberg, com o líder sendo v_1 e o seguidor v_2. Ou seja, para cada escolha do controle v_1, o controle v_2 é escolhido de forma a minimizar o funcional (6). Após obter o controle seguidor ótimo, $v_2 = \mathcal{F}(v_1)$, o qual fica caracterizado em um sistema de otimalidade, o problema de controlabilidade aproximada é resolvido usando o controle independente v_1, que minimiza o funcional (5), pois agora tem-se $y(v_1, v_2) = y(v_1, \mathcal{F}(v_1))$. Para obter a controlabilidade foi usado o Teorema de continuação única de Mizohata (veja [55]) e teoremas de regularidade.

Em [51], Lions estudou o controle hierárquico para um sistema governado pela equação da onda usando a estratégia de Stackelberg e analisando a controlabilidade aproximada com controles na fronteira. Ele considerou um sistema como segue

$$\begin{cases} y_{tt} - \Delta y = 0 & \text{em} \quad Q, \\ y = v_1 1_{\Gamma_1} + v_2 1_{\Gamma_2} & \text{sobre} \quad \Sigma, \\ y(\cdot, 0) = 0 & \text{em} \quad \Omega. \end{cases} \tag{7}$$

Com $Q = \Omega \times (0, T)$, $\Sigma = \Gamma \times (0, T)$ e Ω um conjunto aberto do $\mathbb{R}^n$ com fronteira suave Γ. Onde neste problema 1_{Γ_i} denota a função característica do subconjunto Γ_i de Γ, $i = 1, 2$, v_i é o controle em $L^2(\Gamma_i \times (0, T))$. Além disso, considerou v_1 sendo o líder e v_2 o seguidor.

Agora em [16], Diaz e Lions resolveram o problema de controlabilidade aproximada para um sistema parabólico, como (4), onde consideraram um líder e um número n de seguidores. Neste trabalho, pela primeira vez, usaram a estratégia de Stackelberg associada a idéia de equilíbrio de Nash, [56], a qual consiste em obter n seguidores que minimizem simultaneamente n funcionais custo, similares a (6). Então, eles chamaram esse processo de estratégia de Stackelberg–Nash. Ou seja, nesse trabalho para cada controle líder f foi encontrado um equilíbrio de Nash para os seus funcionais custo J_i, isto é, uma n-úpla $(w_1, \cdots, w_n)$ que resolve o problema de minimização da forma

$$J_i(f;\, w_1, \ldots, w_i, \ldots, w_n) = \min_{\hat{w}_i} J_i(f;\, w_1, \ldots, \hat{w}_i, \ldots, w_n), \quad i = 1, \ldots, n.$$

Esse equilíbrio de Nash fica caracterizado em um sistema de otimalidade, então eles concluem a controlabilidade aproximada aplicando o Teorema de Hahn–Banach e o Teorema de continuação única de Mizohata.

Em [16], [50] e [51] o foco foi a controlabilidade aproximada, já em [3] os autores consideraram sistema distribuído com a equação do calor não linear

$$\begin{cases} y_t - \Delta y + a(x,t) y = F(y) + f 1_{\mathcal{O}} + v_1 1_{\mathcal{O}_1} + v_2 1_{\mathcal{O}_2} & \text{em} \quad Q, \\ y = 0 & \text{sobre} \quad \Sigma, \\ y(\cdot, 0) = y^0 & \text{em} \quad \Omega. \end{cases} \tag{8}$$

Com Ω um domínio limitado do $\mathbb{R}^n$ com fronteira suave Γ. Onde $a \in L^\infty(Q)$, F é uma função contínua localmente lipschitziana e y^0 é dada, $\mathcal{O}$, $\mathcal{O}_1$, $\mathcal{O}_2 \subset \Omega$ são conjuntos abertos e 1_A denota a função característica do conjunto A. Os controles são f, v_1 e v_2, sendo f o líder e v_1, v_2 os seguidores. Os funcionais considerados são

$$J(f) = \frac{1}{2} \iint_{\mathcal{O}\times(0,T)} |f|^2 dxdt \tag{9}$$

e

$$J_i(f; v_1, v_2) = \frac{\alpha_i}{2} \iint_{\mathcal{O}_{i,d}\times(0,T)} |y - y_{i,d}|^2 dxdt + \frac{\mu_i}{2} \iint_{\mathcal{O}_i\times(0,T)} |v_i|^2 dxdt, \tag{10}$$

para $i = 1, 2$. Então, eles resolveram o problema de controlabilidade exata por trajetórias usando controle hierárquico via estratégia de Stackelberg–Nash. A desigualdade de observabilidade para o adjunto do sistema de otimalidade linear foi obtida combinando as desigualdades de Carleman associadas a cada equação do sistema. Depois concluíram o problema de controlabilidade usando o Teorema do ponto fixo de Shauder.

Por fim, em relação a equação de Burgers e Navier–Stokes existem vários trabalhos importantes sobre controlabilidade nula, podemos destacar [26], [27] e [34]. Contudo, ainda não existem trabalhos sobre controle hierárquico e estratégia de Stackelberg–Nash para controlabilidade nula dessas equações.

O objetivo desta tese é analisar o controle hierárquico aplicando a estratégia de Stackelberg–Nash para resolver problemas de controlabilidade exata para as equações do calor e da onda, e ainda, controlabilidade nula para equação de Burgers.

Descrição dos resultados

Nesta tese resolvemos os problemas de controle exato por trajetórias para sistemas como (8) com controles tanto na fronteira como distribuídos. Além disso, vamos resolver o problema de controle exato para um sistema como (7), mas com controles distribuídos. No que segue detalharemos os trabalhos desenvolvidos.

Capítulo 1: Controle hierárquico para controlabilidade exata de equações parabólicas com controles distribuídos e na fronteira.

Seja $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ um domínio limitado com fronteira Γ suficientemente regular. Considere $\mathcal{O}$, $\mathcal{O}_1$, $\mathcal{O}_2 \subset \Omega$ conjuntos abertos não vazios e sejam S, S_1 e S_2 subconjuntos disjuntos fechados de Γ. Dado $T > 0$, consideramos o domínio cilíndrico $Q = \Omega \times (0,T)$ cuja fronteira lateral é $\Sigma = \Gamma \times (0,T)$. Denotaremos por $\nu(x)$ o vetor normal unitário exterior a Ω no ponto $x \in \Gamma$.

Consideramos sistemas da forma

$$\begin{cases} y_t - \Delta y + a(x,t)y = F(y) + f1_{\mathcal{O}} & \text{em} \quad Q, \\ y = v^1\rho_1 + v^2\rho_2 & \text{sobre} \quad \Sigma, \\ y(\cdot,0) = y^0 & \text{em} \quad \Omega, \end{cases} \tag{11}$$

e

$$\begin{cases} p_t - \Delta p + a(x,t)p = F(p) + u^1 1_{\mathcal{O}_1} + u^2 1_{\mathcal{O}_2} & \text{em} \quad Q, \\ p = g1_S & \text{sobre} \quad \Sigma, \\ p(\cdot,0) = p^0 & \text{em} \quad \Omega, \end{cases} \tag{12}$$

onde y^0, p^0, f, g, v^i e u^i, são dadas em espaços apropriados, F é uma função localmente lipschitziana e $\rho_i \in C_0^\infty(S_i)$, $0 \leq \rho \leq 1$. Por 1_A denotamos a função característica do conjunto A.

Os problemas que vamos resolver tem os seguintes objetivos: resolver o problema de controlabilidade exata por trajetórias para os sistemas (11) e (12), impedir que a solução do sistema se distancie de uma certa função dada.

Neste sentido, focando inicialmente no sistema (11), definimos os funcionais custo:

$$J_i(f, v_1, v_2) := \frac{\alpha_i}{2} \iint_{\mathcal{O}_{i,d}\times(0,T)} |y - \xi_{i,d}|^2 \, dx \, dt + \frac{\mu_i}{2} \iint_{S_i \times (0,T)} |v^i|^2 \, d\sigma \, dt, \quad i = 1, 2, \tag{13}$$

onde $\xi_{i,d} = \xi_{i,d}(x,t)$ são dadas em $L^2(\mathcal{O}_{i,d} \times (0,T))$ e α_i, μ_i são constantes positivas. Também definimos o funcional principal

$$J(f) := \frac{1}{2} \iint_{\mathcal{O}\times(0,T)} |f|^2 \, dx \, dt.$$

Então aplicamos a estratégia de Stackelberg–Nash, com f sendo o controle líder e v^1, v^2 os seguidores. Neste sentido, para cada f vamos encontrar um par $(v^1, v^2) = (v^1(f), v^2(f))$ que minimiza simultaneamente os funcionais custo J_i, isto é, resolvem o problema

$$J_1(f; v^1, v^2) = \min_{\hat{v}^1} J_1(f; \hat{v}^1, v^2), \qquad J_2(f; v^1, v^2) = \min_{\hat{v}^2} J_2(f; v^1, \hat{v}^2). \tag{14}$$

Chamamos o par $(v^1(f), v^2(f))$ assim obtido de equilíbrio de Nash para os funcionais J_i.

Lembremos que uma trajetória do sistema (11) é a solução do sistema

$$\begin{cases} \bar{y}_t - \Delta \bar{y} + a(x,t)\bar{y} = F(\bar{y}) & \text{in} \quad Q, \\ \bar{y} = 0 & \text{on} \quad \Sigma, \\ y(\cdot,0) = \bar{y}^0 & \text{in} \quad \Omega. \end{cases} \tag{15}$$

Depois de provar a existência de equilíbrio de Nash, fixada uma trajetória $\bar{y}$, buscamos um controle ótimo $f \in L^2(\mathcal{O} \times (0,T))$, que resolve o problema

$$J(f) = \min_{\hat{f}} J(\hat{f}), \tag{16}$$

sujeito a condição de controlabilidade exata

$$y(\cdot,T) = \bar{y}(\cdot,T) \quad \text{in} \quad \Omega. \tag{17}$$

Em relação ao sistema (12), o líder é o controle g e os seguidores são u^1 e u^2. Os funcionais custo são da forma:

$$K_i(g,u^1,u^2) := \frac{\alpha_i}{2} \iint_{\Sigma_{i,d}} \left| \frac{\partial p}{\partial \nu} - \zeta_{i,d} \right|^2 \rho_{i,d} \, d\sigma \, dt + \frac{\mu_i}{2} \iint_{\mathcal{O}_i \times (0,T)} |u^i|^2 \, dx \, dt, \quad i = 1,2, \tag{18}$$

e

$$K(g) := \frac{1}{2} ||g||^2_{H^{\frac{3}{2},\frac{3}{4}}(S \times (0,T))},$$

onde $\Sigma_{i,d} = \Gamma_{i,d} \times (0,T)$ e $\Gamma_{i,d}$ são subconjuntos fechados não vazios de Γ, $\zeta_{i,d} = \zeta_{i,d}(x,t)$ são funções dadas, $\rho_{i,d} \in C_0^2(\Gamma_{i,d})$ e α_i, μ_i são constantes positivas. Aplicaremos a estratégia de Stackelberg–Nash. O equilíbrio de Nash $(u^1(g),\, u^2(g))$ que resolve simultaneamente o problema de minimização

$$K_1(g;u^1,u^2) = \min_{\hat{u}^1} K_1(g;\hat{u}^1,u^2), \quad K_2(g;u^1,u^2) = \min_{\hat{u}^2} K_2(g;u^1,\hat{u}^2). \tag{19}$$

Temos a seguinte trajetória do sistema (12):

$$\begin{cases} \bar{p}_t - \Delta \bar{p} + a(x,t)\bar{p} = F(\bar{p}) & \text{in} \quad Q, \\ \bar{p} = 0 & \text{on} \quad \Sigma, \\ \bar{p}(\cdot,0) = \bar{p}^0 & \text{in} \quad \Omega. \end{cases} \tag{20}$$

Então, uma vez provada a existência de equilíbrio de Nash para cada controle líder, encontramos um controle $g \in H^{\frac{3}{2},\frac{3}{4}}(S \times (0,T))$ tal que

$$p(\cdot,T) = \bar{p}(\cdot,T) \quad \text{in} \quad \Omega. \tag{21}$$

Observe que o sistema (15) estudado em [3] tem apenas controles distribuídos. Investigamos questões semelhantes a esse trabalho, mas agora a dificuldade é que temos controles tanto na fronteira quanto distribuídos. Resolver problemas de controle hierárquico para controlabilidade exata por trajetórias para equação do calor é a principal contribuição deste capítulo.

Vamos enunciar os principais resultados deste trabalho no que segue.

Teorema (Caso linear): *Suponha $\mathcal{O}_{i,d} \cap \mathcal{O} \neq \emptyset$, $i = 1, 2$. Assuma que uma das seguintes condições é verdadeira:*

$$\mathcal{O}_{1,d} = \mathcal{O}_{2,d} := \mathcal{O}_d, \tag{22}$$

ou

$$\mathcal{O}_{1,d} \cap \mathcal{O} \neq \mathcal{O}_{2,d} \cap \mathcal{O}. \tag{23}$$

Assuma $F \equiv 0$ e as constantes $\mu_i > 0$ ($i = 1, 2$) são suficientemente grandes. Então existe uma função positiva $\rho = \rho(t)$, que decai exponencialmente para zero quando t tende a T, tal que se $\bar{y}$ é a solução de (15) *(com $F \equiv 0$) associada ao estado inicial $\bar{y}^0 \in L^2(\Omega)$ e as funções $\xi_{i,d} \in L^2(\mathcal{O}_{i,d} \times (0, T))$ satisfazem a condição:*

$$\iint_{\mathcal{O}_{i,d}\times(0,T)} \rho^{-2} |\bar{y} - \xi_{i,d}|^2 \, dx \, dt < +\infty, \quad i = 1, 2, \tag{24}$$

então, dado $y^0 \in L^2(\Omega)$, existem um controle $f \in L^2(\mathcal{O} \times (0, T))$ e um equilíbrio de Nash associado $(v^1(f), v^2(f))$ tais que a solução de (11) *correspondente (com $F \equiv 0$) satisfaz* (17).

As figuras seguintes ilustram os casos (22) e (23), respectivamente.

A hipótese (22) é natural, pois desejamos obter (17) sem que a solução de (11) se afaste da função $\xi_{i,d}$, então é bastante razoável que essa função se aproxime da trajetória $\bar{y}$ quando

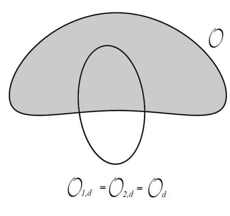

Figura 1: Caso: $\mathcal{O}_{1,d} = \mathcal{O}_{2,d}$.

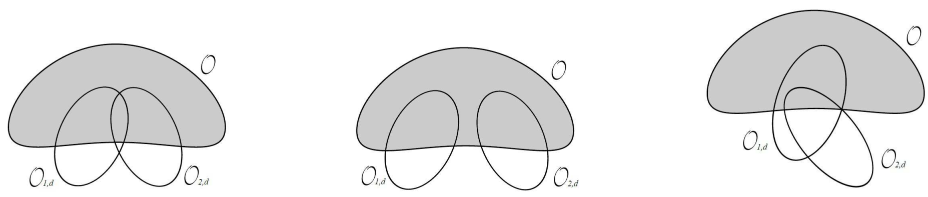

Figura 2: Caso: $\mathcal{O}_{1,d} \cap \mathcal{O} \neq \mathcal{O}_{2,d} \cap \mathcal{O}$.

t tende a T. Além disso, como o sistema é linear os funcionais J_i são convexos, então a condição (14) é equivalente a

$$J_i'(f;v^1,v^2)\cdot\hat{v}^i=0,\quad \forall\hat{v}^i\in L^2(S_i\times(0,T)),\quad i=1,2. \tag{25}$$

Logo, no caso linear um par (v^1,v^2) é um equilíbrio de Nash para os funcionais J_i se, e somente se, satisfaz (25).

Para demonstrar o Teorema no caso linear, transformamos o problema de controlabilidade exata por trajetórias em um problema de controlabilidade nula. Caracterizamos o equilíbrio de Nash em um sistema de otimalidade e obtemos a desigualdade de observabilidade para o sistema adjunto combinando desigualdades de Carleman adequadas.

No caso semilinear, como a convexidade dos J_i é perdida, não temos mais a equivalência entre (14) e (25). Neste caso, usamos a definição de quase equilíbrio de Nash: Um par (v^1,v^2) é um quase equilíbrio de Nash para os funcionais J_i se a condição (25) é satisfeita.

Com esta definição temos os seguintes resultados para o caso semilinear:

Teorema (Caso semilinear): *Suponha* $\mathcal{O}_{i,d}$ *e* μ_i $(i=1,2)$ *nas condições do Teorema do caso linear, seja* $F\in W^{1,\infty}(\mathbb{R})$ *e que as funções* $\xi_{i,d}\in L^2(\mathcal{O}_{i,d}\times(0,T))$ $(i=1,2)$ *tem a propriedade: existe uma função* ρ*, também como no caso linear, tal que se* $\bar{y}$ *é a solução de* (15) *para um estado inicial* $\bar{y}^0\in L^2(\Omega)$ (24) *vale. Então dado* $y^0\in L^2(\Omega)$*, existem um controle* $f\in L^2(\mathcal{O}\times(0,T))$ *e um quase equilíbrio de Nash associado* (v^1,v^2) *tais que a solução de* (11) *correspondente satisfaz* (17).

Para demonstrar este resultado usamos os resultados dos caso linear e aplicamos o Teorema do ponto fixo de Shauder.

Podemos, sob certas condições, obter uma equivalência entre as definições de equilíbrio e quase equilíbrio de Nash. Este é o conteúdo do seguinte resultado:

Teorema (Relação entre equilíbrio e quase equilíbrio): *Suponha* $F\in W^{2,\infty}(\mathbb{R})$ *e* $\xi_{i,d}\in L^\infty(\mathcal{O}_{i,d}\times(0,T))$ $(i=1,2)$*. Se* $y^0\in L^2(\Omega)$ *e* $n\leq 6$*, então existe* $C>0$ *tal que, se* $f\in L^2(\mathcal{O}\times(0,T))$ *e* μ_i *satisfazem*

$$\mu_i\geq C\left(1+\|f\|_{L^2(\mathcal{O}\times(0,T))}\right),$$

as condições (14) *e* (25) *são equivalentes.*

Para demonstrar este resultado basicamente verificamos as condições para que a segunda derivada dos J_i tenha sinal positivo.

Em relação ao sistema (12), temos o seguinte resultado para o caso linear:

Teorema (Caso linear sistema (12)**):** *Suponha*

$$\Gamma_{i,d} \cap S = \emptyset, \quad i = 1, 2, \tag{26}$$

e que as constantes $\mu_i > 0$ $(i = 1, 2)$ *são suficientemente grandes. Então existe uma função positiva* $\bar{\rho} = \bar{\rho}(t)$*, que é nula em* $t = T$*, tal que se* $\bar{p}$ *é a solução de* (20) *(com* $F \equiv 0$*) associada ao estado inicial* $\bar{p}^0 \in V$ *e as funções* $\zeta_{i,d} \in L^2(\Gamma_{i,d} \times (0, T))$ *satisfazem a condição:*

$$\iint_{\Gamma_{i,d}\times(0,T)} \bar{\rho}^{-2} \left| \frac{\partial \bar{p}}{\partial \nu} - \zeta_{i,d} \right|^2 d\sigma\, dt < +\infty, \quad i = 1, 2, \tag{27}$$

então, para todo $p^0 \in V$ *existem um controle* $g \in H^{\frac{3}{2},\frac{3}{4}}(S \times (0, T))$ *e um equilíbrio de Nash* (u^1, u^2) *associado tais que a correspondente solução de* (12) *satisfaz* (21).

A hipótese (27) tem justificativa análoga a (22). E mais, como os funcionais K_i são convexos neste caso temos que um par (u^1, u^2) é um equilíbrio de Nash para os funcionais K_i se, e somente se, satisfazem

$$K_i'(g; u^1, u^2) \cdot \hat{u}^i = 0, \quad \forall\, \hat{u}^i \in L^2(\mathcal{O}_i \times (0, T)), \quad i = 1, 2. \tag{28}$$

Para demonstrar o teorema no caso linear para sistema (12), fazemos um prolongamento do domínio e resolvemos o problema com controles distribuídos.

Usando a definição de quase equilíbrio para os funcionais K_i obtemos o seguinte resultado para o caso semilinear:

Teorema (Caso semilinear sistema (12)**):** *Suponha* $\Gamma_{i,d}$*,* $\mu_i > 0$ $(i = 1, 2)$ *como no caso linear (sistema* (12)*) e* $F \in W^{1,\infty}(\mathbb{R})$*. Então existe uma função* $\bar{\rho}$ *tal que, se* $\bar{p}$ *é a solução de* (20) *para um estado inicial* $\bar{p}^0 \in V$ *e as funções* $\zeta_{i,d} \in L^2(\Gamma_{i,d} \times (0, T))$ $(i = 1,\ 2)$ *são tais que* (27) *vale, para cada* $p^0 \in V$ *dado, existem um controle* $g \in H^{\frac{3}{2},\frac{3}{4}}(S \times (0, T))$ *e um quase equilíbrio de Nash* (u^1, u^2) *associado tais que a correspondente solução de* (12) *satisfaz* (21).

Para demonstrar este resultado também usamos o Teorema do ponto fixo de Shauder.

Por fim, obtemos a equivalência entre equilíbrio e quase equilíbrio de Nash para os funcionais K_i:

Teorema (Relação entre equilíbrio e quase equilíbrio, sistema (12)**):** *Suponha que* $F \in W^{2,\infty}(\mathbb{R})$ *e* $\zeta_{i,d} \in H^{\frac{1}{2},\frac{1}{4}}(\Sigma_{i,d})$ $(i = 1, 2)$*. Se* $p^0 \in V$ *e* $N \leq 6$*, então existe* $C > 0$ *tal que, se* $g \in H^{\frac{3}{2},\frac{3}{4}}(S \times (0, T))$ *e* μ_i *satisfazem*

$$\mu_i \geq C \left(1 + \|g\|_{H^{\frac{3}{2},\frac{3}{4}}(S\times(0,T))} \right),$$

as condições (19) *e* (28) *são equivalentes.*

Capítulo 2: Controle hierárquico para equação da onda.

Seja $\Omega \subset \mathbb{R}^n$ um domínio limitado com fronteira Γ de classe C^2 e suponha $T > 0$. Vamos considerar pequenos conjuntos abertos não vazios e disjuntos $\mathcal{O}$, $\mathcal{O}_1$, $\mathcal{O}_2 \subset \Omega$. Usaremos a notação $Q = \Omega \times (0,T)$, cuja fronteira lateral é $\Sigma = \Gamma \times (0,T)$; por $\nu(x)$ o vetor normal unitário exterior a Ω no ponto $x \in \Gamma$.

Consideramos o seguinte sistema

$$\begin{cases} y_{tt} - \Delta y + a(x,t)y = F(y) + f1_{\mathcal{O}} + v^1 1_{\mathcal{O}_1} + v^2 1_{\mathcal{O}_2} & \text{in} \quad Q, \\ y = 0 & \text{on} \quad \Sigma, \\ y(\cdot,0) = y^0, \ y_t(\cdot,0) = y^1 & \text{in} \quad \Omega, \end{cases} \tag{29}$$

onde $a \in L^\infty(Q)$, $f \in L^2(\mathcal{O} \times (0,T))$, $v^i \in L^2(\mathcal{O}_i \times (0,T))$, $F : \mathbb{R} \to \mathbb{R}$ é uma função localmente lipschitziana, $(y^0, y^1) \in H_0^1(\Omega) \times L^2(\Omega)$ e com 1_A indicamos função característica de A.

Resolvemos um problema com dois objetivos: obter um controle ótimo que resolve o problema de controlabilidade exata e impedir que a solução do sistema se afaste de uma função dada. Para conseguir estes objetivos vamos considerar o controle hierárquico, com f sendo o controle líder v^1, v^2 os seguidores, e aplicaremos a estratégia de Stackelberg–Nash.

Consideramos o funcional principal

$$\frac{1}{2} \iint_{\mathcal{O} \times (0,T)} |f|^2 \, dx \, dt,$$

e os funcionais custo secundários

$$J_i(f, v^1, v^2) := \frac{\alpha_i}{2} \iint_{\mathcal{O}_{i,d} \times (0,T)} |y - y_{i,d}|^2 \, dx \, dt + \frac{\mu_i}{2} \iint_{\mathcal{O}_i \times (0,T)} |v^i|^2 \, dx \, dt, \tag{30}$$

onde $\mathcal{O}_{i,d} \subset \Omega$ são abertos não vazios, $y_{i,d} \in L^2(\mathcal{O}_{i,d} \times (0,T))$ são funções dadas e α_i e μ_i são constantes positivas. A estratégia de Stackelberg–Nash funciona como segue. Para cada escolha de um líder f, buscamos um equilíbrio de Nash para os funcionais J_i, ou seja, um par $(v^1(f), v^2(f)) \in L^2(\mathcal{O}_1 \times (0,T)) \times L^2(\mathcal{O}_2 \times (0,T))$ que resolve simultaneamente o seguinte problema de minimização:

$$J_1(f; v^1, v^2) = \min_{\hat{v}^1} J_1(f; \hat{v}^1, v^2), \quad J_2(f; v^1, v^2) = \min_{\hat{v}^2} J_2(f; v^1, \hat{v}^2). \tag{31}$$

No caso em que os funcionais J_i são convexos, (31) é equivalente a

$$J_i'(f; v^1, v^2) \cdot \hat{v}^i = 0, \ \forall \, \hat{v}^i \in L^2(\mathcal{O}_i \times (0,T)), \ i = 1, 2. \tag{32}$$

Este é caso quando o sistema (29) é linear, no entanto, geralmente isto não acontece no caso não linear.

Dado $(\bar{y}^0, \bar{y}^1) \in H_0^1(\Omega) \times L^2(\Omega)$, uma trajetória para (29) é a solução do sistema

$$\begin{cases} \bar{y}_{tt} - \Delta\bar{y} + a(x,t)\bar{y} = F(\bar{y}) & \text{in} \quad Q, \\ y = 0 & \text{on} \quad \Sigma, \\ y(\cdot,0) = y^0, \ y_t(\cdot,0) = y^1 & \text{in} \quad \Omega. \end{cases} \tag{33}$$

Depois de provar que para cada líder f existe um par equilíbrio de Nash $(v^1, v^2) = (v^1(f), v^2(f))$, fixamos uma trajetória $\bar{y}$ e buscamos um líder $f \in L^2(\mathcal{O} \times (0,T))$ tal que

$$J(f) := \min_{\hat{f}} J(\hat{f}), \tag{34}$$

sujeito a condição de controlabilidade exata

$$y(\cdot,T) = \bar{y}(\cdot,T), \quad y_t(\cdot,T) = \bar{y}_t(\cdot,T) \quad \text{in} \quad \Omega. \tag{35}$$

Em [51], o autor provou resultados sobre controle hierárquico aplicando a estratégia de Stackelberg, no contexto da controlabilidade aproximada para equação da onda, com controles distribuídos. A principal contribuição deste trabalho é entender a estratégia de Stackelberg associada a noção de equilíbrio de Nash, trabalhando no contexto da controlabilidade exata para equação da onda com controles distribuídos.

Antes de enunciar os resultados obtidos neste trabalho façamos algumas considerações. Dado $x_0 \in \mathbb{R}^n \setminus \overline{\Omega}$ definimos o conjunto

$$\Gamma_+ := \{(x - x_0) \cdot \nu(x) > 0\}$$

e as funções $d : \overline{\Omega} \to \mathbb{R}$, com $d(x) = |x - x_0|^2$ para todo $x \in \overline{\Omega}$. Vamos supor que existe $\delta > 0$ tal que

$$\mathcal{O} \supset \mathcal{O}_\delta(\Gamma_+) \cap \Omega, \tag{36}$$

onde

$$\mathcal{O}_\delta(\Gamma_+) = \{x \in \mathbb{R}^n; |x - x'| < \delta, \, x' \in \Gamma_+\}.$$

Devido a velocidade de propagação da equação da onda ser finita, o tempo $T > 0$ tem que ser suficientemente grande. Assim, no que segue vamos supor $T > 2R_1$, onde $R_1 := \max\{\sqrt{d(x)} : \, x \in \overline{\Omega}\}$.

Os resultados obtidos são os seguintes.

Teorema (Caso linear): *Suponha* $F \equiv 0$ *e que as constantes* $\mu_i > 0$ $(i = 1, 2)$ *suficientemente grandes. Então dado* $(y^0, y^1) \in H_0^1 \times L^2(\Omega)$*, existem um controle* $f \in L^2(\mathcal{O} \times (0, T))$ *e um equilíbrio de Nash* $(v^1, v^2) = (v^1(f), v^2(f))$ *tais que a correspondente solução de* (29) *satisfaz* (35).

Para demonstrar este resultado transformamos o problema de controlabilidade exata em um de controlabilidade nula equivalente, depois caracterizamos o equilíbrio de Nash em um sistema de otimalidade. Concluímos a controlabilidade nula usando a desigualdade de observabilidade que obtemos por meio de desigualdades de Carleman apropriadas e da energia.

Considerando o caso semilinear com F localmente lipschitziana, como os funcionais J_i não são convexos, em geral (31) and (32) não são equivalentes. Desse modo, usamos a definição quase equilíbrio de Nash: dado f, um par (v^1, v^2) é um quase equilíbrio de Nash para os funcionais J_i se (32) é satisfeita. Com esta definição temos o seguinte resultado:

Teorema (Caso semilinear): *Suponha que* (36) *vale,* $F \in W^{1,\infty}(\mathbb{R})$ *e os* $\mu_i > 0$ $(i = 1, 2)$ *são suficientemente grandes. Então, dado qualquer par* $(y^0, y^1) \in H_0^1(\Omega) \times L^2(\Omega)$*, existem um controle* $f \in L^2(\mathcal{O} \times (0, T))$ *e um quase equilíbrio de Nash associado* $(v^1, v^2) = (v^1(f), v^2(f))$ *tais que a correspondente solução de* (29) *satisfaz* (35).

Demonstramos este Teorema usando o caso linear, resultados de compacidade e ponto fixo.

Analisamos ainda em que condições as definições de equilíbrio e quase equilíbrio de Nash são equivalentes, a resposta esta no seguinte resultado:

Teorema (Relação entre equilíbrio e quase equilíbrio): *Suponha* $F \in W^{2,\infty}(\mathbb{R})$ *e que* $y_{i,d} \in C^0([0, T]; H_0^1(\mathcal{O}_{i,d})) \cap C^1([0, T]; L^2(\mathcal{O}_{i,d}))$*. Se* $(y^0, y^1) \in H_0^1(\Omega) \times L^2(\Omega)$ *e* $n \leq 8$*, então existe* $C > 0$ *tal que, se* $f \in L^2(\mathcal{O} \times (0, T))$ *e* μ_i *satisfazem*

$$\mu_i \geq C(1 + \|f\|_{L^2(\mathcal{O} \times (0,T))}),$$

as condições (31) *e* (32) *são equivalentes.*

Para demonstrar este resultado analisamos a derivada segunda dos funcionais J_i.

Capítulo 3: Controle hierárquico para a equação de Burgers via estratégia de Stackelberg–Nash.

Seja $T > 0$ e considere os conjuntos abertos não vazios $\mathcal{O}$, $\mathcal{O}_1$, $\mathcal{O}_2 \subset (0,1)$, com $0 \notin \overline{\mathcal{O}}$. Introduzimos o seguinte sistema para a equação de Burgers:

$$\begin{cases} y_t - y_{xx} + yy_x = f1_{\mathcal{O}} + v^1 1_{\mathcal{O}_1} + v^2 1_{\mathcal{O}_2} & \text{em} \quad (0,1)\times(0,T), \\ y(0,t) = y(1,t) = 0 & \text{sobre} \quad (0,T), \\ y(x,0) = y^0(x) & \text{em} \quad (0,1), \end{cases} \tag{37}$$

onde f, v^1, v^2 são os controles e y é o estado e com a notação 1_A representamos a função característica do conjunto A.

Existem vários trabalhos importante sobre a controlabilidade de sistemas como (37), entre os quais podemos citar [26], [34] e [53]. No Capítulo 3 vamos analisar a controlabilidade nula do sistema (33) no contexto do controle hierárquico e aplicando a estratégia de Stackelberg–Nash, com f sendo o líder e v^1, v^2 os seguidores.

Seguindo metodologia dos capítulos anteriores, consideramos os funcionais custo para os seguidores:

$$J_i(f; v^1, v^2) := \frac{\alpha_i}{2} \iint_{\mathcal{O}_{i,d}\times(0,T)} |y - y_{i,d}|^2 \, dx\, dt + \frac{\mu_i}{2} \iint_{\mathcal{O}_i\times(0,T)} |v^i|^2 \, dx\, dt, \tag{38}$$

e o funcional principal

$$J(f) := \frac{1}{2} \iint_{\mathcal{O}\times(0,T)} |f|^2 \, dx\, dt,$$

onde $\mathcal{O}_{i,d} \subset (0,1)$ é um conjunto aberto não vazio, $\alpha_i, \mu_i > 0$ são constantes e $y_{i,d} = y_{i,d}(x,t)$ são funções dadas em $L^2(\mathcal{O}_{i,d} \times (0,T))$.

Para cada líder f escolhido obtemos um par Equilíbrio de Nash associado $(v^1(f), v^2(f))$ para os funcionais J_i.

Como (37) é não linear, para provar a existência e unicidade de equilíbrio de Nash, vamos mostrar que existe um único par $(v^1,\, v^2)$ que satisfaz

$$J_i'(f; v^1, v^2) \cdot \hat{v}^i = 0 \quad \forall\, \hat{v}^i \in L^2(\mathcal{O}_i \times (0,T)), \tag{39}$$

e

$$\left\langle J''(f; v^1, v^2), \hat{v}^i \right\rangle > 0 \quad \forall\, \hat{v}^i \in L^2(\mathcal{O}_i \times (0,T)). \tag{40}$$

Depois de provar que existe um único par de Nash para cada controle líder em $L^2(\mathcal{O}\times(0,T))$, buscamos um controle $f \in L^2(\mathcal{O} \times (0,T))$ tal que

$$J(f) = \min_{\hat{f}} J(\hat{f}) \tag{41}$$

e a correspondente solução de (37) satisfaz a condição de controlabilidade nula:

$$y(\cdot,T) = 0 \quad \text{in} \quad (0,1). \tag{42}$$

Agora vamos apresentar os resultados do Capítulo 3. O primeiro resultado afirma sobre a existência e unicidade de equilíbrio de Nash.

Teorema (Existência e unicidade de equilíbrio de Nash): *Suponha μ_i suficientemente grande. Então, para cada $f \in L^2(\mathcal{O} \times (0,T))$ dado, existe um único equilíbrio de Nash $(v^1(f), v^2(f))$ para os funcionais J_i.*

O seguinte resultado garante a controlabilidade nula local com dado inicial em $H_0^1(0,1)$:

Teorema (caso $y^0 \in H_0^1(0,1)$): *Suponha $y^0 \in H_0^1(0,1)$ com $||y^0||_{H_0^1(0,1)} \leq r$ para algum $r > 0$ e $\mathcal{O}_{i,d} \cap \mathcal{O} \neq \emptyset$, $i = 1,2$. Assuma uma das seguintes condições valem:*

$$\mathcal{O}_{1,d} = \mathcal{O}_{2,d} := \mathcal{O}_d, \tag{43}$$

ou

$$\mathcal{O}_{1,d} \cap \mathcal{O} \neq \mathcal{O}_{2,d} \cap \mathcal{O}. \tag{44}$$

Se as constantes $\mu_i > 0$ $(i = 1,2)$ são suficientemente grandes e existe uma função positiva $\rho = \rho(t)$, que decai exponencialmente para 0 quando $t \to T^-$, tal que, as funções $y_{i,d} \in L^2(\mathcal{O}_{i,d} \times (0,T))$ $(i = 1,2)$ satisfazem

$$\iint_{\mathcal{O}_{i,d}\times(0,T)} \rho^{-2}\, y_{i,d}^2 \, dx\, dt < +\infty, \quad i = 1,2, \tag{45}$$

então, existe um controle $f \in L^2(\mathcal{O}\times(0,T))$ e um equilíbrio de Nash $(v^1(f), v^2(f))$ associado tal que tem-se (41) e a correspondente solução de (37) satisfaz (42).

Aqui temos uma interpretação para a hipótese (44) parecida com a (23) no caso da equação do calor. Como desejamos que o estado seja nulo no tempo $T > 0$, sem que nos afastemos de $y_{i,d}$, é esperado que estas funções fiquem perto de se anular quando t tende a T.

Supor que y^0 é limitada em $H_0^1(0,1)$ é uma condição razoável, pois devido aos resultados em [26] e [34], sabemos que para o sistema (37) não podemos ter controlabilidade nula global.

Além disso, de [26] segue que para $y^0 \in L^2(0,1)$ com $||y^0||_{L^2(0,1)} \leq r$, $r > 0$, temos um tempo mínimo para controlabilidade nula $T(r) > 0$, com

$$C_0 \log(1/r)^{-1} \leq T(r) \leq C_1 \log(1/r)^{-1},$$

onde C_0, $C_1 > 0$ são constantes adequadas.

Em relação à controlabilidade nula local com dado inicial em $L^2(0,1)$ temos o resultado:

Teorema (caso $y^0 \in L^2(0,1)$): *Suponha $y^0 \in L^2(0,1)$ com $||y^0||_{L^2(0,1)} \leq r$ para algum $r > 0$ e $T \geq T(r)$. Assumindo, como no teorema do caso $y^0 \in H_0^1(0,1)$, que uma das condições (43), (44) é satisfeita, μ_i suficientemente grande e que existe uma função ρ tal que a hipótese (45) vale, então, existe um controle $f \in L^2(\mathcal{O} \times (0,T))$ e um equilíbrio de Nash associado $(v^1(f), v^2(f))$ tal que correspondente solução de (37) satisfaz (42).*

Questões em aberto e trabalhos futuros

Problema parabólico com todos os controles na fronteira.

Considere o problema linear

$$\begin{cases} z_t - \Delta z + a(x,t)z = 0 & \text{in} \quad Q, \\ z = v1_S + v^1 1_{S_1} + v^2 1_{S_2} & \text{on} \quad \Sigma, \\ z(\cdot, 0) = z^0 & \text{in} \quad \Omega, \end{cases} \tag{46}$$

Podemos pensar em funcionais custo como os J_i ou K_i definidos no Capítulo 1. Então, seguindo as ideias do Capítulo 1 obtemos um equilíbrio de Nash para cada líder e um sistema de otimalidade. No entanto, para a controlabilidade nula aparece a dificuldade de obter desigualdades de Carleman para provar a desigualdade de observabilidade. A controlabilidade nula para o sistema (46) usando a estratégia de Stackelberg–Nash é uma questão em aberto.

Controle hierárquico aplicando a estratégia de Stackelberg–Nash para o sistema de Navier–Stokes

O sistema (37) pode ser considerado como uma versão unidimensional simplificada do sistema de Navier–Stokes. Seja Ω um domínio limitado do $\mathbb{R}^N$ ($N = 2$ ou $N = 3$), cuja fronteira Γ é suficientemente regular. Considere o sistema de Navier–Stokes

$$\begin{cases} y_t - \Delta y + (y \cdot \nabla)y + \nabla p = f1_{\mathcal{O}} + v^1 1_{\mathcal{O}_1} + v^2 1_{\mathcal{O}_2} & \text{in} \quad Q, \\ \nabla \cdot y = 0 & \text{in} \quad Q, \\ y = 0 & \text{on} \quad \Sigma, \\ y(\cdot, 0) = y^0 & \text{in} \quad \Omega, \end{cases} \tag{47}$$

onde $T > 0$ é dado, $Q = \Omega \times (0, T)$, $\Sigma = \Gamma \times (0, T)$ e os conjuntos $\mathcal{O}$, $\mathcal{O}_1$, $\mathcal{O}_2 \subset \Omega$ abertos não vazios.

Consideramos y^0 no espaço

$$H := \{z \in L^2(\Omega)^N : \nabla \cdot z = 0 \quad \text{in} \quad \Omega, \quad z \cdot \nu \quad \text{on} \quad \Gamma\},$$

onde $\nu(x)$ denota o vetor normal unitário exterior a Ω no ponto $x \in \Gamma$.

A controlabilidade do sistema (47) tem sido bastante estudada nos últimos anos, veja por exemplo, [27] e [34], onde podemos encontrar resultados sobre controlabilidade com controles distribuídos e na fronteira, respectivamente.

No contexto do controle hierárquico, em [2] os autores analizaram o sistema (47), aplicando a estratégia de Stackelberg–Nash, onde eles já conseguiram resultados sobre existência e unicidade de equilíbrio de Nash para funcionais similares aos J_i definidos em (38). Mas controlabilidade nula para (47) ainda é uma questão em aberto.

Controle hierárquico para equação da onda com controles na fronteira.

No Capítulo 2 resolvemos o problema de controle hierárquico para a controlabilidade exata da equação da onda, com um líder e dois seguidores, com todos os controles distribuídos. Em [50], o autor analisou o controle hierárquico com um líder e um seguidor, ambos na fronteira, onde resolveu o problema de controlabilidade aproximada para equação da onda. Um problema que surgiu nessa direção foi o da controlabilidade exata com um líder e pelo menos dois seguidores, com todos os controles na fronteira.

Considere um sistema do tipo:

$$\begin{cases} y_{tt} - \Delta y + a(x,t)y = 0 & \text{in} \quad Q, \\ y = f1_{\Gamma_0} + v^1 1_{\Gamma_1} + v^2 1_{\Gamma_2} & \text{on} \quad \Sigma, \\ y(\cdot, 0) = y^0, \; y_t(\cdot, 0) = y^1 & \text{in} \quad \Omega, \end{cases} \tag{48}$$

Então, a parte de encontrar um equilíbrio de Nash para cada líder é análoga ao que é feito no Capítulo 2, mas a dificuldade aparece para demonstrar a desigualdade de observabilidade. Esta ainda é uma questão em aberto.

Capítulo 1

Hierarchic control for exact controllability of parabolic equations with distributed and boundary controls

Hierarchic control for the exact controllability of parabolic equations with distributed and boundary controls

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Abstract. We present some results concerning the exact controllability of parabolic equations in the context of hierarchical control through Stackelberg–Nash strategies. We analyze two cases: in the first one, the main control (the leader) acts in the interior of the domain and the secondary controls (the followers) act on small parts of boundary; in the second case, we consider a leader acting on a part of the boundary while the followers are of the distributed kind. In both cases, for each leader, an associated Nash equilibrium pair is found; then, we prove the existence of a leader that leads the system exactly to a prescribed (but arbitrary) trajectory. We consider linear and semilinear problems.

1.1 Introduction

Let $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ be a bounded domain with boundary Γ of class C^2. Let $\mathcal{O}, \mathcal{O}_1, \mathcal{O}_2 \subset \Omega$ be (small) nonempty open sets and let S, S_1, and S_2 be nonempty closed and disjoints subsets of Γ. Given $T > 0$, let us consider the cylinder $Q = \Omega \times (0, T)$ with lateral boundary $\Sigma = \Gamma \times (0, T)$. We will denote by $\nu(x)$ the outward unit normal to Ω at the point $x \in \Gamma$.

We will consider parabolic systems of the form

$$\begin{cases} y_t - \Delta y + a(x,t)y = F(y) + f 1_{\mathcal{O}} & \text{in} \quad Q, \\ y = v^1 \rho_1 + v^2 \rho_2 & \text{on} \quad \Sigma, \\ y(\cdot, 0) = y^0 & \text{in} \quad \Omega, \end{cases} \tag{1.1}$$

and

$$\begin{cases} p_t - \Delta p + a(x,t)p = F(p) + u^1 1_{\mathcal{O}_1} + u^2 1_{\mathcal{O}_2} & \text{in} \quad Q, \\ p = g 1_S & \text{on} \quad \Sigma, \\ p(\cdot, 0) = p^0 & \text{in} \quad \Omega, \end{cases} \tag{1.2}$$

where y^0, p^0, f, g, v^i, and u^i are given in appropriate spaces, $F : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz-continuous function and the $\rho_i \in C_0^2(S_i)$ with $0 \leq \rho_i \leq 1$. In this paper, the notation 1_A denotes the characteristic function of the set A.

We will analyze the exact controllability to the trajectories of (1.1) and (1.2) following hierarchic control techniques, as introduce by J.–L. Lions [50]. More precisely, we will apply the Stackelberg–Nash strategy, which combines hierarchical ingredients of the Stackelberg kind and non-cooperative Nash optimization methods.

The hierarchical control of hyperbolic equations has been studied in [50] with two controls (one leader and one follower). On the other hand, satisfactory results with distributed controls have been obtained in [3], [16] and [37].

The main goal in this paper is to analyze the exact controllability (to the trajectories) for the systems (1.1) and (1.2) in the context of hierarchical control, applying the Stackelberg–Nash strategy. In order to explain the methodology we will be initially concerned with linear system (1.1).

Let $\mathcal{O}_{1,d}$ and $\mathcal{O}_{2,d}$ be non-empty open subsets of Ω and let us define the cost functionals

$$J_i(f;v^1,v^2) = \frac{\alpha_i}{2}\iint_{\mathcal{O}_{i,d}\times(0,T)} |y-\xi_{i,d}|^2\,dx\,dt + \frac{\mu_i}{2}\iint_{S_i\times(0,T)} |v^i|^2\,d\sigma\,dt, \quad i=1,2, \tag{1.3}$$

where $\xi_{i,d}=\xi_{i,d}(x,t)$ are given in $L^2(\mathcal{O}_{i,d}\times(0,T))$ and α_i, μ_i are positive constants. Let us also introduce the main functional

$$J(f) := \frac{1}{2}\iint_{\mathcal{O}\times(0,T)} |f|^2\,dx\,dt.$$

For each choice of the leader f, we look for controls v^1 and v^2, depending on f, that provide an "optimal" couple for the cost functionals J_1 and J_2 in the following sense:

$$J_1(f;v^1,v^2) = \min_{\hat{v}^1} J_1(f;\hat{v}^1,v^2), \quad J_2(f;v^1,v^2) = \min_{\hat{v}^2} J_2(f;v^1,\hat{v}^2). \tag{1.4}$$

This pair (v^1,v^2) is called a *Nash equilibrium.*

Let us consider a trajectory of (1.1), that is, a function $\bar{y}=\bar{y}(x,t)$ satisfying

$$\begin{cases} \bar{y}_t - \Delta\bar{y} + a(x,t)\bar{y} = F(\bar{y}) & \text{in} \quad Q, \\ \bar{y} = 0 & \text{on} \quad \Sigma, \\ y(\cdot,0) = \bar{y}^0 & \text{in} \quad \Omega. \end{cases} \tag{1.5}$$

Assuming that a Nash equilibrium exists for each f, we then look for a leader such that

$$J(f) = \min_{\hat{f}} J(\hat{f}), \tag{1.6}$$

subject to condition of controllability constraint

$$y(\cdot,T) = \bar{y}(\cdot,T) \quad \text{in} \quad \Omega. \tag{1.7}$$

In this paper, we will use the Hilbert space

$$W(Q) := \left\{ z \in L^2(0,T;H^1(\Omega)) : \ z_t \in L^2(0,T;H^{-1}(\Omega)) \right\},$$

which is equipped with the norm

$$||z||_{W(Q)} = \left(||z||^2_{L^2(0,T;H^1(\Omega))} + ||z_t||^2_{L^2(0,T;H^{-1}(\Omega))} \right)^{1/2}.$$

We will also use the Hilbert spaces $H^{r,s}(Q) = L^2(0,T;H^r(\Omega)) \cap H^s(0,T;L^2(\Omega))$ for numbers $r,\ s \geq 0$, with norms

$$||z||_{H^{r,s}(Q)} := \left(||z||^2_{L^2(0,T;H^r(\Omega))} + ||z||^2_{H^s(0,T;L^2(\Omega))} \right)^{1/2}$$

and their boundary counter parts $H^{r,s}(\Sigma) := L^2(0,T;H^r(\Gamma)) \cap H^s(0,T;L^2(\Gamma))$; for a detailed description of these spaces and their properties, see [52]. Finally, we introduce the space

$$V := \{u \in H^1(\Omega) : u = 0 \text{ on } \Gamma \setminus S\}.$$

In relation to system (1.2), the secondary cost functionals are defined as follows:

$$K_i(g;u^1,u^2) := \frac{\alpha_i}{2} \iint_{\Sigma_{i,d}} \left| \frac{\partial p}{\partial \nu} - \zeta_{i,d} \right|^2 \rho_{i,d} \, d\sigma \, dt + \frac{\mu_i}{2} \iint_{\mathcal{O}_i \times (0,T)} |u^i|^2 \, dx \, dt, \quad i = 1,2, \tag{1.8}$$

where $\Sigma_{i,d} = \Gamma_{i,d} \times (0,T)$ and the $\Gamma_{i,d}$ are nonempty closed subsets of Γ, the $\zeta_{i,d} = \zeta_{i,d}(x,t)$ are given functions, $\alpha_i,\ \mu_i$ are positive constants and $\rho_{i,d} \in C^2_0(\Gamma_{i,d})$ with $\Gamma'_{i,d} \subset\subset \Gamma_{i,d} \subset \Gamma$ and

$$0 \leq \rho_{i,d} \leq 1, \qquad \rho_{i,d} = 1 \quad \text{in} \quad \Gamma'_{i,d}, \qquad \rho_{i,d} = 0 \quad \text{in} \quad \Gamma \setminus \Gamma_{i,d}.$$

. In this case, the main functional is the following:

$$K(g) := \frac{1}{2} ||g||^2_{H^{3/2,3/4}(S \times (0,T))}.$$

For each leader g, we will find a Nash equilibrium for the cost functionals K_i, that is, a couple $(u^1,\ u^2)$ such that

$$K_1(g;u^1,u^2) = \min_{\hat{u}^1} K_1(g;\hat{u}^1,u^2), \quad K_2(g;u^1,u^2) = \min_{\hat{u}^2} K_2(g;u^1,\hat{u}^2). \tag{1.9}$$

Let the following trajectory for (1.2) be given:

$$\begin{cases} \bar{p}_t - \Delta \bar{p} + a(x,t)\bar{p} = F(\bar{p}) & \text{in} \quad Q, \\ \bar{p} = 0 & \text{on} \quad \Sigma, \\ \bar{p}(\cdot,0) = \bar{p}^0 & \text{in} \quad \Omega. \end{cases} \tag{1.10}$$

Then, we look for a control $g \in H^{3/2,3/4}(S \times (0,T))$ verifying

$$K(g) = \min_{\hat{g}} K(\hat{g}), \tag{1.11}$$

subject to exact controllability condition

$$p(\cdot,T) = \bar{p}(\cdot,T) \quad \text{in} \quad \Omega. \tag{1.12}$$

These problems are motivated by applications found in various fields. For instance, they can play a relevant role in environmental sciences. Thus, we can view the solution $y = y(x,t)$ to (1.1) as the concentration of a chemical product in a lake Ω. The set $\mathcal{O}$ can be regarded as a part of the lake where we can apply a control f which tries to approach y to a desired state $\bar{y}$ at time T. But, at the same time, we do not want y to be too far from $\xi_{i,d}$ in $\mathcal{O}_{i,d}$, for $i = 1$, 2 and, to this purpose, we apply controls v^i on some parts S_i of the shore of lake. A similar interpretation can be given in the context of (1.2).

1.1.1 Main results

First, we will consider the system (1.1) with $F \equiv 0$. In this linear case, we have the following result:

Theorem 1 *Suppose that $\mathcal{O}_{i,d} \cap \mathcal{O} \neq \emptyset$, $i = 1,2$. Assume that one of the following conditions holds: either*

$$\mathcal{O}_{1,d} = \mathcal{O}_{2,d} \quad (\textit{and then we set } \mathcal{O}_d := \mathcal{O}_{i,d}) \quad \textit{and} \quad \xi_{1,d} = \xi_{2,d} = \xi_d \tag{1.13}$$

or

$$\mathcal{O}_{1,d} \cap \mathcal{O} \neq \mathcal{O}_{2,d} \cap \mathcal{O}. \tag{1.14}$$

If the constants $\mu_i > 0$ $(i = 1,2)$ are large enough, there exists a positive function $\rho = \rho(t)$ blowing up at $t = T$ such that, if $\bar{y}$ is the unique solution to (1.5) (with $F \equiv 0$) associated to the initial state $\bar{y}^0 \in L^2(\Omega)$ and the $\xi_{i,d} \in L^2(\mathcal{O}_{i,d} \times (0,T))$ satisfy

$$\iint_{\mathcal{O}_{i,d}\times(0,T)} \rho^2 |\bar{y} - \xi_{i,d}|^2 \, dx \, dt < +\infty, \quad i = 1,2, \tag{1.15}$$

for any $y^0 \in L^2(\Omega)$ there exist a control $f \in L^2(\mathcal{O}\times(0,T))$ and an associated Nash equilibrium (v^1, v^2) such that the solution to (1.1) satisfies (1.7).

The assumptions in (3.10) are natural, because we want to get (1.7) with a state y not too far from the $\xi_{i,d}$ and, consequently, it is reasonable to impose that the $\xi_{i,d}$ approach $\bar{y}$

as t goes to T. Note that, in this linear case, the cost functionals J_i are convex. So, (1.4) is equivalent to

$$J_i'(f;v^1,v^2)\cdot\hat{v}^i=0 \quad \forall \hat{v}^i\in L^2(S_i\times(0,T)), \quad i=1,2. \tag{1.16}$$

Let us now consider more general semilinear systems of the form (1.1). Note that, now, we cannot guarantee the convexity of the J_i. This motivates the following definition:

Definition 1 *The pair (v^1,v^2) is a Nash quasi-equilibrium for the functionals J_i if* (1.16) *is satisfied.*

With this definition in mind, we have the following results:

Theorem 2 *Suppose that the $\mathcal{O}_{i,d}$ and μ_i satisfy the assumptions in Theorem 1 and $F\in W^{1,\infty}(\mathbb{R})$. There exists a function ρ as in Theorem 1 such that, if $\bar{y}$ is the unique solution to* (1.5) *associated to the initial state $\bar{y}^0\in L^2(\Omega)$ and the $\xi_{i,d}\in L^2(\mathcal{O}_{i,d}\times(0,T))$ are such that* (3.10) *holds, then for any $y^0\in L^2(\Omega)$ there exist controls $f\in L^2(\mathcal{O}\times(0,T))$ and associated Nash quasi-equilibria (v^1,v^2) such that the corresponding solutions to* (1.1) *satisfy* (1.7).

In the semilinear case, there are some situations where the definitions of Nash equilibrium and Nash quasi-equilibrium are in fact equivalent. This is shown in the following result:

Proposition 1 *Assume that $F\in W^{2,\infty}(\mathbb{R})$ and the $\xi_{i,d}\in L^\infty(\mathcal{O}_{i,d}\times(0,T))$ $(i=1,2)$. Suppose that $y^0\in L^2(\Omega)$ and $n\le 6$. There exists $C>0$ such that, if $f\in L^2(\mathcal{O}\times(0,T))$ and*

$$\mu_i\ge C\left(1+\|f\|_{L^2(\mathcal{O}\times(0,T))}\right), \quad i=1,\,2,$$

then (1.4) *and* (1.16) *are equivalent.*

Let us now turn to systems of the kind (1.2). Assuming that $F\equiv 0$, we get the following result:

Theorem 3 *Suppose that*

$$\Gamma_{i,d}\cap S=\emptyset, \quad i=1,2, \tag{1.17}$$

and the constants $\mu_i>0$ $(i=1,2)$ are large enough. Then, there exists a positive function $\bar{\rho}=\bar{\rho}(t)$ blowing up at $t=T$ such that, if $\bar{p}$ is the solution to (1.10) *(with $F\equiv 0$) associated to the initial state $\bar{p}^0\in V$ and the $\zeta_{i,d}\in H^{1/2,1/4}(\Sigma_{i,d})$ satisfy*

$$\iint_{\Sigma_{i,d}}\bar{\rho}^2\left|\frac{\partial\bar{p}}{\partial\nu}-\zeta_{i,d}\right|^2 d\sigma\,dt<+\infty, \quad i=1,2, \tag{1.18}$$

then, for any $p^0\in V$, there exist a control $g\in H^{3/2,3/4}(S\times(0,T))$ and an associated Nash equilibrium (u^1,u^2) such that the corresponding solution to (1.2) *satisfy* (1.12).

As in the case of (3.10), the assumption (1.18) means that $\partial\bar{p}/\partial\nu$ approaches $\zeta_{i,d}$ on $\Gamma_{i,d} \times (0,T)$ as t goes to T. In this case, again, the functionals K_i are convex. So, (1.9) is equivalent to

$$K_i'(g; u^1, u^2) \cdot \hat{u}^i = 0, \quad \forall\, \hat{u}^i \in L^2(\mathcal{O}_i \times (0,T)), \quad i = 1, 2. \tag{1.19}$$

Thus, a pair (u^1, u^2) is a Nash equilibrium if and only if it satisfies (1.19).

As before, in the semilinear case we lose the convexity of the K_i. This implies that (1.9) and (1.19) are not equivalent. Accordingly, we give the following

Definition 2 *The pair (u^1, u^2) is a Nash quasi-equilibrium for the functionals K_i (1.19) is satisfied.*

The following result holds.

Theorem 4 *Suppose that the $\Gamma_{i,d}$ and the $\mu_i > 0$ are as in Theorem 3 and $F \in W^{1,\infty}(\mathbb{R})$. There exists a function $\bar{\rho}$ as in Theorem 3 such that, if $\bar{p}$ is the solution to (1.10) associated to the initial state $\bar{p}^0 \in V$ and the $\zeta_{i,d} \in H^{1/2,1/4}(\Sigma_{i,d})$ are such that (1.18) holds, then for each $p^0 \in V$, there exist controls $g \in H^{3/2,3/4}(S \times (0,T))$ and associated Nash quasi-equilibria (u^1, u^2) such that the corresponding solutions to (1.2) satisfy (1.12).*

As in Proposition 8, we can analyze the equivalence between the Nash equilibrium and Nash quasi-equilibrium for the functionals K_i. This analysis leads to the following result:

Proposition 2 *Let us assume that $F \in W^{2,\infty}(\mathbb{R})$ and the $\zeta_{i,d} \in H^{1/2,1/4}(\Sigma_{i,d})$ $(i = 1, 2)$. Suppose that $p^0 \in V$ and $n \leq 6$. There exists $C > 0$ such that, if $g \in H^{3/2,3/4}(S \times (0,T))$ and*

$$\mu_i \geq C\left(1 + \|g\|_{H^{3/2,3/4}(S\times(0,T))}\right), \quad i = 1,\, 2,$$

then (1.9) and (1.19) are equivalent.

The rest of this paper is organized as follows. In Section 3.19, we analyze the Stackelberg–Nash exact controllability of (1.1), that we divide in two cases, corresponding to the linear and semilinear situations. In the linear case, we prove Theorem 1; in the semilinear case, we establish Theorem 2 using a fixed-point argument. In Section 3.20, we investigate the control of (1.2), with arguments similar to those in Section 2. Finally, Section 3.21 is devoted to present some additional comments.

1.2 The exact controllability with a distributed leader and two boundary followers

1.2.1 The linear case

The purpose of this section is to prove Theorem 1. We will do this in the next three sections.

For (1.1) (with $F \equiv 0$), we can reduce the problem of exact controllability to the trajectory $\bar{y}$ to a null controllability problem. Indeed, let us introduce the new variable $z := y - \bar{y}$, where $\bar{y}$ is the solution to (1.5). Then z is the solution to the system

$$\begin{cases} z_t - \Delta z + a(x,t)z = f1_{\mathcal{O}} & \text{in} \quad Q, \\ z = v^1\rho_1 + v^2\rho_2 & \text{on} \quad \Sigma, \\ z(\cdot,0) = z^0 & \text{in} \quad \Omega, \end{cases} \tag{1.20}$$

with $z^0 := y^0 - \bar{y}^0$ and (1.7) is equivalent to

$$z(\cdot,T) = 0 \quad \text{in} \quad \Omega. \tag{1.21}$$

We can rewrite the functionals J_i in (1.3) as follows:

$$J_i(f;v^1,v^2) = \frac{\alpha_i}{2}\iint_{\mathcal{O}_{i,d}\times(0,T)} |z - z_{i,d}|^2\,dx\,dt + \frac{\mu_i}{2}\iint_{S_i\times(0,T)} |v^i|^2\,d\sigma\,dt, \quad i = 1,2, \tag{1.22}$$

where the $z_{i,d} := \xi_{i,d} - \bar{y}$, $i = 1,2$.

Existence and uniqueness of Nash equilibrium

Let $f \in L^2(\mathcal{O}\times(0,T))$ be fixed. Let us consider the spaces $\mathcal{H}_i := L^2(S_i \times (0,T))$ and $\mathcal{H} := \mathcal{H}_1 \times \mathcal{H}_2$. Then, since the J_i are convex, (v^1,v^2) is a Nash equilibrium if and only if

$$\alpha_i \iint_{\mathcal{O}_{i,d}\times(0,T)} (z - z_{i,d})\hat{w}^i\,dx\,dt + \mu_i \iint_{S_i\times(0,T)} v^i\hat{v}^i\,d\sigma\,dt = 0 \quad \forall \hat{v}^i \in \mathcal{H}_i, \quad i = 1,2, \tag{1.23}$$

where $\hat{w}^i$ is the solution to

$$\begin{cases} \hat{w}^i_t - \Delta\hat{w}^i + a(x,t)\hat{w}^i = 0 & \text{in} \quad Q, \\ \hat{w}^i = \hat{v}^i\rho_i & \text{on} \quad \Sigma, \\ \hat{w}^i(\cdot,0) = 0 & \text{in} \quad \Omega. \end{cases} \tag{1.24}$$

Let us introduce the linear operator $L_i \in \mathcal{L}(\mathcal{H}_i; L^2(Q))$, with $L_i \hat{v}^i = \hat{w}^i$, where $\hat{w}^i$ is the solution to (1.24) associated to $\hat{v}^i$. Let $\bar{z}$ be the solution to the system

$$\begin{cases} \bar{z}_t - \Delta \bar{z} + a(x,t)\bar{z} = f1_{\mathcal{O}} & \text{in} \quad Q, \\ \bar{z} = 0 & \text{on} \quad \Sigma, \\ \bar{z}(\cdot, 0) = z^0 & \text{in} \quad \Omega. \end{cases}$$

Then the solution to (1.20) can be written in the form $z = L_1 v^1 + L_2 v^2 + \bar{z}$. Therefore, we see from (1.23) that (v^1, v^2) is a Nash equilibrium if and only if

$$\alpha_i L_i^*((L_1 v^1 + L_2 v^2)1_{\mathcal{O}_{i,d}}) + \mu_i v^i = \alpha_i L_i^*((z_{i,d} - \bar{z})1_{\mathcal{O}_{i,d}}) \quad \text{in} \quad \mathcal{H}_i, \quad i = 1,2. \tag{1.25}$$

Here, $L_i^* \in \mathcal{L}(L(Q); \mathcal{H}_i)$ is the adjoint of L_i for $i = 1,2$. Let us define the operator $\mathbb{L} : \mathcal{H} \to \mathcal{H}$ with

$$\mathbb{L}(v^1, v^2) := (\alpha_1 L_1^*((L_1 v^1 + L_2 v^2)1_{\mathcal{O}_{1,d}}) + \mu_1 v^1, \alpha_2 L_2^*((L_1 v^1 + L_2 v^2)1_{\mathcal{O}_{2,d}}) + \mu_2 v^2) \quad \forall (v^1, v^2) \in \mathcal{H}.$$

Then $\mathbb{L} \in \mathcal{L}(\mathcal{H}; \mathcal{H})$ and we can choose μ_i large enough to have

$$(\mathbb{L}(v^1, v^2), (v^1, v^2))_{\mathcal{H}} \geqslant \mu_0 ||(v^1, v^2)||^2_{\mathcal{H}} \quad \forall (v^1, v^2) \in \mathcal{H} \tag{1.26}$$

where

$$\mu_0 = \max_{i=1,2} \left\{ \mu_i - \left(\frac{\alpha_1 + \alpha_2}{2} \right) \delta_i \right\}, \quad \delta_i = \min_{i=1,2} \left\{ ||L_i||^2_{\mathcal{L}(\mathcal{H}_i, L^2(\mathcal{O}_{i,d}))}, ||L_i||^2_{\mathcal{L}(\mathcal{H}_i, L^2(\mathcal{O}_{3-i,d}))} \right\}.$$

Now, let us consider the following bilinear form on $\mathcal{H}$:

$$A((v^1, v^2), (\hat{v}^1, \hat{v}^2)) := (\mathbb{L}(v^1, v^2), (\hat{v}^1, \hat{v}^2))_{\mathcal{H}} \quad \forall \left(v^1, v^2\right), \left(\hat{v}^1, \hat{v}^2\right) \in \mathcal{H}.$$

From the definition of $\mathbb{L}$ and the inequalities (1.26), it follows that $A(\cdot, \cdot)$ is continuous and coercive. Hence by Lax-Milgram's Theorem, for each $\Phi \in \mathcal{H}'$ there exists exactly one (v^1, v^2) satisfying

$$A((v^1, v^2), (\hat{v}^1, \hat{v}^2)) = \langle \Phi, (\hat{v}^1, \hat{v}^2) \rangle_{\mathcal{H}' \times \mathcal{H}} \quad \forall (\hat{v}^1, \hat{v}^2) \in \mathcal{H}, \quad (v^1, v^2) \in \mathcal{H}. \tag{1.27}$$

In particular, if $\Phi \in \mathcal{H}'$ is given by

$$\langle \Phi, (\hat{v}^1, \hat{v}^2) \rangle_{\mathcal{H}' \times \mathcal{H}} := ((\alpha_1 L_1^*((z_{1,d} - \bar{z})1_{\mathcal{O}_{1,d}}), \alpha_2 L_2^*((z_{2,d} - \bar{z})1_{\mathcal{O}_{2,d}})), (\hat{v}^1, \hat{v}^2))_{\mathcal{H}} \quad \forall \left(\hat{v}^1, \hat{v}^2\right) \in \mathcal{H}.$$

we get (1.25).

Thus, the following result is proved:

Proposition 3 *If*

$$\mu_i - \left(\frac{\alpha_1+\alpha_2}{2}\right)\delta_i > 0, \quad i=1,2,$$

then, for each $f \in L^2(\mathcal{O}\times(0,T))$*, there exists a unique Nash equilibrium* (v^1,v^2) *for the functionals* J_i*.*

The optimality system

This section is devoted to obtain the optimality system corresponding to a Nash equilibrium. Thus, let $f \in L^2(\mathcal{O}\times(0,T))$ be given and let (v^1,v^2) be an associated Nash equilibrium. Let us introduce the adjoint system to (1.24):

$$\begin{cases} -\phi^i_t - \Delta\phi^i + a(x,t)\phi^i = \alpha_i(z-z_{i,d})1_{\mathcal{O}_{i,d}} & \text{in} \quad Q, \\ \phi^i = 0 & \text{on} \quad \Sigma, \\ \phi^i(\cdot,T) = 0 & \text{in} \quad \Omega. \end{cases} \tag{1.28}$$

Then, for any $\hat{v}^i \in \mathcal{H}_i$, if we set $\hat{w}^i := L_i\hat{v}^i$, the following identity is found from (1.24) and (1.28):

$$\iint_{\mathcal{O}_{i,d}\times(0,T)} \alpha_i(z-z_{i,d})\hat{w}^i\,dx\,dt = -\iint_\Sigma \hat{v}^i\rho_i\frac{\partial\phi^i}{\partial\nu}\,d\sigma\,dt.$$

By (1.23), we also have

$$v^i = \frac{1}{\mu_i}\frac{\partial\phi^i}{\partial\nu}\rho_i \quad \text{on} \quad \Sigma, \quad i=1,2.$$

Consequently, we get the optimality system

$$\begin{cases} z_t - \Delta z + a(x,t)z = f1_{\mathcal{O}} & \text{in} \quad Q, \\ -\phi^i_t - \Delta\phi^i + a(x,t)\phi^i = \alpha_i(z-z_{i,d})1_{\mathcal{O}_{i,d}} & \text{in} \quad Q, \\ z = \displaystyle\sum_{i=1}^{2}\frac{1}{\mu_i}\frac{\partial\phi^i}{\partial\nu}\rho_i, \quad \phi^i = 0 & \text{on} \quad \Sigma, \\ z(\cdot,0) = z^0, \ \phi^i(\cdot,T) = 0 & \text{in} \quad \Omega. \end{cases} \tag{1.29}$$

Our task is to find a control $f \in L^2(\mathcal{O}\times(0,T))$ such that the solution to (3.15) satisfies (2.10). Before that, we will analyze the well-posedness of this system.

The following result holds:

Proposition 4 *Suppose that* $f \in L^2(\mathcal{O}\times(0,T))$*,* $z^0 \in L^2(\Omega)$*, the* $z_{i,d} \in L^2(\mathcal{O}_{i,d}\times(0,T))$ *and the* μ_i *are large enough. Then* (3.15) *has a unique solution in the class*

$$(z,\,\phi^1,\,\phi^2) \in W(Q)\times[H^{2,1}(Q)]^2.$$

Demonstração. We will use a fixed point argument. Let us set $\mu_0 := \min\{\mu_1, \mu_2\}$. For each $\bar{z} \in L^2(Q)$, let us consider the system

$$\begin{cases} z_t - \Delta z + a(x,t)z = f1_{\mathcal{O}} & \text{in} \quad Q, \\ -\phi^i_t - \Delta\phi^i + a(x,t)\phi^i = \alpha_i(\bar{z} - z_{i,d})1_{\mathcal{O}_{i,d}} & \text{in} \quad Q, \\ z = \displaystyle\sum_{i=1}^{2} \frac{1}{\mu_i}\frac{\partial\phi^i}{\partial\nu}\rho_i, \quad \phi^i = 0 & \text{on} \quad \Sigma, \\ z(\cdot,0) = z^0, \ \phi^i(\cdot,T) = 0 & \text{in} \quad \Omega. \end{cases} \tag{1.30}$$

Notice that $\phi^i \in H^{2,1}(Q)$. Moreover, there exists a constant $C > 0$ such that

$$||\phi^i||^2_{L^2(0,T;H^2(\Omega))} + ||\phi^i_t||^2_{L^2(Q)} \leq C\left(||\bar{z}||^2_{L^2(Q)} + ||z_{i,d}||^2_{L^2(\mathcal{O}_{i,d}\times(0,T))}\right), \quad i = 1, 2. \tag{1.31}$$

On the other hand, from [52, Theorem 2.1, p. 9], we know that there exists a constant $C > 0$ such that

$$\left|\left|\frac{\partial\phi^i}{\partial\nu}\rho_i\right|\right|^2_{H^{1/2,1/4}(\Sigma)} \leq C\left(||\phi^i||^2_{L^2(0,T;H^2(\Omega))} + ||\phi^i_t||^2_{L^2(Q)}\right), \quad i = 1, 2. \tag{1.32}$$

From (1.32) and [52, pp. 83-84], we deduce that $z \in W(Q)$ and, moreover, there exists a constant $C > 0$ such that

$$\begin{aligned} &||z||^2_{L^2(0,T;H^1(\Omega))} + ||z_t||^2_{L^2(0,T;H^{-1}(\Omega))} \\ &\qquad \leq C\left(||z^0||^2_{L^2(\Omega)} + ||f||^2_{L^2(\mathcal{O}_{i,d}\times(0,T))} + \sum_{i=1}^{2}\frac{1}{\mu_i^2}\left|\left|\frac{\partial\phi^i}{\partial\nu}\rho_i\right|\right|^2_{H^{1/2,1/4}(\Sigma)}\right). \end{aligned} \tag{1.33}$$

So, by (1.31)–(1.33), we deduce that

$$\begin{aligned} &||z||^2_{L^2(0,T;H^1(\Omega))} + ||z_t||^2_{L^2(0,T;H^{-1}(\Omega))} \\ &\quad \leq C\left(||z^0||^2_{L^2(\Omega)} + ||f||^2_{L^2(\mathcal{O}_{i,d}\times(0,T))} + \frac{1}{\mu_0}||\bar{z}||^2_{L^2(Q)} + \sum_{i=1}^{2}||z_{i,d}||^2_{L^2(\mathcal{O}_{i,d}\times(0,T))}\right). \end{aligned} \tag{1.34}$$

Let us introduce the mapping $\Lambda : L^2(Q) \to L^2(Q)$, with $\Lambda(\bar{z}) := z$, where (z, ϕ^1, ϕ^2) is the solution to (1.30) corresponding to $\bar{z}$. Note that Λ is a well defined, continuous and affine. Moreover, the previous estimates written for $\Lambda(\bar{z}_1) - \Lambda(\bar{z}_2)$ obviously give

$$||\Lambda(\bar{z}_1) - \Lambda(\bar{z}_2)||_{L^2(Q)} \leq \frac{C}{\mu_0}||\bar{z}_1 - \bar{z}_2||_{L^2(Q)} \quad \bar{z}_1, \bar{z}_2 \in L^2(Q).$$

In other words, if μ_0 is sufficiently large, Λ is a contraction, whence it possesses exactly one fixed-point. This ends the proof. ■

Remark 1 *From the proof of Proposition 4, we see that, if the μ_i are large enough, there exists $C > 0$ such that the solution to (1.20) satisfies*

$$||z||_{L^2(0,T;H^1(\Omega))} + ||z_t||_{L^2(0,T;H^{-1}(\Omega))} \leq C\left(1 + ||f||_{L^2(\mathcal{O}\times(0,T))}\right).$$

Null controllability

The null controllability of (3.15) is reduced to an observability inequality for the solutions to the adjoint system, which in this case is given by

$$\begin{cases} -\psi_t - \Delta\psi + a(x,t)\psi = \displaystyle\sum_{i=1}^{2} \alpha_i \gamma^i 1_{\mathcal{O}_{i,d}} & \text{in} \quad Q, \\ \gamma_t^i - \Delta\gamma^i + a(x,t)\gamma^i = 0 & \text{in} \quad Q, \\ \psi = 0, \quad \gamma^i = \dfrac{1}{\mu_i}\dfrac{\partial\psi}{\partial\nu}\rho_i & \text{on} \quad \Sigma, \\ \psi(\cdot,T) = \psi^T, \quad \gamma^i(\cdot,0) = 0 & \text{in} \quad \Omega. \end{cases} \tag{1.35}$$

Let us explain this. From (3.15) and (1.35), we have

$$\begin{aligned} \sum_{i=1}^{2} \alpha_i \iint_Q z_{i,d}\gamma^i 1_{\mathcal{O}_{i,d}}\,dx\,dt + \int_\Omega z(x,T)\psi^T(x)\,dx \\ - \int_\Omega z^0(x)\psi(x,0)\,dx = \iint_Q f 1_{\mathcal{O}}\psi\,dx\,dt. \end{aligned} \tag{1.36}$$

Therefore, to prove the null controllability is equivalent to prove that, for each $z^0 \in L^2(\Omega)$, there exists a control f such that

$$\iint_{\mathcal{O}\times(0,T)} f\psi\,dx\,dt = -\int_\Omega z^0(x)\psi(x,0)\,dx + \sum_{i=1}^{2} \alpha_i \iint_Q z_{i,d}\gamma^i 1_{\mathcal{O}_{i,d}}\,dx\,dt \quad \forall\, \psi^T \in L^2(\Omega).$$

In the following result, we deduce an estimate that ensures the existence and uniqueness of f.

Proposition 5 *Suppose that $\mathcal{O}_{i,d} \cap \mathcal{O} \neq \emptyset$, $i = 1,2$. Assume that one of the following conditions holds:*

$$\mathcal{O}_{1,d} = \mathcal{O}_{2,d} = \mathcal{O}_d \quad \textit{and} \quad z_{1,d} = z_{2,d} = z_d \tag{1.37}$$

or

$$\mathcal{O}_{1,d} \cap \mathcal{O} \neq \mathcal{O}_{2,d} \cap \mathcal{O}. \tag{1.38}$$

Then, there exist $C > 0$ and a weight $\rho = \rho(t)$, blowing up at $t = T$, and depending on $||a||_{L^\infty(Q)}$, such that, for any $\psi^T \in L^2(\Omega)$, the solution $(\psi, \gamma^1, \gamma^2)$ to (1.35), satisfies, respectively,

$$\int_\Omega |\psi(x,0)|^2\,dx + \iint_{\mathcal{O}_{i,d}\times(0,T)} \rho^{-2}|\alpha_1\gamma^1 + \alpha_2\gamma^2|^2\,dx\,dt \leq C \iint_{\mathcal{O}\times(0,T)} |\psi|^2\,dx\,dt. \tag{1.39}$$

or

$$\int_\Omega |\psi(x,0)|^2\,dx + \sum_{i=1}^{2} \iint_{\mathcal{O}_{i,d}\times(0,T)} \rho^{-2}|\gamma^i|^2\,dx\,dt \leq C \iint_{\mathcal{O}\times(0,T)} |\psi|^2\,dx\,dt. \tag{1.40}$$

The proof of Proposition 11 is given below. Let us see how this result leads to the null controllability of (3.15).

For each $\varepsilon > 0$, let us introduce the functional

$$F_\varepsilon(\psi^T) := \frac{1}{2}\iint_{\mathcal{O}\times(0,T)} |\psi|^2\,dx\,dt + \varepsilon||\psi^T||_{L^2(\Omega)} + \int_\Omega z^0(x)\psi(x,0)\,dx \\ - \sum_{i=1}^{2}\alpha_i \iint_{\mathcal{O}_{i,d}\times(0,T)} z_{i,d}\,\gamma^i\,dx\,dt.$$

Obviously $F_\varepsilon : L^2(\Omega) \to \mathbb{R}$ is continuous and strictly convex. We also have, by (3.40), that F_ε is coercive. Therefore, F_ε possesses a unique minimizer, that will be denoted by ψ_ε^T. If $\psi_\varepsilon^T = 0$, arguing as in [21], we deduce that the control $f = 0$ furnishes a state satisfying

$$||z_\varepsilon(\cdot,T)||_{L^2(\Omega)} \le \varepsilon. \tag{1.41}$$

If $\psi_\varepsilon^T \neq 0$,

$$F'_\varepsilon(\psi_\varepsilon^T)\cdot\psi^T = 0 \quad \forall \psi^T \in L^2(\Omega), \tag{1.42}$$

whence, denoting by $(\psi_\varepsilon, \gamma_\varepsilon^1, \gamma_\varepsilon^2)$ the solution to (1.35) corresponding to ψ_ε^T, one has

$$\iint_{\mathcal{O}\times(0,T)} \psi_\varepsilon\psi\,dx\,dt + \varepsilon\left(\frac{\psi_\varepsilon^T}{||\psi_\varepsilon^T||_{L^2(\Omega)}}, \psi^T\right) + \int_\Omega z^0(x)\psi(x,0)\,dx \\ - \sum_{i=1}^{2}\alpha_i \iint_{\mathcal{O}_{i,d}\times(0,T)} z_{i,d}\,\gamma^i\,dx\,dt = 0 \quad \forall \psi^T \in L^2(\Omega). \tag{1.43}$$

Let us set $f_\varepsilon := \psi_\varepsilon 1_{\mathcal{O}\times(0,T)}$ and let us denote by z_ε the associated state. Then, comparing (1.43) and (1.36), we obtain:

$$\int_\Omega \left(z_\varepsilon(x,T) + \varepsilon\frac{\psi_\varepsilon^T(x)}{||\psi_\varepsilon^T||_{L^2(\Omega)}}\right)\psi^T(x)\,dx = 0 \quad \forall \psi^T \in L^2(\Omega)$$

and, again, one has (1.41).

From (3.40) and (1.43), it follows that

$$||f_\varepsilon||_{L^2(\mathcal{O}\times(0,T))} \le C\left(\sum_{i=1}^{2}\iint_{\mathcal{O}_{i,d}\times(0,T)} \rho^2|z_{i,d}|^2\,dx\,dt + ||z^0||^2_{L^2(\Omega)}\right)^{1/2}.$$

Consequently the controls f_ε are uniformly bounded in $L^2(\mathcal{O}\times(0,T))$ and, therefore, it can be assumed that $f_\varepsilon \to f$ weakly in $L^2(\mathcal{O}\times(0,T))$. In this way, one also has

$$z_\varepsilon \to z \quad \text{weakly} \quad \text{in} \quad W(Q),$$
$$\phi_\varepsilon^i \to \phi^i \quad \text{weakly} \quad \text{in} \quad H^{2,1}(Q), \quad i = 1,2,$$

where (z, ϕ^1, ϕ^2) is the solution to (3.15) associated to f. In particular, we have the weak convergence of $(z_\varepsilon(\cdot, T))$ in $L^2(\Omega)$. Thus, from (1.41), we get $z(\cdot, T) = 0$ and the null controllability to (3.15) is proved.

This ends the proof of Theorem 1.

Now, let us turn to Proposition 11 to establish its proof.

Proof of Proposition 11.

Case 1: Let us assume that (3.38) holds. Then, there exists a nonempty open set ω with $\omega \subset\subset \mathcal{O}_d \cap \mathcal{O}$. Let $\eta^0 \in C^2(\bar{\Omega})$ be a function verifying

$$\eta^0 > 0 \ \text{ in } \ \Omega, \quad \eta^0 = 0 \ \text{ on } \ \Gamma, \quad |\nabla \eta^0| > 0 \ \text{ in } \ \overline{\Omega} \backslash \omega.$$

The proof of the existence of such function can be found in [34]. Let $l \in C^\infty([0, T])$ be a function satisfying

$$l(t) > 0 \quad \forall t \in (0, T), \quad l(t) = t \quad \forall t \in \left[0, \frac{T}{4}\right], \quad l(t) = T - t \quad \forall t \in \left[\frac{3T}{4}, T\right]$$

and let us introduce the following weight functions:

$$\varphi(x, t) := \frac{e^{\lambda(\eta^0(x) + m_1)}}{l^k(t)}, \quad \alpha(x, t) := \frac{e^{\lambda(\eta^0(x) + m_1)} - e^{\lambda(||\eta^0||_{L^\infty(\Omega)} + m_2)}}{l^k(t)}, \tag{1.44}$$

with $k \geq 4$ and $\lambda \geq 1$. The constants m_1 and m_2 are chosen such that $m_1 \leq m_2$ and

$$|\alpha_t| \leq C \varphi^{\frac{k+1}{k}} \quad \forall \lambda \geq 1, \tag{1.45}$$

where $C > 0$.

Let us also introduce the following notation:

$$I_1(\pi) := \iint_Q (s^{-1} \varphi^{-1} |\nabla \pi|^2 + s \lambda^2 \varphi |\pi|^2) e^{2s\alpha} \, dx \, dt$$

and

$$I_2(\pi) := \iint_Q \left[s^{-1} \varphi^{-1} (|\pi_t|^2 + |\Delta \pi|^2) + s \lambda^2 \varphi |\nabla \pi|^2 + s^3 \lambda^4 \varphi^3 |\psi|^2\right] e^{2s\alpha} \, dx \, dt.$$

In view of the density of $H_0^1(\Omega)$ in $L^2(\Omega)$, it can be assumed that $\psi^T \in H_0^1(\Omega)$. Thus, according to Proposition 4, system (1.35) possesses a unique solution $(\psi, \gamma^1, \gamma^2) \in H^{2,1}(Q) \times [W(Q)]^2$. Consequently $h := \alpha_1 \gamma^1 + \alpha_2 \gamma^2 \in W(Q)$. Then, applying the Carleman estimates in [39] and [34] to the equations fulfilled by h and ψ, respectively, we obtain

$$\begin{aligned} I_1(h) \leq C \Bigg\{ & s^{-1/2} \sum_{i=1}^{2} \frac{\alpha_i^2}{\mu_i^2} \left|\left| \varphi^{-1/4} \frac{\partial \psi}{\partial \nu} \rho_i e^{s\alpha} \right|\right|^2_{H^{1/2,1/4}(\Sigma)} \\ & + s \lambda^2 \iint_{\omega \times (0,T)} \varphi e^{2s\alpha} |h|^2 \, dx \, dt \Bigg\} \end{aligned} \tag{1.46}$$

and

$$I_2(\psi) \le C\left\{ \iint_Q e^{2s\alpha}|h|^2\,dx\,dt + s^3\lambda^4 \iint_{\omega\times(0,T)} \varphi^3 e^{2s\alpha}|\psi|^2\,dx\,dt \right\}. \tag{1.47}$$

Thanks to (3.38), we have that, in $\mathcal{O}_d \times (0,T)$, $h = -\psi_t - \Delta\psi + a(x,t)\psi$. Let ω' be an open set such that $\omega \subset\subset \omega' \subset\subset \mathcal{O}_d \cap \mathcal{O}$ and let ζ be a function in $C_0^2(\omega')$ satisfying

$$\zeta = 1 \quad \text{in} \quad \omega \quad \text{and} \quad 0 \le \zeta \le 1. \tag{1.48}$$

From (1.35) and the definition of h, integrating by parts and using (1.45) and (1.48), we obtain

$$\begin{aligned} \iint_{\omega\times(0,T)} s\lambda^2\varphi e^{2s\alpha}|h|^2\,dx\,dt &\le s\lambda^2 \iint_{\omega'\times(0,T)} \zeta\varphi e^{2s\alpha} h(-\psi_t - \Delta\psi + a(x,t)\psi)\,dx\,dt \\ &\le C\left\{ s^3\lambda^4 \iint_{\omega'\times(0,T)} \varphi^3 e^{2s\alpha} h\psi\,dx\,dt \right. \\ &+ s^2\lambda^3 \iint_{\omega'\times(0,T)} \varphi^2 e^{2s\alpha}|\nabla h|\psi\,dx\,dt \\ &\left. + s\lambda^2 \iint_{\omega'\times(0,T)} \varphi e^{2s\alpha}(h_t - \Delta h + a(x,t)h)\psi\,dx\,dt \right\}. \end{aligned} \tag{1.49}$$

In view the definition of h, the last integral is zero. Thus, recalling the definition of $I_1(h)$, we find that

$$\iint_{\omega\times(0,T)} s\lambda^2\varphi e^{2s\alpha}|h|^2\,dx\,dt \le C\left\{ \varepsilon I_1(h) + \frac{1}{\varepsilon} s^5\lambda^6 \iint_{\omega'\times(0,T)} \varphi^5 e^{2s\alpha}|\psi|^2\,dx\,dt \right\}.$$

Let us take again $\mu_0 = \min\{\mu_1, \mu_2\}$. Arguing as in [40, p. 364], we get

$$\begin{aligned} s^{-1/2} \sum_{i=1}^{2} \frac{\alpha_i^2}{\mu_i^2} \left\| \varphi^{-1/4} \frac{\partial\psi}{\partial\nu} \rho_i e^{s\alpha} \right\|^2_{H^{1/2,1/4}(\Sigma)} &\le \frac{C}{\mu_0^2} \left\{ \iint_Q s^{\frac{3}{2}} \varphi^2 e^{2s\alpha}|\psi|^2\,dx\,dt \right. \\ &\left. + s^{-1/2} \iint_{\mathcal{O}_d\times(0,T)} e^{2s\alpha}|h|^2\,dx\,dt \right\}. \end{aligned} \tag{1.50}$$

Therefore, if the μ_i and s are large enough, from (1.46), (1.47) and (1.50), the following is found:

$$I_1(h) + I_2(\psi) \le C \iint_{\omega'\times(0,T)} \varphi^5 e^{2s\alpha}|\psi|^2\,dx\,dt. \tag{1.51}$$

At this point, we will consider the following new weight functions:

$$\bar\varphi(x,t) := \frac{e^{\lambda(\eta^0(x)+m_1)}}{\bar l^k(t)}, \quad \bar\alpha(x,t) := \frac{e^{\lambda(\eta^0(x)+m_1)} - e^{\lambda(\|\eta^0\|_{L^\infty(\Omega)}+m_2)}}{\bar l^k(t)}, \tag{1.52}$$

where $\bar l : [0,T] \to \mathbb{R}$ is defined as follows

$$\bar l(t) = \begin{cases} l(t_0), & \text{if} \quad t \in [0,t_0], \\ l(t), & \text{if} \quad t \in [t_0,T], \end{cases} \tag{1.53}$$

and $t_0 \in (T/4, 3T/4)$ is such that $l(t_0)$ is a local maximum. The constants λ, k, m_1, and m_2 are as in (1.44) and (1.45).

Arguing as in [27, Lemma 1], using the equations in (1.35) and the inequality (1.51), we deduce that there exists a constant $C > 0$ such that

$$\begin{aligned} ||\psi(0)||^2_{L^2(\Omega)} + \iint_Q \bar{\varphi} e^{2s\bar{\alpha}} |\nabla \psi|^2 \, dx \, dt + \iint_Q \bar{\varphi}^3 e^{2s\bar{\alpha}} |\psi|^2 \, dx \, dt \, + \, \bar{I}_1(h) \\ \leq C \iint_{\omega' \times (0,T)} \bar{\varphi}^5 e^{2s\bar{\alpha}} |\psi|^2 \, dx \, dt, \end{aligned} \tag{1.54}$$

where $\bar{I}_1(h)$ is the analogous to $I_1(h)$, with $\bar{\alpha}$ and $\bar{\varphi}$ instead of α and φ, respectively. With this, we get (1.39).

Case 2: Now, let us assume that (3.39) holds. Let $\widetilde{\mathcal{O}} \subset\subset \mathcal{O}$ be a nonempty connected open set. By (3.39), there exist nonempty open sets $\omega_i \subset\subset \mathcal{O}_{i,d} \cap \widetilde{\mathcal{O}}$ $(i = 1,\ 2)$, such that $\omega_1 \cap \omega_2 = \emptyset$. Hence, we have to analyze the following situations:

$$\omega_1 \cap \mathcal{O}_{2,d} = \emptyset \quad \text{and} \quad \omega_2 \cap \mathcal{O}_{1,d} = \emptyset \tag{1.55}$$

and

$$\omega_i \subset \mathcal{O}_{j,d} \quad \text{and} \quad \omega_j \cap \mathcal{O}_{i,d} = \emptyset, \quad \text{with} \quad (i,j) = (1,2) \quad \text{or} \quad (i,j) = (2,1). \tag{1.56}$$

There exist functions $\eta^1, \eta^2 \in C^2(\overline{\Omega})$ such that

$$\begin{cases} \eta^i > 0 \quad \text{in} \quad \Omega, \quad \eta^i = 0 \quad \text{on} \quad \Gamma, \quad |\nabla \eta^i| > 0 \quad \text{in} \quad \overline{\Omega} \setminus \omega_i, \\ \eta^1 = \eta^2 \quad \text{in} \quad \Omega \setminus \widetilde{\mathcal{O}} \quad \text{and} \quad ||\eta^1||_\infty = ||\eta^2||_\infty. \end{cases} \tag{1.57}$$

The proof of the existence of these functions is given in [1]. Accordingly, let us introduce the following weight functions:

$$\varphi_i(x,t) = \frac{e^{\lambda(\eta^i(x)+m_1)}}{l^k(t)}, \qquad \sigma_i(x,t) = \frac{e^{\lambda(\eta^i(x)+m_1)} - e^{\lambda(||\eta^i||_{L^\infty(\Omega)}+m_2)}}{l^k(t)}, \tag{1.58}$$

where the constants λ, k, m_1, and m_2 are as in (1.44) and (1.45). Also, let us set

$$I_1^i(\gamma^j) := \iint_Q (s^{-1}\varphi_i^{-1}|\nabla \gamma^j|^2 + s\lambda^2 \varphi_i |\gamma^j|^2) e^{2s\sigma_i} \, dx \, dt. \tag{1.59}$$

By [40], the following inequality holds for $i = 1, 2$,

$$\begin{aligned} I_1^i(\gamma^i) \leq C \Bigg\{ s^{-1/2} \frac{1}{\mu_i^2} \left|\left| \varphi_i^{-1/4} \frac{\partial \psi}{\partial \nu} \rho_i e^{s\sigma_i} \right|\right|^2_{H^{1/2,1/4}(\Sigma)} \\ + \iint_{\omega_i \times (0,T)} s\lambda^2 \varphi_i |\gamma^i|^2 e^{2s\sigma_i} \, dx \, dt \Bigg\}. \end{aligned} \tag{1.60}$$

Let $\hat{\zeta}$ be a function in $C^2(\overline{\Omega})$ such that

$$\hat{\zeta} = 0 \quad \text{in} \quad \widetilde{\mathcal{O}}, \quad \hat{\zeta} = 1 \quad \text{in} \quad \Omega \setminus \overline{\mathcal{O}}.$$

Then it is clear that $\hat{\psi} = \hat{\zeta}\psi$ is the solution to the system

$$\begin{cases} -\hat{\psi}_t - \Delta\hat{\psi} + a(x,t)\hat{\psi} = \displaystyle\sum_{i=1}^{2} \alpha_i \hat{\zeta}\gamma^i 1_{\mathcal{O}_{i,d}} - 2\nabla\cdot(\psi\nabla\hat{\zeta}) + \psi\Delta\hat{\zeta} & \text{in} \quad Q, \\ \hat{\psi} = 0 & \text{on} \quad \Sigma, \\ \hat{\psi}(\cdot,T) = \hat{\zeta}\psi^T & \text{in} \quad \Omega. \end{cases} \tag{1.61}$$

By [1], we have the existence of a positive constant $C > 0$ such that

$$R^1(\hat{\psi}) \le C\left\{ s^{-1}\lambda^{-1}\sum_{i=1}^{2}\iint_{\mathcal{O}_{i,d}\times(0,T)} \varphi_1^{-1}e^{2s\sigma_1}|\hat{\zeta}\gamma^i|^2\,dx\,dt + s^{-1}\lambda^{-1}\iint_Q \varphi_1^{-1}e^{2s\sigma_1}|\psi\Delta\hat{\zeta}|^2\,dx\,dt \right.$$
$$\left. + s\lambda\sum_{i=1}^{n}\iint_Q \varphi_1 e^{2s\sigma_1}\left|\psi\frac{\partial\hat{\zeta}}{\partial x_i}\right|^2\,dx\,dt + s^2\lambda^3\iint_{\omega_1\times(0,T)} \varphi_1^2 e^{2s\sigma_1}|\hat{\psi}|^2\,dx\,dt \right\},$$

where

$$R^i(\hat{\psi}) := \lambda\iint_Q e^{2s\sigma_i}|\nabla\hat{\psi}|^2\,dx\,dt + s^2\lambda^3\iint_Q \varphi_i^2 e^{2s\sigma_i}|\hat{\psi}|^2\,dx\,dt. \tag{1.62}$$

On the other hand, from the definition of $\hat{\zeta}$, we have

$$\begin{aligned} R^1(\hat{\psi}) \le C\Bigg\{ & s\lambda\iint_{\mathcal{O}\times(0,T)} \varphi_1 e^{2s\sigma_1}|\psi|^2\,dx\,dt + s^{-1}\lambda^{-1}\iint_{\mathcal{O}_{1,d}\times(0,T)} \varphi_1^{-1}e^{2s\sigma_1}|\gamma^1|^2\,dx\,dt \\ & + s^{-1}\lambda^{-1}\iint_{\mathcal{O}_{2,d}\times(0,T)} \varphi_1^{-1}e^{2s\sigma_1}|\hat{\zeta}\gamma^2|^2\,dx\,dt \\ & + s^2\lambda^3\iint_{\omega_1\times(0,T)} \varphi_1^2 e^{2s\sigma_1}|\hat{\psi}|^2\,dx\,dt \Bigg\} \end{aligned} \tag{1.63}$$

and, since $\hat{\zeta} = 1$ in $\Omega \setminus \overline{\mathcal{O}}$, we also obtain

$$R^1(\hat{\psi}) \ge s^2\lambda^3\iint_Q \varphi_1^2 e^{2s\sigma_1}|\psi|^2\,dx\,dt - s^2\lambda^3\iint_{\mathcal{O}\times(0,T)} \varphi_1^2 e^{2s\sigma_1}|\psi|^2\,dx\,dt. \tag{1.64}$$

Let us first assume that (1.55) holds. Combining (1.60), (1.63) and (1.64), it follows that

$$\begin{aligned} & I_1^1(\gamma^1) + I_1^2(\gamma^2) + s^2\lambda^3\iint_Q \varphi_1^2 e^{2s\sigma_1}|\psi|^2\,dx\,dt \\ & \le C\Bigg\{ s^{-1/2}\sum_{i=1}^{2}\frac{1}{\mu_i^2}\left\|\varphi_i^{-1/4}\frac{\partial\psi}{\partial\nu}\rho_i e^{s\sigma_i}\right\|^2_{H^{1/2,1/4}(S_i'\times(0,T))} + s\lambda^2\sum_{i=1}^{2}\iint_{\omega_i\times(0,T)} \varphi_i e^{2s\sigma_i}|\gamma^i|^2\,dx\,dt \\ & \quad + s^{-1}\lambda^{-1}\iint_{\mathcal{O}_{1,d}\times(0,T)} \varphi_1^{-1}e^{2s\sigma_1}|\gamma^1|^2\,dx\,dt + s^{-1}\lambda^{-1}\iint_{\mathcal{O}_{2,d}\times(0,T)} \varphi_1^{-1}e^{2s\sigma_1}|\hat{\zeta}\gamma^2|^2\,dx\,dt \\ & \quad + s^2\lambda^3\iint_{\mathcal{O}\times(0,T)} \varphi_1^2 e^{2s\sigma_1}|\psi|^2\,dx\,dt \Bigg\}. \end{aligned} \tag{1.65}$$

Since that $\eta_1 = \eta_2 = 0$ on Σ, and $||\eta^1||_\infty = ||\eta^2||_\infty$, we have $\sigma_1 = \sigma_2$ and $\phi_1 = \phi_2$ on Σ. Hence, arguing as in [40], we get

$$\begin{aligned} & s^{-1/2}\sum_{i=1}^{2}\frac{1}{\mu_i^2}\left|\left|\varphi_i^{-1/4}\frac{\partial\psi}{\partial\nu}\rho_i e^{s\sigma_i}\right|\right|^2_{H^{1/2,1/4}(\Sigma)} \\ & \quad \le \frac{C}{\mu_0^2}\left\{s^{3/2}\iint_Q \varphi_1^2 e^{2s\sigma_1}|\psi|^2\,dx\,dt + s^{-1/2}\sum_{i=1}^{2}\iint_{\mathcal{O}_{i,d}\times(0,T)}\varphi_i^{-1/2}e^{2s\sigma_i}|\gamma^i|^2\,dx\,dt\right\}, \end{aligned} \tag{1.66}$$

where, again, $\mu_0 = \min\{\mu_1, \mu_2\}$. For s and λ large enough, the third and fourth terms in the right-hand side in (1.65) and the right-hand side in (1.66) can be absorbed by left-hand side in the (1.65). Thus,

$$\begin{aligned} & I_1^1(\gamma^1) + I_1^2(\gamma^2) + s^2\lambda^3\iint_Q \varphi_1^2 e^{2s\sigma_1}|\psi|^2\,dx\,dt \\ & \le C\left\{s\lambda^2\sum_{i=1}^{2}\iint_{\omega_i\times(0,T)}\varphi_i e^{2s\sigma_i}|\gamma^i|^2\,dx\,dt + s^2\lambda^3\iint_{\mathcal{O}\times(0,T)}\varphi_1^2 e^{2s\sigma_1}|\psi|^2\,dx\,dt\right\}. \end{aligned} \tag{1.67}$$

In order to eliminate the first term in the right-hand side of (1.67), let us introduce new open sets $\tilde{\omega}_i \subset \Omega$ such that $\omega_i \subset\subset \tilde{\omega}_i \subset\subset \mathcal{O}_{i,d} \cap \widetilde{\mathcal{O}}$ and $\tilde{\omega}_i \cap \mathcal{O}_{3-i,d} = \emptyset$ and let us define functions $\delta_i \in C_0^2(\tilde{\omega}_i)$, with

$$\delta_i = 1 \quad \text{in} \quad \omega_i, \quad 0 \le \delta_i \le 1. \tag{1.68}$$

Then, by (1.35) and (1.59), for s and λ large enough, we have

$$\begin{aligned} s\lambda^2\iint_{\omega_i\times(0,T)}\varphi_i e^{2s\sigma_i}|\gamma^i|^2\,dx\,dt & \le s\lambda^2\iint_{\tilde{\omega}_i\times(0,T)}\delta_i\varphi_i e^{2s\sigma_i}|\gamma^i|^2\,dx\,dt \\ & \le \frac{s\lambda^2}{\alpha_i}\iint_{\tilde{\omega}_i\times(0,T)}\delta_i\varphi_i e^{2s\sigma_i}\gamma^i\left(-\psi_t - \Delta\psi + a(x,t)\psi\right)\,dx\,dt \\ & \le C\left\{\varepsilon\, I_1^i(\gamma^i) + \frac{1}{\varepsilon}s^5\lambda^6\iint_{\tilde{\omega}_1\times(0,T)}\varphi_1^5 e^{2s\sigma_1}|\psi|^2\,dx\,dt\right\} \end{aligned} \tag{1.69}$$

for $i = 1, 2$. Thus, by (1.67) and (1.69), we see that the following inequality holds:

$$\begin{aligned} & I_1^1(\gamma^1) + I_1^2(\gamma^2) + s^2\lambda^3\iint_Q \varphi_1^2 e^{2s\sigma_1}|\psi|^2\,dx\,dt \\ & \le Cs^5\lambda^6\iint_{\mathcal{O}\times(0,T)}(\varphi_1^5 e^{2s\sigma_1} + \varphi_2^5 e^{2s\sigma_2})|\psi|^2\,dx\,dt. \end{aligned} \tag{1.70}$$

Let us consider the weight functions

$$\bar{\varphi}_i(x,t) = \frac{e^{\lambda(\eta^i(x)+m_1)}}{\bar{l}^k(t)}, \qquad \bar{\sigma}_i(x,t) = \frac{e^{\lambda(\eta^i(x)+m_1)} - e^{\lambda(||\eta^i||_{L^\infty(\Omega)}+m_2)}}{\bar{l}^k(t)}, \tag{1.71}$$

where $\bar{l}$ is defined in (1.53). Let us denote by $\bar{I}_1^i$ an expression similar to (1.59), where we replace σ_i and φ_i by $\bar{\sigma}_i$ and $\bar{\varphi}_i$, respectively. Then, arguing as we did to get the inequality (1.54), we obtain:

$$\begin{aligned} &||\psi(0)||^2_{L^2(\Omega)} + \bar{I}_1^1(\gamma^1) + \bar{I}_1^2(\gamma^2) + s^2\lambda^3 \iint_Q \bar{\varphi}_1^2 e^{2s\bar{\sigma}_1}|\psi|^2\,dx\,dt \\ &\quad \leq C\, s^5\lambda^6 \iint_{\mathcal{O}\times(0,T)} (\bar{\varphi}_1^5 e^{2s\bar{\sigma}_1} + \bar{\varphi}_2^5 e^{2s\bar{\sigma}_2})|\psi|^2\,dx\,dt. \end{aligned} \tag{1.72}$$

Note that, since that $||\eta^1||_\infty = ||\eta^2||_\infty$, we have

$$\sigma^* := \min_{x\in\overline{\Omega}} \left\{\bar{\sigma}^1(x,t)\right\} = \min_{x\in\overline{\Omega}} \left\{\bar{\sigma}^2(x,t)\right\} \quad \forall t\in[0,T). \tag{1.73}$$

In this way, from (1.72), we get

$$||\psi(0)||^2_{L^2(\Omega)} + \sum_{i=1}^2 \iint_Q e^{2s\sigma^*}|\gamma^i|^2\,dx\,dt \leq C \iint_{\mathcal{O}\times(0,T)} \left(\bar{\varphi}_1^5 e^{2s\bar{\sigma}_1} + \bar{\varphi}_2^5 e^{2s\bar{\sigma}_2}\right)|\psi|^2\,dx\,dt$$

and we deduce the inequality (3.40) with $\rho(t) = e^{-s\sigma^*(t)}$ for all $t\in(0,T)$.

Let us now assume that (1.56) holds. To fix ideas, let us take $(i,j) = (2,1)$.

For $h := \alpha_1\gamma_1 + \alpha_2\gamma_2$, we have

$$I_1^2(h) = s^{-1}\iint_Q \varphi_2^{-1} e^{2s\sigma_2}|\nabla h|^2\,dx\,dt + s\lambda^2 \iint_Q \varphi_2 e^{2s\sigma_2}|h|^2\,dx\,dt \tag{1.74}$$

whence,

$$\begin{aligned} I_1^2(h) \leq C &\left\{ s^{-1/2}\sum_{i=1}^2 \frac{\alpha_i^2}{\mu_i^2}\left|\left|\varphi_2^{-1/4}\frac{\partial\psi}{\partial\nu}\rho_i e^{s\sigma_2}\right|\right|^2_{H^{1/2,1/4}(\Sigma)} \right. \\ &\left. + s\lambda^2 \iint_{\omega_2\times(0,T)} \varphi_2 e^{2s\sigma_2}|h|^2\,dx\,dt \right\}. \end{aligned} \tag{1.75}$$

Combining (1.60) (with i=1), (1.63), and (1.75), it follows that

$$\begin{aligned} &I_1^1(\gamma^1) + I_1^2(h) + R^1(\hat{\psi}) \leq C\left\{ s^{-1/2}\frac{1}{\mu_1^2}\left|\left|\varphi_1^{-1/4}\frac{\partial\psi}{\partial\nu}\rho_1 e^{s\sigma_1}\right|\right|^2_{H^{1/2,1/4}(\Sigma)} \right. \\ &+ s\lambda^2 \iint_{\omega_1\times(0,T)} \varphi_1 e^{2s\sigma_1}|\gamma^1|^2\,dx\,dt \;+\; s^{-1/2}\sum_{i=1}^2 \frac{\alpha_i^2}{\mu_i^2}\left|\left|\varphi_2^{-1/4}\frac{\partial\psi}{\partial\nu}\rho_i e^{s\sigma_2}\right|\right|^2_{H^{1/2,1/4}(\Sigma)} \\ &+ s\lambda^2 \iint_{\omega_2\times(0,T)} \varphi_2 e^{2s\sigma_2}|h|^2\,dx\,dt + s^{-1}\lambda^{-1}\iint_{\mathcal{O}_{1,d}\times(0,T)} \varphi_1^{-1}e^{2s\sigma_1}|\gamma^1|^2\,dx\,dt \\ &\left. + s^{-1}\lambda^{-1}\iint_{\mathcal{O}_{2,d}\times(0,T)} \varphi_1^{-1}e^{2s\sigma_1}|\hat{\zeta}\gamma^2|^2\,dx\,dt + s^2\lambda^3\iint_{\mathcal{O}\times(0,T)} \varphi_1^2 e^{2s\sigma_1}|\psi|^2\,dx\,dt \right\}. \end{aligned} \tag{1.76}$$

Let us analyze each term on the right-hand side of (1.76). By (1.69), we obtain

$$s\lambda^2 \iint_{\omega_1\times(0,T)} \varphi_1 e^{2s\sigma_1}|\gamma^1|^2\,dx\,dt \leq C\left\{\varepsilon I_1^1(\gamma^1) + \frac{1}{\varepsilon}s^5\lambda^6 \iint_{\tilde{\omega}_1\times(0,T)} \varphi_1^5 e^{2s\sigma_1}|\psi|^2\,dx\,dt\right\}. \tag{1.77}$$

Analogously to (1.49) or (1.69), considering the nonempty conected open set $\tilde{\omega}_2 \subset\subset \mathcal{O}_{2,d} \cap \mathcal{O}$ with $\omega_2 \subset\subset \tilde{\omega}_2$ and the function $\tilde{\delta}_2 \in C_0^2(\tilde{\omega}_2)$ satisfying (1.68) for $i = 2$,

$$\tilde{\delta}_2 = 1 \quad \text{in} \quad \omega_2, \quad 0 \le \tilde{\delta}_2 \le 1, \tag{1.78}$$

we obtain:

$$s\lambda^2 \iint_{\omega_2 \times (0,T)} \varphi_2 e^{2s\sigma_2} |h|^2 \, dx \, dt \le C \left\{ \varepsilon I_1^2(h) + \frac{1}{\varepsilon} \, s^5 \lambda^6 \iint_{\tilde{\omega}_2 \times (0,T)} \varphi_2^5 e^{2s\sigma_2} |\psi|^2 \, dx \, dt \right\}. \tag{1.79}$$

Moreover, note that $\varphi_1 = \varphi_2$ and $\sigma_1 = \sigma_2$ on Σ. Using the definition of $\hat{\zeta}$, we see that $\partial\hat{\psi}/\partial\nu = \hat{\zeta}\partial\psi/\partial\nu$ on Σ. Hence, arguing as in [39], we get from (1.61) the inequalities

$$\begin{aligned}
& s^{-1/2} \frac{1}{\mu_1^2} \left\| \varphi_1^{-1/4} \frac{\partial \psi}{\partial \nu} \rho_1 e^{s\sigma_1} \right\|^2_{H^{1/2,1/4}(\Sigma)} + s^{-1/2} \sum_{i=1}^{2} \frac{\alpha_i^2}{\mu_i^2} \left\| \varphi_2^{-1/4} \frac{\partial \psi}{\partial \nu} \rho_i e^{s\sigma_2} \right\|^2_{H^{1/2,1/4}(\Sigma)} \\
& \quad \le \frac{C}{\mu_0^2} \left\{ s^{-1/2} \iint_Q \varphi_1^{-1/2} e^{2s\sigma_1} \left| \sum_{i=1}^{2} \alpha_i \hat{\zeta} \gamma^1 1_{\mathcal{O}_{i,d}} - 2\nabla \cdot (\psi \nabla \hat{\zeta}) + \psi \Delta \hat{\zeta} \right|^2 dx \, dt \right. \\
& \qquad \qquad \left. + s^{3/2} \iint_Q \varphi_1^2 e^{2s\sigma_1} |\hat{\psi}|^2 \, dx \, dt \right\} \\
& \quad \le \frac{C}{\mu_0^2} \left\{ \varepsilon R^1(\hat{\psi}) + s^{3/2} \iint_Q \varphi_1^2 e^{2s\sigma_1} |\psi|^2 \, dx \, dt + s^{-1/2} \iint_{\mathcal{O}_{1,d} \times (0,T)} \varphi_1^{-1/2} e^{2s\sigma_1} |\gamma^1|^2 \, dx \, dt \right. \\
& \qquad \qquad \left. + s^{-1/2} \iint_{\mathcal{O}_{2,d} \times (0,T)} \varphi_1^{-1/2} e^{2s\sigma_1} |\hat{\zeta} \gamma^2|^2 \, dx \, dt \right\}.
\end{aligned}$$

Then, assuming that $\mu_0^2 > C$, we get

$$\begin{aligned}
& s^{-1/2} \frac{1}{\mu_1^2} \left\| \varphi_1^{-1/4} \frac{\partial \psi}{\partial \nu} \rho_1 e^{s\sigma_1} \right\|^2_{H^{1/2,1/4}(\Sigma)} + s^{-1/2} \sum_{i=1}^{2} \frac{\alpha_i^2}{\mu_i^2} \left\| \varphi_2^{-1/4} \frac{\partial \psi}{\partial \nu} \rho_i e^{s\sigma_2} \right\|^2_{H^{1/2,1/4}(\Sigma)} \\
& \quad \le \varepsilon R^1(\hat{\psi}) + s^{3/2} \iint_Q \varphi_1^2 e^{2s\sigma_1} |\psi|^2 \, dx \, dt + s^{-1/2} \iint_{\mathcal{O}_{1,d} \times (0,T)} \varphi_1^{-1/2} e^{2s\sigma_1} |\gamma^1|^2 \, dx \, dt \\
& \qquad + s^{-1/2} \iint_{\mathcal{O}_{2,d} \times (0,T)} \varphi_1^{-1/2} e^{2s\sigma_1} |\hat{\zeta} \gamma^2|^2 \, dx \, dt.
\end{aligned} \tag{1.80}$$

Since $\eta_1 = \eta_2$ in supp $(\hat{\zeta}) \cap \mathcal{O}_{2,d}$, for s and λ large enough, we have

$$\begin{aligned}
s^{-1/2} \iint_{\mathcal{O}_{2,d} \times (0,T)} \varphi_1^{-1/2} e^{2s\sigma_1} |\hat{\zeta} \gamma^2|^2 \, dx \, dt & \le C \, s^{-1/2} \iint_{\mathcal{O}_{2,d} \times (0,T)} \varphi_2^{-1/2} e^{2s\sigma_2} |\hat{\zeta} h|^2 \, dx \, dt \\
& \quad + C \, s^{-1/2} \iint_{\mathcal{O}_{2,d} \times (0,T)} \varphi_1^{-1/2} e^{2s\sigma_1} |\hat{\zeta} \gamma^1|^2 \, dx \, dt \\
& \le \varepsilon \, \{ I_1^2(h) + I_1^1(\gamma^1) \},
\end{aligned} \tag{1.81}$$

$$s^{-1/2} \iint_{\mathcal{O}_{1,d} \times (0,T)} \varphi_1^{-1/2} e^{2s\sigma_1} |\gamma^1|^2 \, dx \, dt \; \le \; \varepsilon \, I_1^1(\gamma^1) \tag{1.82}$$

and finally, combining (1.76)–(1.77) and (1.79)–(1.82), it follows that

$$\begin{aligned} I_1^1(\gamma^1) + I_1^2(h) + R^1(\hat{\psi}) \leq C \Big\{ & s^2\lambda^3 \iint_{\mathcal{O}\times(0,T)} \varphi_1^2 e^{2s\sigma_1} |\psi|^2 \,dx\,dt + s^{3/2} \iint_Q \varphi_1^2 e^{2s\sigma_1} |\psi|^2 \,dx\,dt \\ & + s^5\lambda^6 \iint_{\mathcal{O}\times(0,T)} \left(\varphi_1^5 e^{2s\sigma_1} + \varphi_2^5 e^{2s\sigma_2}\right) |\psi|^2 \,dx\,dt \Big\}. \end{aligned}$$

Thus, by (1.64), we find that, if s and λ are sufficiently large, then

$$\begin{aligned} & I_1^1(\gamma^1) + I_1^2(h) + s^2\lambda^3 \iint_Q \varphi_1^2 e^{2s\sigma_1} |\psi|^2 \,dx\,dt \\ & \leq C\, s^5\lambda^6 \iint_{\mathcal{O}\times(0,T)} \left(\varphi_1^5 e^{2s\sigma_1} + \varphi_2^5 e^{2s\sigma_2}\right) |\psi|^2 \,dx\,dt. \end{aligned} \tag{1.83}$$

Now, taking $\bar{\varphi}_i$ and $\bar{\sigma}_i$ as in (1.71), and arguing as in (1.72), we get

$$\begin{aligned} & ||\psi(\cdot,0)||^2_{L^2(\Omega)} + \bar{I}_1^1(\gamma^1) + \bar{I}_1^2(h) + s^2\lambda^3 \iint_Q \bar{\varphi}_1^2 e^{2s\bar{\sigma}_1} |\psi|^2 \,dx\,dt \\ & \leq C\, s^5\lambda^6 \iint_{\mathcal{O}\times(0,T)} (\bar{\varphi}_1^5 e^{2s\bar{\sigma}} + \bar{\varphi}_2^5 e^{2s\bar{\sigma}_2}) |\psi|^2 \,dx\,dt. \end{aligned} \tag{1.84}$$

Considering the function $\bar{\sigma}$ defined in (1.73), since $h = \alpha_1\gamma^1 + \alpha_2\gamma^2$, we see from (1.84) that

$$||\psi(0)||^2_{L^2(\Omega)} + \sum_{i=1}^{2} \iint_Q e^{2s\bar{\sigma}(t)} |\gamma^i|^2 \,dx\,dt \leq C \iint_{\mathcal{O}\times(0,T)} (\bar{\varphi}_1^5 e^{2s\bar{\sigma}_1} + \bar{\varphi}_2^5 e^{2s\bar{\sigma}_2}) |\psi|^2 \,dx\,dt$$

and, again, we get the inequality (3.40) with $\rho(t) = e^{-s\sigma^*(t)}$ for all $t \in (0,T)$.

This concludes the proof. ■

1.2.2 The semilinear case

This section is devoted to prove Theorem 2 and Proposition 8. To this purpose, we will first derive the optimality system characterizing a Nash quasi-equilibrium.

The optimality system in the semilinear case

Let us consider the semilinear system (1.1). According to Definition 1, if a pair (v^1, v^2) is a Nash quasi-equilibrium, then

$$\alpha_i \iint_{\mathcal{O}_{i,d}\times(0,T)} (y - \xi_{i,d})\hat{y}^i \,dx\,dt + \mu_i \iint_{S_i\times(0,T)} v^i \hat{v}^i \,dx\,dt = 0 \quad \forall \hat{v}^i \in \mathcal{H}_i, \quad i = 1,2, \tag{1.85}$$

where $\hat{y}^i$ is the solution to

$$\begin{cases} \hat{y}^i_t - \Delta\hat{y}^i + a(x,t)\hat{y}^i = F'(y)\hat{y}^i & \text{in} \quad Q, \\ \hat{y}^i = \hat{v}^i \rho_i & \text{on} \quad \Sigma, \\ \hat{y}^i(\cdot,0) = 0 & \text{in} \quad \Omega. \end{cases} \tag{1.86}$$

The adjoint system to (1.86) is given by

$$\begin{cases} -\phi^i_t - \Delta\phi^i + a(x,t)\phi^i = \alpha_i(y-\xi_{i,d})1_{\mathcal{O}_{i,d}} + F'(y)\phi^i & \text{in} \quad Q, \\ \phi^i = 0 & \text{on} \quad \Sigma, \\ \phi^i(\cdot,T) = 0 & \text{in} \quad \Omega. \end{cases} \tag{1.87}$$

Therefore, we get from (1.85)–(1.87) that

$$\iint_{S_i\times(0,T)} \left(-\frac{\partial\phi^i}{\partial\nu} + \mu_i v^i\right) \hat{v}^i \, d\sigma \, dt = 0 \quad \forall \hat{v}^i \in \mathcal{H}_i \tag{1.88}$$

and, therefore,

$$v^i = \frac{1}{\mu_i}\frac{\partial\phi^i}{\partial\nu}\bigg|_{S_i\times(0,T)}, \quad i = 1,\, 2. \tag{1.89}$$

Consequently, if (v^1, v^2) is a Nash quasi-equilibrium for the functionals J_i ($i = 1,\, 2$), it must be given by (1.89), where the ϕ^i satisfy together with y, the following optimality system:

$$\begin{cases} y_t - \Delta y + a(x,t)y = F(y) + f1_{\mathcal{O}} & \text{in} \quad Q, \\ -\phi^i_t - \Delta\phi^i + a(x,t)\phi^i = \alpha_i(y-\xi_{i,d})1_{\mathcal{O}_{i,d}} + F'(y)\phi^i & \text{in} \quad Q, \\ y = \displaystyle\sum_{i=1}^{2} \frac{1}{\mu_i}\frac{\partial\phi^i}{\partial\nu}\rho_i, \quad \phi^i = 0 & \text{on} \quad \Sigma, \\ y(\cdot,0) = \tilde{y}^0, \quad \phi^i(\cdot,T) = 0 & \text{in} \quad \Omega. \end{cases} \tag{1.90}$$

Proof of Theorem 2

We will proof Theorem 2 using arguments similar to those in [3]. Let us take $z = y - \bar{y}$, where $\bar{y}$ is the solution to (1.5) and let us rewrite (1.90) in the form

$$\begin{cases} z_t - \Delta z + a(x,t)z = G(x,t;z)z + f1_{\mathcal{O}} & \text{in} \quad Q, \\ -\phi^i_t - \Delta\phi^i + a(x,t)\phi^i = \alpha_i(z-z_{i,d})1_{\mathcal{O}_{i,d}} + F'(z+\bar{y})\phi^i & \text{in} \quad Q, \\ z = \displaystyle\sum_{i=1}^{2} \frac{1}{\mu_i}\frac{\partial\phi^i}{\partial\nu}\rho_i, \quad \phi^i = 0 & \text{on} \quad \Sigma, \\ z(\cdot,0) = z^0, \quad \phi^i(\cdot,T) = 0 & \text{in} \quad \Omega, \end{cases} \tag{1.91}$$

where $z_{i,d} = \xi_{i,d} - \bar{y}$, $z^0 = y^0 - \bar{y}^0$, and

$$G(x,t;z) = \int_0^1 F'\left(\bar{y} + \tau z\right)\, d\tau. \tag{1.92}$$

Let us consider, for each $z \in L^2(Q)$ and each $f \in L^2(\mathcal{O} \times (0,T))$, the linear system

$$\begin{cases} w_t - \Delta w + a(x,t)w = G(x,t;z)w + f1_{\mathcal{O}} & \text{in} \quad Q, \\ -\phi^i_t - \Delta \phi^i + a(x,t)\phi^i = \alpha_i(w - z_{i,d})1_{\mathcal{O}_{i,d}} + F'(z+\bar{y})\phi^i & \text{in} \quad Q, \\ w = \displaystyle\sum_{i=1}^{2} \frac{1}{\mu_i}\frac{\partial \phi^i}{\partial \nu}\rho_i, \quad \phi^i = 0 & \text{on} \quad \Sigma, \\ w(\cdot,0) = z^0, \quad \phi^i(\cdot,T) = 0 & \text{in} \quad \Omega. \end{cases} \tag{1.93}$$

Since $F \in W^{1,\infty}(\mathbb{R})$, there exists a constant $M > 0$ such that

$$|G(x,t;s)| + |F'(s)| \leq M \quad \forall\,(x,t,s) \in Q \times \mathbb{R}. \tag{1.94}$$

From (1.93), arguing as in the proof of Proposition 3 and recalling Remark 2, we see that there exists a constant $C > 0$ such that

$$||w||_{L^2(0,T;H^1(\Omega))} + ||w_t||_{L^2(0,T;H^{-1}(\Omega))} \leq C\left(1 + ||f||_{L^2(\mathcal{O}\times(0,T))}\right). \tag{1.95}$$

Let us fix z, let us denote by $(w_z, \phi^1_z, \phi^2_z)$ the solution to (1.93) corresponding to z and let us consider the adjoint system

$$\begin{cases} -\psi_{z,t} - \Delta \psi_z + a(x,t)\psi_z = G(x,t;z)\psi_z + \displaystyle\sum_{i=1}^{2} \alpha_i \gamma^i_z 1_{\mathcal{O}_{i,d}} & \text{in} \quad Q, \\ \gamma^i_{z,t} - \Delta\gamma^i_z + a(x,t)\gamma^i_z = F'(z+\bar{y})\gamma^i_z & \text{in} \quad Q, \\ \psi_z = 0, \quad \gamma^i_z = \dfrac{1}{\mu_i}\dfrac{\partial \psi_z}{\partial \nu}\rho_i & \text{on} \quad \Sigma, \\ \psi_z(\cdot,T) = \psi^T, \quad \gamma^i_z(\cdot,0) = 0 & \text{in} \quad \Omega. \end{cases} \tag{1.96}$$

Then from (1.93) and (1.96), we see that

$$\sum_{i=1}^{2} \alpha_i \iint_{\mathcal{O}_{i,d}\times(0,T)} \gamma^i_z\, z_{i,d}\, dx\, dt + \int_\Omega w_z(x,T)\psi^T(x)\, dx - \int_\Omega z^0(x)\psi_z(x,0)\, dx = \iint_Q f1_{\mathcal{O}}\psi_z\, dx\, dt.$$

Hence, the desired null controllability property holds if and only if

$$\sum_{i=1}^{2} \alpha_i \iint_{\mathcal{O}_{i,d}\times(0,T)} \gamma^i_z z_{i,d}\, dx\, dt - \int_\Omega z^0(x)\psi_z(x,0)\, dx = \iint_Q f1_{\mathcal{O}}\psi_z\, dx\, dt.$$

As in the linear case, we can define the functional

$$\begin{aligned} F_{z,\varepsilon}(\psi^T) := & \iint_{\mathcal{O}\times(0,T)} |\psi_z|^2\, dx\, dt + \varepsilon||\psi^T||_{L^2(\Omega)} \\ & + \int_\Omega z^0(x)\psi_z(x,0)\, dx - \sum_{i=1}^{2} \alpha_i \iint_{\mathcal{O}_{i,d}\times(0,T)} z_{i,d}\gamma^i_z\, dx\, dt. \end{aligned}$$

Arguing as in Proposition 11, we deduce the following observability inequality:

$$||\psi_z(\cdot,0)||^2_{L^2(\Omega)} + \sum_{i=1}^{2} \iint_Q \rho^{-2}|\gamma_z^i|^2\,dx\,dt \le C \iint_{\mathcal{O}\times(0,T)} |\psi_z|^2\,dx\,dt, \tag{1.97}$$

where $C > 0$ is independent of z. This inequality allows to minimize $F_{z,\varepsilon}$ in $L^2(\Omega)$. Then, taking the limit as ε goes to zero, we get a (unique) leader $f_z \in L^2(\mathcal{O}\times(0,T))$ such that the associate solution $(w_z, \phi_z^1, \phi_z^2)$ to (1.93), satisfies

$$w_z(\cdot, T) = 0 \quad \text{in} \quad \Omega. \tag{1.98}$$

Also,

$$||f_z||_{L^2(\mathcal{O}\times(0,T))} \le C \quad \forall z \in L^2(Q). \tag{1.99}$$

At this point, we can apply a standard fixed-point argument to the mapping $z \mapsto w_z$ and deduce that (1.91) is null-controllable. Hence, exact controllability to the trajectories holds for (1.90) and Theorem 2 is proved.

Equilibria and quasi-equilibrium

In this section we will prove Proposition 8.

We will argue as in [3]. Let us suppose that the hypothesis in the Theorem 2 are satisfied, $F \in W^{2,\infty}(\mathbb{R})$ and $f \in L^2(\mathcal{O}\times(0,T))$ and let us consider the associated Nash quasi-equilibrium pair (v^1, v^2). For all $w^1, w^2 \in \mathcal{H}_1$ and any $s \in \mathbb{R}$, we denote by y^s the solution to the system

$$\begin{cases} y_t^s - \Delta y^s + a(x,t)y^s = F(y^s) + f1_{\mathcal{O}} & \text{in} \quad Q, \\ y^s = (v^1 + sw^1)\rho_1 + v^2\rho_2 & \text{on} \quad \Sigma, \\ y^s(\cdot,0) = y^0 & \text{in} \quad \Omega. \end{cases} \tag{1.100}$$

Let us use the notation $y := y^s|_{s=0}$. Then

$$\begin{aligned} &D_1J_1(f; v^1 + sw^1, v^2)\cdot w^2 - D_1J_1(f; v^1, v^2)\cdot w^2 \\ &= \alpha_1 \iint_{\mathcal{O}_{1,d}\times(0,T)} (y^s - \xi_{1,d})q^s\,dx\,dt - \alpha_1 \iint_{\mathcal{O}_{1,d}\times(0,T)} (y - \xi_{1,d})q\,dx\,dt \\ &\quad + s\mu_1 \iint_{S_1\times(0,t)} w^1w^2\,d\sigma\,dt, \end{aligned} \tag{1.101}$$

where q^s is the derivative of y^s in respect to the first follower in the direction w^2. Thus, q^s is the solution to

$$\begin{cases} q_t^s - \Delta q^s + a(x,t)q^s = F'(y^s)q^s & \text{in} \quad Q, \\ q^s = w^2\rho_1 & \text{on} \quad \Sigma, \\ q^s(\cdot,0) = 0 & \text{in} \quad \Omega. \end{cases} \tag{1.102}$$

In (1.101), we also have used the notation $q; = q^s|_{s=0}$.

Let us introduce the adjoint of (1.102):

$$\begin{cases} -\phi_t^s - \Delta\phi^s + a(x,t)\phi^s = \alpha_1(y^s - \xi_{1,d})1_{\mathcal{O}_{1,d}} + F'(y^s)\phi^s & \text{in} \quad Q, \\ \phi^s = 0 & \text{on} \quad \Sigma, \\ \phi^s(\cdot,T) = 0 & \text{in} \quad \Omega. \end{cases} \tag{1.103}$$

Then, from (1.102) and (1.103), we get the identity

$$\alpha_1 \iint_{\mathcal{O}_{1,d}\times(0,T)} (y^s - \xi_{1,d})q^s \, dx \, dt = \iint_\Sigma w^2 \rho_1 \frac{\partial \phi^s}{\partial \nu} \, d\sigma \, dt. \tag{1.104}$$

Replacing in (1.101), we get:

$$\begin{aligned} &D_1J_1(f; v^1 + sw^1, v^2)\cdot w^2 - D_1J_1(f; v^1, v^2) \\ &\quad = \iint_\Sigma \left(\frac{\partial\phi^s}{\partial\nu} - \frac{\partial\phi}{\partial\nu}\right)\rho_1 w^2 \, d\sigma \, dt + s\mu_1 \iint_{S_1\times(0,T)} w^1w^2 \, d\sigma \, dt. \end{aligned} \tag{1.105}$$

Note that the following limits exist:

$$h = \lim_{s\to 0}\frac{1}{s}(y^s - y), \quad \eta = \lim_{s\to 0}\frac{1}{s}(\phi^s - \phi), \quad \text{and} \quad \frac{\partial\eta}{\partial\nu} = \lim_{s\to 0}\frac{1}{s}\left(\frac{\partial\phi^s}{\partial\nu} - \frac{\partial\phi}{\partial\nu}\right).$$

Note also that the couple (h,η) solves the system

$$\begin{cases} h_t - \Delta h + a(x,t)h = F'(y)h & \text{in} \quad Q, \\ -\eta_t - \Delta\eta + a(x,t)\eta = \alpha_1 h 1_{\mathcal{O}_{1,d}} + F''(y)h\phi + F'(y)\eta & \text{in} \quad Q, \\ h = w^1\rho_1, \quad \eta = 0 & \text{on} \quad \Sigma, \\ h(\cdot,0) = 0, \quad \eta(\cdot,T) = 0 & \text{in} \quad \Omega. \end{cases} \tag{1.106}$$

Then, from (1.105) and (1.106), we see that

$$D_1^2J_1(f; v^1, v^2)\cdot(w^1,w^2) = \iint_\Sigma \frac{\partial\eta}{\partial\nu}\rho_1 w^2 \, d\sigma \, dt + \mu_1 \iint_{S_1\times(0,T)} w^1w^2 \, d\sigma \, dt \quad \forall w^1, w^2 \in \mathcal{H}_1$$

and, in particular,

$$D_1^2J_1(f; v^1, v^2)\cdot(w^1,w^1) = \iint_\Sigma \frac{\partial\eta}{\partial\nu}\rho_1 w^1 \, d\sigma \, dt + \mu_1 \iint_{S_1\times(0,T)} |w^1|^2 \, d\sigma \, dt \quad \forall w^1 \in \mathcal{H}_1. \tag{1.107}$$

Let us show that there exists a positive constant C_1 depending on $||a||_\infty$, $||F''||_\infty$, M and $||y^0||_{L^2(\Omega)}$, such that

$$\left|\iint_{S_1\times(0,T)} \frac{\partial\eta}{\partial\nu} w^1 \, d\sigma \, dt\right| \le C_1\left(1 + ||f||_{L^2(\mathcal{O}\times(0,T))}\right)||w^1||^2_{\mathcal{H}_1} \quad \forall w^1 \in \mathcal{H}_1. \tag{1.108}$$

In fact, for any given $w^1 \in \mathcal{H}_1$; the corresponding solution to (1.106) satisfies $h \in H^{1/2,1/4}(Q)$. Moreover, there exists a constant $C > 0$, independent of w^1, such that

$$||h||_{H^{1/2,1/4}(Q)} \leq C||w^1||_{\mathcal{H}_1}. \tag{1.109}$$

From (1.106), we also obtain

$$\iint_{S_1\times(0,T)} \frac{\partial \eta}{\partial \nu} w^1 \, d\sigma \, dt = \iint_Q \left(\alpha_1 h^2 1_{\mathcal{O}_{1,d}} + F''(y) h^2 \phi\right) \, dx \, dt. \tag{1.110}$$

Let us check that the right-hand side of (1.110) is bounded by the right-hand side of (1.108). To this end, let us obtain r and s such that

$$\phi \in L^r(0,T; L^s(\Omega)) \quad \text{and} \quad h \in L^{2r'}(0,T; L^{2s'}(\Omega)),$$

where r' and s' are conjugate of r and s, respectively.

Since $h \in H^{1/2,1/4}(Q)$, by interpolation we get that $h \in L^a(0,T; L^b(\Omega))$, with

$$\frac{1}{a} = \frac{1+\theta}{4} \quad \text{and} \quad \frac{1}{b} = \frac{a(n-2)+4}{2an},$$

where $\theta \in (0,1)$. Accordingly, taking $a = 2r'$ and $b = 2s'$, it follows that appropriate values of r and s are

$$r = \frac{a}{a-2} \quad \text{and} \quad s = \frac{an}{2a-4}.$$

On the other hand, since $v^i \rho_i \in H^{1/2,1/4}(\Sigma)$ and $y^0 \in L^2(\Omega)$, we have $y \in L^2(0,T; H^1(\Omega)) \cap L^\infty(0,T; L^2(\Omega))$ and, by interpolation, we deduce that $y \in L^{\bar{a}}(0,T; L^{\bar{b}}(\Omega))$, where

$$\frac{1}{\bar{a}} = \frac{\bar{\theta}}{2} \quad \text{and} \quad \frac{1}{\bar{b}} = \frac{\bar{a}(n-2)+4}{2\bar{a}n},$$

with $\bar{\theta} \in (0,1)$. Thus, from the regularity results for the solution to the heat equation, we get

$$\phi \in L^{\bar{a}}(0,T; w^{2,\bar{b}}(\Omega)) \hookrightarrow L^{\bar{a}}(0,T; L^{\bar{b}n/(n-2\bar{b})}(\Omega)) = L^{\bar{a}}(0,T; L^{2\bar{a}n/(\bar{a}(n-6)+4)}(\Omega)).$$

Therefore, taking $\bar{a} = r$, it follows that $\phi \in L^r(0,T; L^{\frac{2an}{a(n-2)-8}}(\Omega))$ and, in order to have $\phi \in L^r(0,T; L^s(\Omega))$, we need

$$\frac{an}{2a-4} \leq \frac{2an}{a(n-2)-8},$$

which is true if and only if $n \leq 6$. In this case, we can guarantee by (1.90), (1.100), (1.103), (1.110), that

$$\begin{aligned}\left|\iint_{S_1\times(0,T)} \frac{\partial\eta}{\partial\nu} w^1\, d\sigma\, dt\right| &\leq C\left(\alpha_1 ||h||^2_{L^2(\mathcal{O}_d\times(0,T))} + ||F''||_\infty ||h||^2_{L^{2r'}(0,T;L^{2s'}(\Omega))} ||\phi||_{L^r(0,T;L^s(\Omega))}\right)\\ &\leq C\left(1+||\phi||_{L^r(0,T;L^s(\Omega))}\right)||w^1||^2_{\mathcal{H}_1}\\ &\leq C\left(1+||y||_{L^2(Q)}\right)||w^1||^2_{\mathcal{H}_1}\\ &\leq C\left(1+||f||_{L^2(\mathcal{O}\times(0,T))}\right)||w^1||^2_{\mathcal{H}_1},\end{aligned}$$

which proves (1.108).

As a consequence of (1.107) and (1.108), it follows that

$$D_1^2 J_1(f;v^1,v^2)\cdot(w^1,w^1) \geq [\mu_1 - C_1(1+||f||_{L^2(\mathcal{O}\times(0,T))})]||w^1||^2_{\mathcal{H}_1} \quad \forall w^1 \in \mathcal{H}_1. \tag{1.111}$$

Analogously, we can prove that there exists another constant $C_2 > 0$ such that

$$D_2^2 J_2(f;v^1,v^2)\cdot(w^2,w^2) \geq [\mu_2 - C_2(1+||f||_{L^2(\mathcal{O}\times(0,T))})]||w^2||^2_{\mathcal{H}_2} \quad \forall w^2 \in \mathcal{H}_2. \tag{1.112}$$

Note that the constants C_i are independent of μ_i $(i=1,2)$. Hence, for μ_i large enough, we conclude that $\mu_i - C_i\left(1+||f||_{L^2(\mathcal{O}\times(0,T))}\right)$ is a positive number, v^1 (resp. v^2) minimizes $J_1(\cdot,v^2)$ (resp. $J_2(v^1,\cdot)$) and the pair (v^1,v^2) is in fact a Nash equilibrium.

This ends the proof.

1.3 Hierarchical control with a boundary leader and two distributed followers

1.3.1 The linear case

The aim of this section is to prove Theorem 3. We will do this in the next tree sections.

Let us set $z = p - \bar{p}$, where p is the solutions to (1.2) and $\bar{p}$ is the solution to (1.10) with $F \equiv 0$. Then, z is the solution to the system

$$\begin{cases} z_t - \Delta z + a(x,t)z = u^1 1_{\mathcal{O}_1} + u^2 1_{\mathcal{O}_2} & \text{in} \quad Q,\\ z = g1_S & \text{on} \quad \Sigma,\\ z(\cdot,0) = z^0 & \text{in} \quad \Omega,\end{cases} \tag{1.113}$$

where $z^0 = p^0 - \bar{p}^0$. Thus, (1.12) is equivalent to

$$z(\cdot,T) = 0 \quad \text{in} \quad \Omega. \tag{1.114}$$

We can rewrite the cost functionals K_i $(i = 1,\ 2)$, given in (1.8), as follows:

$$K_i(g; u^1, u^2) := \frac{\alpha_i}{2} \iint_{\Sigma_{i,d}} \left| \frac{\partial z}{\partial \nu} - z_{i,d} \right|^2 \rho_{i,d}\, d\sigma\, dt + \frac{\mu_i}{2} \iint_{\mathcal{O}_i \times (0,T)} |u^i|^2\, dx\, dt, \quad i = 1, 2,$$

where $z_{i,d} = \zeta_{i,d} - \partial \bar{p}/\partial \nu$.

Existence and uniqueness of Nash equilibrium

Let $g \in H^{3/2,3/4}(S \times (0,T))$ be fixed. Let us consider the spaces $\mathbf{H}_i = L^2(\mathcal{O}_i \times (0,T))$ and $\mathbf{H} = \mathbf{H}_1 \times \mathbf{H}_2$. Then, since the K_i are convex, (1.9) and (1.19) are equivalent and (u^1, u^2) is a Nash equilibrium for the K_i if and only if

$$\alpha_i \iint_{\Sigma_{i,d}} \left(\frac{\partial z}{\partial \nu} - z_{i,d} \right) \frac{\partial \hat{z}^i}{\partial \nu} \rho_{i,d}\, d\sigma\, dt + \mu_i \iint_{\mathcal{O}_i \times (0,T)} u^i \hat{u}^i\, dx\, dt = 0 \quad \forall \hat{u}^i \in \mathbf{H}_i, \tag{1.115}$$

where $\hat{z}^i$ solves

$$\begin{cases} \hat{z}^i_t - \Delta \hat{z}^i + a(x,t)\hat{z}^i = \hat{u}^i 1_{\mathcal{O}_i} & \text{in} \quad Q, \\ \hat{z}^i = 0 & \text{on} \quad \Sigma, \\ \hat{z}^i(\cdot, 0) = 0 & \text{in} \quad \Omega. \end{cases} \tag{1.116}$$

By [52], for each $\hat{u}^i \in \mathbf{H}_i$ there exists a unique solution $\hat{z}^i \in H^{2,1}(Q)$ to (1.116). Let us define the operators $L_i : \mathbf{H}_i \to H^{2,1}(Q)$, with $L_i(\hat{u}^i) = \hat{z}^i$, where $\hat{z}^i$ is the solution to (1.116). Let $\bar{z}$ be the solution to the system

$$\begin{cases} \bar{z}_t - \Delta \bar{z} + a(x,t)\bar{z} = 0 & \text{in} \quad Q, \\ \bar{z} = g 1_S & \text{on} \quad \Sigma, \\ \bar{z}(\cdot, 0) = z^0 & \text{in} \quad \Omega. \end{cases}$$

Then $z = L_1(u^1) + L_2(u^2) + \bar{z}$ and, by (1.115), it follows that

$$\alpha_i \iint_{\Sigma_{i,d}} \left(\gamma_1 (L_1 u^1 + L_2 u^2 + \bar{z}) - z_{i,d} \right) \gamma_1 (L_i(\hat{u}^i)) \rho_{i,d}\, d\sigma\, dt + \mu_i \iint_{\mathcal{O}_i \times (0,T)} u^i \hat{u}^i\, dx\, dt = 0 \quad \forall \hat{u}^i \in \mathbf{H}_i,$$

where γ_1 denotes the normal derivative trace operator.

Using arguments similar to those in Section 1.2.1, the following result can be deduced:

Proposition 6 *If*

$$\mu_i - ||\gamma_1||^2 ||L_i||^2 \left(\frac{3\alpha_i + \alpha_{3-i}}{2} \right) > 0, \quad i = 1, 2,$$

then, for each $g \in H^{3/2,3/4}(S \times (0,T))$, there exists a unique Nash equilibrium (u^1, u^2).

In addition to this, we have the existence of a positive constant C such that

$$||z||_{L^2(0,T;H^2(\Omega))} + ||z_t||_{L^2(Q)} \leq C \left(1 + ||g||_{H^{3/2,3/4}(S \times (0,T))} \right), \tag{1.117}$$

where z is the solution of (1.113).

Optimality system

Let $g \in H^{3/2,3/4}(S \times (0,T))$ be given, let $(u^1, u^2) = (u^1(g), u^2(g))$ be the associated Nash equilibrium and let z be the corresponding solution to (1.113). Let us consider the adjoint systems

$$\begin{cases} -\phi^i_t - \Delta\phi^i + a(x,t)\phi^i = 0 & \text{in} \quad Q, \\ \phi^i = -\alpha_i \left(\dfrac{\partial z}{\partial \nu} - z_{i,d} \right) \rho_{i,d} & \text{on} \quad \Sigma, \\ \phi^i(\cdot, T) = 0 & \text{in} \quad \Omega, \end{cases} \tag{1.118}$$

where $(1.118)_2$ is motivated by (1.115). From (1.115), (1.116) and (1.118) we have

$$\iint_{\mathcal{O}_i \times (0,T)} \left(\phi^i + \mu_i u^i \right) \hat{u}^i \, dx \, dt = 0 \quad \forall \hat{u}^i \in L^2(\mathcal{O}_i \times (0,T)).$$

Hence, the Nash equilibrium is characterized by the identities

$$u^i = -\frac{1}{\mu_i} \phi^i \bigg|_{\mathcal{O}_i \times (0,T)} \tag{1.119}$$

and the optimality system is

$$\begin{cases} z_t - \Delta z + a(x,t) z = -\displaystyle\sum_{i=1}^{2} \frac{1}{\mu_i} \phi^i 1_{\mathcal{O}_i} & \text{in} \quad Q, \\ -\phi^i_t - \Delta\phi^i + a(x,t)\phi^i = 0 & \text{in} \quad Q, \\ z = g 1_S, \quad \phi^i = -\alpha_i \left(\dfrac{\partial z}{\partial \nu} - z_{i,d} \right) \rho_{i,d} & \text{on} \quad \Sigma, \\ z(\cdot, 0) = z^0, \quad \phi^i(\cdot, T) = 0 & \text{in} \quad \Omega. \end{cases} \tag{1.120}$$

Since $z^0 \in V$ and $g \in H^{3/2,3/4}(S \times (0,T))$, we can proceed as in Proposition 4 and ensure that system (1.120) possesses a unique solution in the class $(z, \phi^1, \phi^2) \in H^{2,1}(Q) \times [W(Q)]^2$.

Null controllability

Now, let us prove that (1.120) is null-controllable. This will achieve the proof of Theorem 3.

Let us introduce a bounded regular domain $\widetilde{\Omega}$, with boundary $\widetilde{\Gamma}$ of class C^2, such that $\Omega \subset \widetilde{\Omega}$ and $\widetilde{\Gamma} \cap \Gamma = \Gamma \setminus S$. Let $\widetilde{Q} = \widetilde{\Omega} \times (0,T)$ be a new cylinder with lateral boundary $\widetilde{\Sigma} = \widetilde{\Gamma} \times (0,T)$. By $\tilde{z}^0$ we denote the extension of z^0 to $\widetilde{\Omega}$. Let $\mathcal{O}$ be the nonempty open set satisfying $\mathcal{O} \subset\subset \widetilde{\Omega} \setminus \overline{\Omega}$. We will consider the following extended system

$$\begin{cases} \tilde{z}_t - \Delta \tilde{z} + a(x,t)\tilde{z} = \tilde{g}1_{\mathcal{O}} - \displaystyle\sum_{i=1}^{2} \frac{1}{\mu_i}\tilde{\phi}^i 1_{\mathcal{O}_i} & \text{in} \quad \widetilde{Q}, \\ -\tilde{\phi}^i_t - \Delta\tilde{\phi}^i + a(x,t)\tilde{\phi}^i = 0 & \text{in} \quad \widetilde{Q}, \\ \tilde{z} = 0, \quad \tilde{\phi}^i = -\alpha_i \left(\dfrac{\partial \tilde{z}}{\partial \nu} - z_{i,d} \right) \rho_{i,d} & \text{on} \quad \widetilde{\Sigma}, \\ \tilde{z}(\cdot, 0) = \tilde{z}^0, \quad \tilde{\phi}^i(\cdot, T) = 0 & \text{in} \quad \widetilde{\Omega}, \end{cases} \tag{1.121}$$

where $\tilde{g} \in L^2(\mathcal{O} \times (0,T))$. We will prove that there exists a control $\tilde{g} \in L^2(\mathcal{O} \times (0,T))$ such that the solution to (1.121) satisfy (1.114) in $\widetilde{\Omega}$.

As in Section 3.19, the null controllability of (1.121) is a consequence of an observability inequality for the adjoint system

$$\begin{cases} -\tilde{\psi}_t - \Delta\tilde{\psi} + a(x,t)\tilde{\psi} = 0 & \text{in} \quad \widetilde{Q}, \\ \tilde{\gamma}^i_t - \Delta\tilde{\gamma}^i + a(x,t)\tilde{\gamma}^i = \dfrac{1}{\mu_i}\tilde{\psi}1_{\mathcal{O}_i} & \text{in} \quad \widetilde{Q}, \\ \tilde{\psi} = \displaystyle\sum_{i=1}^{2} \alpha_i \frac{\partial \tilde{\gamma}^i}{\partial \nu}\rho_{i,d}, \quad \tilde{\gamma}^i = 0 & \text{on} \quad \widetilde{\Sigma}, \\ \tilde{\psi}(\cdot, T) = \tilde{\psi}^T, \quad \tilde{\gamma}^i(\cdot, 0) = 0 & \text{in} \quad \widetilde{\Omega}. \end{cases} \tag{1.122}$$

For any $\tilde{\psi}^T \in L^2(\widetilde{\Omega})$, this system has a unique solution in the class $(\tilde{\psi}, \tilde{\gamma}^1, \tilde{\gamma}^2) \in W(\widetilde{Q}) \times [H^{2,1}(Q)]^2$ and the desired estimate is given in the following result:

Proposition 7 *Suppose that $\Gamma_{i,d} \cap S = \emptyset$. There exist a constant $C > 0$ and a weight function $\bar{\rho} = \bar{\rho}(t)$, such that*

$$||\tilde{\psi}(0)||^2_{L^2(\widetilde{\Omega})} + \sum_{i=1}^{2} \iint_{\Sigma_{i,d}} \bar{\rho}^2 \left| \frac{\partial \tilde{\gamma}^i}{\partial \nu} \right|^2 d\sigma\, dt \leq \iint_{\mathcal{O}\times(0,T)} |\tilde{\psi}|^2\, dx\, dt \tag{1.123}$$

for all $\tilde{\psi}^T \in L^2(\widetilde{\Omega})$ where $(\tilde{\psi}, \tilde{\gamma}^1, \tilde{\gamma}^2)$ is the solution of (1.122) associated to $\tilde{\psi}^T$.

With the help of Proposition 7 and arguing as in the Section 1.2.1, we deduce the desired null controllability result for (1.121). Then, taking $g = \tilde{z}\big|_{S\times(0,T)}$, we have that $g \in H^{3/2,3/4}(S \times (0,T))$ and the corresponding solution (z, ϕ^1, ϕ^2) of the system (1.120) verifies $z(\cdot, T) = 0$.

To conclude the proof of the Theorem 3, it remains to prove Proposition 7.

Proof of Proposition 7.

Let us consider $\omega \subset \mathcal{O}$ a nonempty open set and a function $\eta^0 \in C^2(\overline{\widetilde{\Omega}})$ such that

$$\eta^0 > 0 \quad \text{in} \quad \widetilde{\Omega}, \quad \eta^0 = 0 \quad \text{on} \quad \widetilde{\Gamma}, \quad |\nabla \eta^0| > 0 \quad \text{in} \quad \overline{\widetilde{\Omega}\backslash\omega}.$$

The proof of the existence of η^0 can be found in [34].Then, we define weight functions $\alpha = \alpha(x,t)$ and $\varphi = \varphi(x,t)$ as in (1.44), with $(x,t) \in \widetilde{Q}$. Let us also write:

$$\tilde{I}_1(\pi) := \iint_{\widetilde{Q}} (s^{-1}\varphi^{-1}|\nabla\pi|^2 + s\lambda^2\varphi|\pi|^2)e^{2s\alpha}\,dx\,dt$$

and

$$\tilde{I}_2(\pi) := \iint_{\widetilde{Q}} \left[s^{-1}\varphi^{-1}(|\pi_t|^2 + |\Delta\pi|^2) + s\lambda^2\varphi|\nabla\pi|^2 + s^3\lambda^4\varphi^3|\psi|^2\right] e^{2s\alpha}\,dx\,dt.$$

Applying Carleman estimates in [40] to the equation satisfied by $\tilde{\psi}$, we obtain

$$I_1(\tilde{\psi}) \leq C\left\{ s^{-1/2}\sum_{i=1}^{2}\left|\left|\varphi^{-1/4}e^{s\alpha}\frac{\partial\tilde{\gamma}^i}{\partial\nu}\rho_{i,d}\right|\right|^2_{H^{1/2,1/4}(\Sigma_{i,d})} + s\iint_{\omega\times(0,T)} \varphi\, e^{2s\alpha}\,|\tilde{\psi}|^2\,dx\,dt\right\} \quad (1.124)$$

and by [34], we have for $\tilde{\gamma}^i$

$$I_2(\tilde{\gamma}^i) \leq C\left\{\frac{1}{\mu_i}\iint_{Q} e^{2s\alpha}|\tilde{\psi}|^2\,dx\,dt + s^3\lambda^4\iint_{\omega\times(0,T)} \varphi^3 e^{2s\alpha}|\tilde{\gamma}^i|^2\,dx\,dt\right\}. \quad (1.125)$$

Let us set a function $\tilde{\zeta} \in C^2(\widetilde{\Omega})$, with

$$\tilde{\zeta} = 1 \quad \text{in } \omega \quad \text{and} \quad \tilde{\zeta} = 0 \quad \text{in } \widetilde{\Omega}\setminus\mathcal{O}.$$

Then, $\hat{\gamma}^i := \tilde{\zeta}\,\tilde{\gamma}^i$ is the solution to system

$$\begin{cases} \hat{\gamma}^i_t - \Delta\hat{\gamma}^i + a(x,t)\hat{\gamma}^i = \nabla\tilde{\zeta}\cdot\nabla\tilde{\gamma}^i - \tilde{\gamma}^i\,\Delta\tilde{\zeta} & \text{in } \widetilde{Q}, \\ \hat{\gamma}^i = 0 & \text{on } \widetilde{\Sigma}, \\ \hat{\gamma}^i(\cdot,0) = 0 & \text{in } \widetilde{\Omega} \end{cases}$$

and, from energy inequality, we get

$$||\hat{\gamma}^i||_{C^0([0,T]\times\omega)} = 0,$$

then, since that $\tilde{\gamma}^i = \hat{\gamma}^i$ in ω, we see that last integral in (1.125) is zero.

Proceeding as in [39], we have

$$\begin{aligned} & s^{-1/2}\left|\left|\varphi^{-1/4}e^{s\alpha}\frac{\partial\tilde{\gamma}^i}{\partial\nu}\rho_{i,d}\right|\right|^2_{H^{1/2,1/4}(\Sigma_{i,d})} \\ & \leq C\left\{\frac{C_1}{\mu_i^2}\,s^{-1/2}\iint_{\widetilde{Q}}\varphi^{-1/2}e^{2s\alpha}\,|\tilde{\psi}|^2\,dx\,dt + s^{3/2}\iint_{\widetilde{Q}}\varphi^2 e^{2s\alpha}\,|\tilde{\gamma}^i|^2\,dx\,dt\right\}. \end{aligned} \quad (1.126)$$

Thus, arguing as in the proof of Proposition 11, for μ_i large enough we get a constant $C > 0$ such that

$$||\tilde{\psi}(0)||^2_{L^2(\widetilde{\Omega})} + \sum_{i=1}^{2}\iint_{\Sigma_{i,d}}\bar{\varphi}^{-1/4}e^{2s\hat{\alpha}}\left|\frac{\partial\tilde{\gamma}^i}{\partial\nu}\right|^2 d\sigma\,dt \leq C\iint_{\omega\times(0,T)}\bar{\varphi}\,e^{2s\bar{\alpha}}\,|\tilde{\psi}|^2\,dx\,dt, \quad (1.127)$$

where $\bar{\alpha}$, $\bar{\varphi}$ are defined in (1.52), and $\hat{\alpha}(t) = \min\{\bar{\alpha}(x,t) : \ x \in \overline{\overline{\Omega}}\}$ for all $t \in (0,T)$. So, the inequality (1.123) follows from (1.127) with $\bar{\rho}(t) = \bar{\varphi}^{-1/4} e^{-s\hat{\alpha}(t)}$ for all $t \in (0,T)$. This ends the proof. ■

1.3.2 The semilinear case

In this section we will prove Theorem 4 and Proposition 2. The proofs follow the same ideas in Section 1.2.2.

Semilinear optimality system

We will obtain a semilinear optimality system which characterize the Nash quasi-equilibrium for the functionals K_i. A pair $(u^1, u^2) \in \mathbf{H}$ is a Nash quasi-equilibrium if and only if satisfies (1.19), that is,

$$\alpha_i \iint_{\Sigma_{i,d}} \left(\frac{\partial p}{\partial \nu} - \zeta_{i,d} \right) \frac{\partial \hat{p}^i}{\partial \nu} \rho_{i,d} \, d\sigma \, dt + \mu_i \iint_{\mathcal{O}_i \times (0,T)} u^i \hat{u}^i \, dx \, dt = 0 \quad \forall \hat{u}^i \in \mathbf{H}_i, \tag{1.128}$$

where $\hat{p}^i$ is the solution of the system

$$\begin{cases} \hat{p}^i_t - \Delta \hat{p}^i + a(x,t)\hat{p}^i = F'(p)\hat{p}^i + \hat{u}^i 1_{\mathcal{O}_i} & \text{in} \quad Q, \\ \hat{p}^i = 0 & \text{on} \quad \Sigma, \\ \hat{p}^i(\cdot, 0) = 0 & \text{in} \quad \Omega. \end{cases} \tag{1.129}$$

Motivated by (1.128), we define the adjoint systems for (1.129) as follows:

$$\begin{cases} -\phi^i_t - \Delta \phi^i + a(x,t)\phi^i = F'(p)\phi^i & \text{in} \quad Q, \\ \phi^i = -\alpha_i \left(\dfrac{\partial p}{\partial \nu} - \zeta_{i,d} \right) \rho_{i,d} & \text{on} \quad \Sigma, \\ \phi^i(\cdot, T) = 0 & \text{in} \quad \Omega. \end{cases} \tag{1.130}$$

By (1.128), (1.129) and (1.130) we deduce that

$$\iint_{\mathcal{O}_i \times (0,T)} \left(\phi^i + \mu_i u^i \right) \hat{u}^i \, dx \, dt = 0 \quad \forall \hat{u} \in \mathbf{H}_i. \tag{1.131}$$

This furnishes the following characterization Nash quasi-equilibrium:

$$u^i = -\frac{1}{\mu_i} \phi^i \bigg|_{\mathcal{O}_i \times (0,T)}, \quad i = 1, 2. \tag{1.132}$$

Therefore, we get the following optimality system:

$$\begin{cases} p_t - \Delta p + a(x,t)p = F(p) - \displaystyle\sum_{i=1}^{2} \frac{1}{\mu_i}\phi^i_{\mathcal{O}_i} & \text{in} \quad Q, \\ -\phi^i_t - \Delta\phi^i + a(x,t)\phi^i = F'(p)\phi^i & \text{in} \quad Q, \\ p = g1_S, \quad \phi^i = -\alpha_i\left(\dfrac{\partial p}{\partial\nu} - \zeta_{i,d}\right)\rho_{i,d} & \text{on} \quad \Sigma, \\ p(\cdot,0) = p^0, \quad \phi^i(\cdot,T) = 0 & \text{in} \quad \Omega. \end{cases} \tag{1.133}$$

Sketch of the proof of Theorem 4

To prove Theorem 4, we use again the ideas in [3]. Considering the change of variables $z = p - \bar{p}$, where p is the solution to (1.2) and $\bar{p}$ is the solution to (1.10), we can rewrite (1.133) as follows:

$$\begin{cases} z_t - \Delta z + a(x,t)z = G(x,t;z)z - \displaystyle\sum_{i=1}^{2} \frac{1}{\mu_i}\phi^i_{\mathcal{O}_i} & \text{in} \quad Q, \\ -\phi^i_t - \Delta\phi^i + a(x,t)\phi^i = F'(z+\bar{p})\phi^i & \text{in} \quad Q, \\ z = g1_S, \quad \phi^i = -\alpha_i\left(\dfrac{\partial z}{\partial\nu} - z_{i,d}\right)\rho_{i,d} & \text{on} \quad \Sigma, \\ z(\cdot,0) = z^0, \quad \phi^i(\cdot,T) = 0 & \text{in} \quad \Omega, \end{cases} \tag{1.134}$$

where $z^0 = p^0 - \bar{p}^0$, $z_{i,d} = \zeta_{i,d} - \partial\bar{p}/\partial\nu$, and

$$G(x,t;z) := \int_0^1 (F'(\bar{p}) + \tau z)\, d\tau.$$

For each $z \in L^2(Q)$ and each $g \in H^{3/2,3/4}(S \times (0,T))$, let us consider the linear system

$$\begin{cases} w_t - \Delta w + a(x,t)w = G(x,t;z)w - \displaystyle\sum_{i=1}^{2} \frac{1}{\mu_i}\phi^i_{\mathcal{O}_i} & \text{in} \quad Q, \\ -\phi^i_t - \Delta\phi^i + a(x,t)\phi^i = F'(z+\bar{p})\phi^i & \text{in} \quad Q, \\ w = g1_S, \quad \phi^i = -\alpha_i\left(\dfrac{\partial w}{\partial\nu} - z_{i,d}\right)\rho_{i,d} & \text{on} \quad \Sigma, \\ w(\cdot,0) = z^0, \quad \phi^i(\cdot,T) = 0 & \text{in} \quad \Omega. \end{cases} \tag{1.135}$$

Since $F \in W^{1,\infty}(\mathbb{R})$, there exists $M > 0$ such that

$$|G(x,t;s)| + |F(s)| \leq M, \quad \forall (x,t,s) \in Q \times \mathbb{R}. \tag{1.136}$$

Let us fix z and let us denote by $(w_z, \phi^1_z, \phi^2_z)$ the associated solution to (1.135). By (1.117), there exists a constant $C > 0$ such that

$$||w_z||_{L^2(0,T;H^2(\Omega))} + ||w_{z,t}||_{L^2(Q)} \leq \left(1 + ||g||_{H^{3/2,3/4}(S\times(0,T))}\right). \tag{1.137}$$

Now, as in the linear case, we can extend the system (1.135). This way, we get the following system:

$$\begin{cases} \tilde{w}_{z,t} - \Delta\tilde{w}_z + \tilde{a}(x,t)\tilde{w}_z = \tilde{G}(x,t;z)\tilde{w}_z - \displaystyle\sum_{i=1}^{2}\frac{1}{\mu_i}\phi^i_{\mathcal{O}_i} + \tilde{g}1_{\mathcal{O}} & \text{in} \quad \widetilde{Q}, \\ -\tilde{\phi}^i_t - \Delta\tilde{\phi}^i + a(x,t)\tilde{\phi}^i = F'(z+\bar{p})\tilde{\phi}^i & \text{in} \quad \widetilde{Q}, \\ \tilde{w}_z = 0, \ \ \tilde{\phi}^i = -\alpha_i\left(\dfrac{\partial w_z}{\partial\nu} - z_{i,d}\right)\rho_{i,d} & \text{on} \quad \widetilde{\Sigma}, \\ \tilde{w}_z(\cdot,0) = \tilde{z}^0, \ \ \tilde{\phi}^i(\cdot,T) = 0 & \text{in} \quad \widetilde{\Omega}. \end{cases} \tag{1.138}$$

Let us consider the corresponding adjoint system:

$$\begin{cases} -\tilde{\psi}_{z,t} - \Delta\tilde{\psi}_z + \tilde{a}(x,t)\tilde{\psi}_z = \tilde{G}(x,t;z)\tilde{\psi}_z & \text{in} \quad \widetilde{Q}, \\ \tilde{\gamma}^i_{z,t} - \Delta\tilde{\gamma}^i_z + a(x,t)\tilde{\gamma}^i_z = F'(z+\bar{p})\tilde{\gamma}^i_z + \dfrac{1}{\mu_i}\tilde{\psi}_z 1_{\mathcal{O}_i} & \text{in} \quad \widetilde{Q}, \\ \tilde{\psi}_z = \displaystyle\sum_{i=1}^{2}\alpha_i\frac{\partial\gamma^i_z}{\partial\nu}\rho_{i,d}, \ \ \tilde{\gamma}^i_z = 0 & \text{on} \quad \widetilde{\Sigma}, \\ \tilde{\psi}_z(\cdot,T) = \tilde{\psi}^T_z, \ \ \tilde{\gamma}^i_z(\cdot,0) = 0 & \text{in} \quad \widetilde{\Omega}. \end{cases} \tag{1.139}$$

Then, by (1.138) and (1.139), we have

$$(\tilde{w}_z(T), \tilde{\psi}^T_z) - (\tilde{z}^0, \tilde{\psi}_z(0)) - \sum_{i=1}^{2}\alpha_i \iint_{\Sigma_{i,d}} z_{i,d}\frac{\partial\gamma^i_z}{\partial\nu}\rho_{i,d}\,d\sigma\,dt = \iint_{\mathcal{O}\times(0,T)} \tilde{g}\,\tilde{\psi}_z\,dx\,dt,$$

and the null controllability property holds if and only if, for each z^0, there exists a control $\tilde{g} \in L^2(\mathcal{O}\times(0,T))$ such that

$$-(\tilde{z}^0, \tilde{\psi}_z(0)) - \sum_{i=1}^{2}\alpha_i \iint_{\Sigma_{i,d}} z_{i,d}\frac{\partial\gamma^i_z}{\partial\nu}\rho_{i,d}\,d\sigma\,dt = \iint_{\mathcal{O}\times(0,T)} \tilde{\psi}_z f\,dx\,dt \quad \forall\tilde{\psi}^T_z \in L^2(\widetilde{\Omega}).$$

The remainder of the proof is very similar to the final part of the proof of Theorem 2. For brevity, it will be omitted.

Equilibria and quasi-equilibria with distributed followers

In this section, we will prove Proposition 2, we will follow the ideas of the Section 1.2.2. Let us assume $F \in W^{2,\infty}(\mathbb{R})$, let g be the leader control and (u^1, u^2) the associated Nash quasi-equilibrium pair. For all $w^1 \in \mathbf{H}_1$ and $s \in \mathbb{R}$, we denote by p^s the solution to

$$\begin{cases} p^s_t - \Delta p^s + a(x,t)p^s = F(p^s) + (u^1 + sw^1)1_{\mathcal{O}_1} + u^2 1_{\mathcal{O}_2} & \text{in} \quad Q, \\ p^s = g1_S & \text{on} \quad \Sigma, \\ p^s(\cdot,0) = p^0 & \text{in} \quad \Omega, \end{cases} \tag{1.140}$$

and we set $p := p^s\big|_{s=0}$. For all $w^2 \in \mathbf{H}_1$ we have

$$\begin{aligned}
&D_1K_1(g; u^1 + sw^1, u^2)\cdot w^2 - D_1K_1(g; u^1, u^2)\cdot w^2 \\
&= \alpha_1 \iint_{\Sigma_{1,d}} \left(\frac{\partial p^s}{\partial \nu} - \zeta_{1,d}\right)\frac{\partial q^s}{\partial \nu}\rho_{i,d}\, d\sigma\, dt - \alpha_1 \iint_{\Sigma_{1,d}} \left(\frac{\partial p}{\partial \nu} - \zeta_{1,d}\right)\frac{\partial q}{\partial \nu}\rho_{i,d}\, d\sigma\, dt \\
&\quad + \alpha_1 \iint_{\mathcal{O}_1\times(0,T)} w^1 w^2\, dx\, dt,
\end{aligned} \tag{1.141}$$

where q^s, with $q := q^s\big|_{s=0}$, is the solution to the system

$$\begin{cases} q^s_t - \Delta q^s + a(x,t)q^s = F'(p^s)q^s + w^2 1_{\mathcal{O}_1} & \text{in} \quad Q, \\ q^s = 0 & \text{on} \quad \Sigma, \\ q^s(\cdot,0) = 0 & \text{in} \quad \Omega. \end{cases} \tag{1.142}$$

Let us introduce the associated adjoint system:

$$\begin{cases} -\phi^s_t - \Delta \phi^s + a(x,t)\phi^s = F'(p^s)\phi^s & \text{in} \quad Q, \\ \phi^s = -\alpha_1 \left(\dfrac{\partial p^s}{\partial \nu} - \zeta_{1,d}\right)\rho_{i,d} & \text{on} \quad \Sigma, \\ \phi^s(\cdot,T) = 0 & \text{in} \quad \Omega, \end{cases} \tag{1.143}$$

where we also set $\phi := \phi\big|_{s=0}$. Then, by (1.141), (1.142) and (1.143), we deduce that

$$\alpha_1 \iint_{\Sigma_{1,d}} \left(\frac{\partial p^s}{\partial \nu} - \zeta_{1,d}\right)\frac{\partial q^s}{\partial \nu}\rho_{i,d}\, d\sigma\, dt = \iint_{\mathcal{O}_1\times(0,T)} \phi^s w^2\, dx\, dt. \tag{1.144}$$

Hence substituting (1.144) in (1.141), we have

$$\begin{aligned}
&D_1K_1(g; u^1 + sw^1, u^2)\cdot w^2 - D_1K_1(g; u^1, u^2)\cdot w^2 \\
&= \iint_{\mathcal{O}_1\times(0,T)} (\phi^s - \phi)w^2\, dx\, dt + \mu_1 s \iint_{\mathcal{O}_1\times(0,T)} w^1 w^2\, dx\, dt.
\end{aligned} \tag{1.145}$$

Note that

$$-(\phi^s - \phi)_t - \Delta(\phi^s - \phi) + a(x,t)(\phi^s - \phi) = [F'(p^s) - F'(p)]\phi^s + F'(p)(\phi^s - \phi)$$

and

$$(\phi^s - \phi)\big|_{\Sigma} = -\alpha_1 \left(\frac{\partial p^s}{\partial \nu} - \frac{\partial p}{\partial \nu}\right)\rho_{i,d}.$$

Therefore, the following limits exist:

$$h = \lim_{s\to 0}\frac{1}{s}(p^s - p), \quad \eta = \lim_{s\to 0}\frac{1}{s}(\phi^s - \phi)$$

and the couple (h,η) solves the system

$$\begin{cases} h_t - \Delta h + a(x,t)h = F'(p)h + w^1 1_{\mathcal{O}_1} & \text{in} \quad Q, \\ -\eta_t - \Delta\eta + a(x,t)\eta = F''(p)h\phi + F'(p)\eta & \text{in} \quad Q, \\ h = 0, \quad \eta = -\alpha_1 \dfrac{\partial h}{\partial \nu}\rho_{i,d} & \text{on} \quad \Sigma, \\ h(\cdot,0) = 0, \quad \eta(\cdot,T) = 0 & \text{in} \quad \Omega. \end{cases} \tag{1.146}$$

Thus, by (1.145) and (1.146), we see that

$$D_1^2 K_1(g;u^1,u^2)\cdot(w^1,w^2) = \iint_{\mathcal{O}_1\times(0,T)} \eta\, w^2\,dx\,dt + \mu_1 \iint_{\mathcal{O}_1\times(0,T)} w^1 w^2\,dx\,dt \quad \forall w^2 \in \mathbf{H}_1$$

and, in particular, for all $w^1 \in \mathbf{H}_1$, we have

$$D_1^2 K_1(g;u^1,u^2)\cdot(w^1,w^1) = \iint_{\mathcal{O}_1\times(0,T)} \eta\, w^1\,dx\,dt + \mu_1 \iint_{\mathcal{O}_1\times(0,T)} |w^1|^2\,dx\,dt. \tag{1.147}$$

By multiplying $(1.146)_1$ by η and integrating in Q, using integration by parts, we get

$$\iint_{\mathcal{O}_1\times(0,T)} \eta w^1 dxdt = \iint_Q F''(p)h^2\phi dxdt + \alpha_1 \iint_{\Sigma_{1,d}} \left|\frac{\partial h}{\partial \nu}\right|^2 dxdt. \tag{1.148}$$

Arguing as in the proof of (1.108), we deduce that, if $n \le 6$, there exist $C > 0$ such that

$$\left|\iint_{\mathcal{O}_1\times(0,T)} \eta\, w^1\,dx\,dt\right| \le C\left(1 + ||g||^2_{H^{\frac{3}{2},\frac{3}{4}}(S\times(0,T))}\right) ||w^1||^2_{\mathbf{H}_1}. \tag{1.149}$$

This way, from (1.147) and (1.149), one also has

$$D_1^2 K_1(g;u^1,u^2)\cdot(w^1,w^1) \ge \left[\mu_1 - C\left(1 + ||g||_{H^{\frac{3}{2},\frac{3}{4}}(S\times(0,T))}\right)\right] ||w^1||^2_{\mathbf{H}_1} \quad \forall w^1 \in \mathbf{H}_1.$$

Analogously, we get

$$D_2^2 K_2(g;u^1,u^2)\cdot(w^2,w^2) \ge \left[\mu_2 - C\left(1 + ||g||_{H^{\frac{3}{2},\frac{3}{4}}(S\times(0,T))}\right)\right] ||w^2||^2_{\mathbf{H}_2} \quad \forall w^2 \in \mathbf{H}_2.$$

In both inequalities, the constant $C > 0$ is independent of μ_i. Then, taking μ_i large enough, we get a constant $C > 0$ such that

$$D_i^2 K_i(g;u^1,u^2)\cdot(w^i,w^i) \ge C\,||w^i||^2_{\mathbf{H}_i} \quad \forall w^i \in \mathbf{H}_i.$$

We deduce at once that (u^1,u^2) is a Nash equilibrium for the functionals K_i and this ends the proof.

1.4 Final comments

1.4.1 The case where all controls are on the boundary

The third situation that we can analyze corresponds to take all the controls (the leader and the followers) on the boundary. Thus, we can consider the linear system

$$\begin{cases} z_t - \Delta z + a(x,t)z = 0 & \text{in} \quad Q, \\ z = v1_S + v^1 1_{S_1} + v^2 1_{S_2} & \text{on} \quad \Sigma, \\ z(\cdot,0) = z^0 & \text{in} \quad \Omega, \end{cases} \tag{1.150}$$

where $S,\ S_1,\ S_2 \in \Gamma$ are non-empty closed and disjoints subsets. We can introduce cost functionals as in (1.3). Then, arguing as in Sections 1.2.1 and 1.3.1, we can prove the existence and uniqueness of a Nash equilibrium for each leader provided μ_1 ad μ_2 large enough. Furthermore, we get the optimality system

$$\begin{cases} z_t - \Delta z + a(x,t)z = 0 & \text{in} \quad Q, \\ -\phi^i_t - \Delta \phi^i + a(x,t)\phi^i = \alpha_i (z - z_{i,d}) 1_{\mathcal{O}_{i,d}} & \text{in} \quad Q, \\ z = v1_S + \displaystyle\sum_{i=1}^{2} \frac{1}{\mu_i} \frac{\partial \phi^i}{\partial \nu} \rho_i, \quad \phi^i = 0 & \text{on} \quad \Sigma, \\ z(\cdot,0) = z^0, \quad \phi^i(\cdot,T) = 0 & \text{in} \quad \Omega, \end{cases} \tag{1.151}$$

whose adjoint is given as follows:

$$\begin{cases} -\psi_t - \Delta \psi + a(x,t)\psi = \displaystyle\sum_{i=1}^{2} \alpha_i \gamma^i 1_{\mathcal{O}_{i,d}} & \text{in} \quad Q, \\ \gamma^i_t - \Delta \gamma^i + a(x,t)\gamma^i = 0 & \text{in} \quad Q, \\ \psi = 0, \quad \gamma^i = \dfrac{1}{\mu_i} \dfrac{\partial \psi}{\partial \nu} \rho_i & \text{on} \quad \Sigma, \\ \psi(\cdot,T) = \psi^T, \quad \gamma^i(\cdot,0) = 0 & \text{in} \quad \Omega. \end{cases} \tag{1.152}$$

Considering cost functionals as in (1.8), we obtain a similar system. However, in both cases, we find a nontrivial difficulty to obtain the observability inequality (this difficulty appears when we try to combine the Carleman inequalities for ψ and $h = \alpha_1\gamma^1 + \alpha_2\gamma^2$). Accordingly, the Stackelberg–Nash null controllability of (1.151) is an open question.

1.4.2 The assumption on the μ_i

In the proof of Propositions 3 and 6, we assumed that the μ_i are large enough. This assumption allowed to conclude the results thanks to Lax-Milgram's Theorem. However, we can prove the existence and uniqueness of a Nash equilibrium without using Lax-Milgram's Theorem.

For instance, it can be proved that a sequence $\{\mu^n\}$ exists with

$$\mu^1 \geq \mu^2 \geq \cdots \geq \mu^n \geq \cdots, \quad \mu^n > 0 \quad \text{for all } n, \quad \mu^n \to 0$$

such that, for $\mu_1 = \mu_2 = \mu$ and $\mu \neq \mu^n$ for all n, we can assign to any leader control exactly one Nash equilibrium pair.

An interesting question remains: can we prove the null controllability of the associated system for these μ_i? This will be analyzed in a forthcoming paper.

1.4.3 The hypotheses on the $\mathcal{O}_{i,d}$ and $\Gamma_{i,d}$

In this article, we have used (3.8) and (3.9) in Theorem 1 only in the proof of Proposition 11 to allow for the connection between the Carleman inequalities for ψ and $h = \alpha_1\gamma_1 + \alpha_2\gamma_2$, in the first case, and the Carleman inequalities for ψ and γ^i in the second one. A question that remains unknown is wether (1.1)is null-controllable when we assume that $\mathcal{O}_{1,d} \neq \mathcal{O}_{2,d}$ and $\mathcal{O}_{1,d} \cap \mathcal{O} = \mathcal{O}_{2,d} \cap \mathcal{O}$.

On the other hand, the hypothesis $\Gamma_{i,d} \cap S = \emptyset$, $i = 1, 2$, was used only in the proof of Proposition 7. Therefore, another interesting open question concerns the case where $\Gamma_{i,d} \cap S \neq \emptyset$, $i = 1$ or $i = 2$.

1.4.4 Hierarchical control of the wave equation

It makes sense to consider hierarchic controllability problems for the wave equation. For instance, we can consider the system

$$\begin{cases} z_{tt} - \Delta z + a(x,t)z = f1_{\mathcal{O}} + v^1 1_{\mathcal{O}_1} + v^2 1_{\mathcal{O}_2} & \text{in} \quad Q, \\ z = 0 & \text{on} \quad \Sigma, \\ z(\cdot,0) = z^0, \ z_t(\cdot,0) = z^1 & \text{in} \quad \Omega. \end{cases} \tag{1.153}$$

where the $\mathcal{O}$, $\mathcal{O}_1$, $\mathcal{O}_2 \subset \Omega$ are nonempty disjoint open sets. Let us introduce the secondary cost functionals

$$P_i(f; v^1, v^2) := \frac{\alpha_i}{2} \iint_{\mathcal{O}_{i,d}\times(0,T)} |z - z_{i,d}|^2 \, dx \, dt + \frac{\mu_i}{2} \iint_{\mathcal{O}_i\times(0,T)} |v|^2 \, dx \, dt, \quad i = 1, 2,$$

and the main functional

$$P(f) := \frac{1}{2} \iint_{\mathcal{O}\times(0,T)} |f|^2 \, dx \, dt,$$

where the $\mathcal{O}_{i,d} \subset \Omega$ are nonempty open sets and the $z_{i,d} \in L^2(\mathcal{O}_{i,d})$. In this context, an appropriate problem is the following: for any prescribed (z^1, z^2) and (w^1, w^2) in appropriate spaces, for each leader f, we look for a Nash equilibrium (v^1, v^2) associated to the functionals P_i; then, we try to find a control $f \in L^2(\mathcal{O} \times (0,T))$ such that

$$P(f) = \min_{\hat{f}} P(f),$$

subject to

$$z(\cdot, T) = w^1, \quad z_t(\cdot, T) = w^2 \quad \text{in} \quad \Omega.$$

Arguing as in Sections 1.2.1 and 1.3.1, we get the following optimality system:

$$\begin{cases} z_{tt} - \Delta z + a(x,t)z = f1_{\mathcal{O}} - \displaystyle\sum_{i=1}^{2} \frac{1}{\mu_i}\phi^i 1_{\mathcal{O}_i} & \text{in} \quad Q, \\ \phi^i_{tt} - \Delta \phi^i + a(x,t)\phi^i = \alpha_i (z - z_{i,d})1_{\mathcal{O}_{i,d}} & \text{in} \quad Q, \\ z = 0,\ \phi^i = 0 & \text{on} \quad \Sigma, \\ z(\cdot, 0) = z^0,\ z_t(\cdot,0) = z^1,\ \phi^i(\cdot,T) = 0,\ \phi^i_t(\cdot,T) = 0 & \text{in} \quad \Omega. \end{cases} \tag{1.154}$$

The associated adjoint system is

$$\begin{cases} \psi_{tt} - \Delta \psi + a(x,t)\psi = \displaystyle\sum_{i=1}^{2} \alpha_i \gamma^i 1_{\mathcal{O}_{i,d}} & \text{in} \quad Q, \\ \gamma^i_{tt} - \Delta \gamma^i + a(x,t)\gamma^i = -\dfrac{1}{\mu_i}\psi 1_{\mathcal{O}_i} & \text{in} \quad Q, \\ z = 0,\ \phi^i = 0 & \text{on} \quad \Sigma, \\ \psi(\cdot,T) = \psi^T,\ \psi_t(\cdot,T) = \psi^T_1,\ \gamma^i(\cdot,0) = 0,\ \gamma^i_t(\cdot,0) = 0 & \text{in} \quad \Omega. \end{cases} \tag{1.155}$$

Thus, the tasks are in this case to prove an existence and uniqueness for (1.154) and an observability inequality for (1.155). This problem will be also analyzed in a forthcoming paper.

Capítulo 2

Hierarchic control for the wave equation

Hierarchic control for the wave equation

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Abstract. We present some results on the exact controllability of the wave PDE in the context of hierarchic control through Stackelberg-Nash strategy. We consider one leader and two followers. To each leader we associate a Nash equilibrium corresponding to a bi-objective optimal control problem; then we look for a leader that solves the exact controllability problem. We consider linear and semilinear equations; for the latter, we use a fixed point method.

2.1 Statement of the problem

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with boundary Γ of class C^2 and assume that $T > 0$. Let us consider small disjoints open nonempty sets $\mathcal{O}$, $\mathcal{O}_1$, $\mathcal{O}_2 \subset \Omega$. We will use the notation $Q = \Omega \times (0, T)$, which lateral boundary $\Sigma = \Gamma \times (0, T)$; by $\nu(x)$ we denote the outward unit normal to Ω at the point $x \in \Gamma$.

Let us consider the following system

$$\begin{cases} y_{tt} - \Delta y + a(x,t)y = F(y) + f1_{\mathcal{O}} + v^1 1_{\mathcal{O}_1} + v^2 1_{\mathcal{O}_2} & \text{in} \quad Q, \\ y = 0 & \text{on} \quad \Sigma, \\ y(\cdot, 0) = y^0, \ y_t(\cdot, 0) = y^1 & \text{in} \quad \Omega, \end{cases} \tag{2.1}$$

where $a \in L^\infty(Q)$, $f \in L^2(\mathcal{O} \times (0,T))$, $v^i \in L^2(\mathcal{O}_i \times (0,T))$, $F : \mathbb{R} \mapsto \mathbb{R}$ is a locally Lipschitz-continuous function, $(y^0, y^1) \in H_0^1(\Omega) \times L^2(\Omega)$ and the notation 1_A indicates the characteristic function of A.

In [50], Lions analyzed the approximate controllability for a hyperbolic PDE and introduced, in the context of controllability, the concept of hierarchic control, where he considered a main control independent, called the leader, and a control dependent on leader, called the follower. In this paper, we will analyze a related problem for (3.1), where f is the leader and v^1 and v^2 are the followers. We will apply the Stackelberg-Nash rule, which combines the strategies of cooperative optimization of Stackelberg and the non-cooperative strategy of Nash.

In [16] and [37], the hierarchic control of a parabolic PDE and the Stokes systems have been analyzed and used to solve a approximate controllability problem. In [3] and [6], the hierarchic control of a parabolic PDE has been used to provide exact controllability (to the trajectories); the problem was analyzed in [3] with distributed leader and follower controls, while [6] deals with distributed and boundary controls.

The main goal this article is to analyze the hierarchic control of (3.1) and, in particular, to prove that the Stackelberg-Nash strategy allows to solve the exact controllability problem.

Let us define the following main cost functional

$$\frac{1}{2}\iint_{\mathcal{O}\times(0,T)} |f|^2\,dx\,dt,$$

and the secondary cost functionals

$$J_i(f,v^1,v^2) := \frac{\alpha_i}{2}\iint_{\mathcal{O}_{i,d}\times(0,T)} |y-y_{i,d}|^2\,dx\,dt + \frac{\mu_i}{2}\iint_{\mathcal{O}_i\times(0,T)} |v^i|^2\,dx\,dt, \tag{2.2}$$

where the $\mathcal{O}_{i,d}\subset\Omega$ are nonempty open sets, $y_{i,d}\in L^2(\mathcal{O}_{i,d}\times(0,T))$ are given functions and α_i and μ_i are positive constants.

Now, let us describe the Stackelberg-Nash strategy. Thus, for each choice of the leader f, we try to find a Nash equilibrium pair for cost functionals J_i; that is, we look for controls $v^1\in L^2(\mathcal{O}_1\times(0,T))$ and $v^2\in L^2(\mathcal{O}_1\times(0,T))$ depending on f, satisfying simultaneously

$$J_1(f;v^1,v^2) = \min_{\hat{v}^1} J_1(f;\hat{v}^1,v^2), \quad J_2(f;v^1,v^2) = \min_{\hat{v}^2} J_2(f;v^1,\hat{v}^2). \tag{2.3}$$

If the functionals J_i are convex, (2.3) is equivalent to

$$J_i'(f;v^1,v^2)\cdot\hat{v}^i = 0,\ \forall\,\hat{v}^i\in L^2(\mathcal{O}_i\times(0,T)),\ i=1,2. \tag{2.4}$$

Note that this is the case if the PDE in (3.1) is linear, but this is not true in general.

Given $(\bar{y}^0,\bar{y}^1)$ in $H_0^1(\Omega)\times L^2(\Omega)$, a trajectory to (3.1) is a solution to the system

$$\begin{cases} \bar{y}_{tt}-\Delta\bar{y}+a(x,t)\bar{y}=F(\bar{y}) & \text{in} \quad Q,\\ y=0 & \text{on} \quad \Sigma,\\ y(\cdot,0)=y^0,\ y_t(\cdot,0)=y^1 & \text{in} \quad \Omega. \end{cases} \tag{2.5}$$

After proving that, for each leader f, there exists at least one Nash equilibrium $(v^1,v^2) = (v^1(f),v^2(f))$ and after fixing a trajectory $\bar{y}$, we will look for a leader $f\in L^2(\mathcal{O}\times(0,T))$, such that,

$$J(f) := \min_{\hat{f}} J(\hat{f}), \tag{2.6}$$

subject to the exact controllability condition

$$y(\cdot,T)=\bar{y}(\cdot,T),\quad y_t(\cdot,T)=\bar{y}_t(\cdot,T)\text{ in }\Omega. \tag{2.7}$$

As an application we can consider in the exact controllability of a flexible structure that we wish to make attain a desired state at time T in such a way that the state does not separate too much from $y_{i,d}$ in $\mathcal{O}_{i,d}$ along $(0,T)$.

2.1.1 Main results

Let $x_0\in\mathbb{R}^n\setminus\overline{\Omega}$ be given and let us consider the following set

$$\Gamma_+ := \{(x-x_0)\cdot\nu(x)>0\}$$

and the function $d:\overline{\Omega}\mapsto\mathbb{R}$, with $d(x)=|x-x_0|^2$ for all $x\in\overline{\Omega}$. We will impose of the following assumption: there exists $\delta>0$ such that

$$\mathcal{O}\supset\mathcal{O}_\delta(\Gamma_+)\cap\Omega, \tag{2.8}$$

where

$$\mathcal{O}_\delta(\Gamma_+)=\{x\in\mathbb{R}^n;|x-x'|<\delta,\ x'\in\Gamma_+\}.$$

Due to the finite propagation speed of the solutions to the wave PDE, the time $T>0$ has to be large enough. Therefore, we will also assume in the sequel that the estimate $T>2R_1$, where $R_1:=\max\{\sqrt{d(x)}:\ x\in\overline{\Omega}\}$.

In the linear case, for $F\equiv 0$, we have the following result on the exact controllability of (3.1):

Theorem 5 *Suppose that $F\equiv 0$ and the constants $\mu_i>0$ $(i=1,2)$ are sufficiently large. Then, for any $(y^0,y^1)\in H_0^1\times L^2(\Omega)$, there exist a control $f\in L^2(\mathcal{O}\times(0,T))$ and an associated Nash equilibrium pair $(v^1,v^2)=(v^1(f),v^2(f))$ such that the corresponding solution to* (3.1) *satisfies* (2.7).

In the linear case, the functionals J_i are convex whence, as we had already observed, a pair (v^1,v^2) is a Nash equilibrium if and only if satisfies (2.4). However, in the semilinear case, with F being a locally Lipschitz continuous function, we do not have the convexity of the J_i. This motivates the following weaker definition:

Definition 3 *Let f be given. The pair (v^1,v^2) is a Nash quasi-equilibrium for the functionals J_i associated to f if* (2.4) *is satisfied.*

The following results hold in the semilinear case:

Theorem 6 *Assume that (2.8) holds, $F \in W^{1,\infty}(\mathbb{R})$ and the $\mu_i > 0$ $(i = 1, 2)$ are sufficiently large. Then, for any $(y^0, y^1) \in H_0^1(\Omega) \times L^2(\Omega)$, there exist a control $f \in L^2(\mathcal{O} \times (0, T))$ and an associated Nash quasi-equilibrium pair $(v^1, v^2) = (v^1(f), v^2(f))$ such that the corresponding solutions to* (3.1) *satisfies* (2.7).

In the semilinear case, we can prove that, under certain conditions, the definitions of Nash equilibrium and Nash quasi-equilibrium are equivalent. More precisely, we have the following result:

Proposition 8 *Assume that*

$$F \in W^{2,\infty}(\mathbb{R}) \quad \textit{and} \quad y_{i,d} \in C^0([0, T]; H_0^1(\mathcal{O}_{i,d})) \cap C^1([0, T]; L^2(\mathcal{O}_{i,d})).$$

Suppose that $(y^0, y^1) \in H_0^1(\Omega) \times L^2(\Omega)$ and $n \leq 8$. Then, there exists $C > 0$ such that, if $f \in L^2(\mathcal{O} \times (0, T))$ and the μ_i satisfy

$$\mu_i \geq C\left(1 + \|f\|_{L^2(\mathcal{O}\times(0,T))}\right),$$

the conditions (2.3) and (2.4) are equivalent.

The contents of this article is organized as follows. In Section 2.2, we analyze the linear system, we prove the existence and uniqueness of a Nash equilibrium pair for each leader, we obtain an associated optimality system and we prove Theorem 7 using a standard technique based on a observability estimate, which we get using Carleman inequalities. In Section 2.3, we analyze the semilinear case, we deduce another optimality system, we prove the Theorem 8 using fixed point techniques and we prove Proposition 8 by adapting some arguments from [3] and [6]. Finally, in Section 2.4, we present some additional comments.

2.2 The linear case

In this section we consider the linear case, ($F = 0$ in (3.1)). The goal is to prove of the Theorem 7.

Since the system is linear, we have that the exact controllability is equivalent to null controllability. In fact, we can introduce the change of variable $z = y - \bar{y}$, where $\bar{y}$ is the solution to system (3.1); then, we see that z is the solution to

$$\begin{cases} z_{tt} - \Delta z + a(x,t)z = f1_{\mathcal{O}} + v^1 1_{\mathcal{O}_1} + v^2 1_{\mathcal{O}_2} & \text{in} \quad Q, \\ z = 0 & \text{on} \quad \Sigma, \\ z(\cdot, 0) = z^0, \;\; z_t(\cdot, 0) = z^1 & \text{in} \quad \Omega, \end{cases} \tag{2.9}$$

where $z^0 = y^0 - \bar{y}^0 \in H_0^1(\Omega)$ and $z^1 = y^1 - \bar{y}^1 \in L^2(\Omega)$. We will apply the Stackelberg-Nash strategy to solve the null controllability to system (3.12), that is, we will look for a leader f and an associated Nash equilibrium $(v^1(f), v^2(f))$ such that the solution to (3.12) satisfies

$$(z(\cdot, T), z_t(\cdot, T)) = (0, 0). \tag{2.10}$$

With this change of variables, the cost functionals become

$$J_i(f, v^1, v^2) := \frac{\alpha_i}{2} \iint_{\mathcal{O}_{i,d}\times(0,T)} |z - z_{i,d}|^2 dxdt + \frac{\mu_i}{2} \iint_{\mathcal{O}_i\times(0,T)} |v^i|^2 \, dx \, dt, \tag{2.11}$$

where $z_{i,d} = y_{i,d} - \bar{y}$. We will solve the null controllability problem for (3.12) with f and $(v^1(f), v^2(f))$ in the following three subsections.

2.2.1 The existence and uniqueness of a Nash equilibrium

Let us prove that for each f, a unique Nash pair $(v^1(f), v^2(f))$ exists. The proof is implied by Lax-Milgram's Theorem, following some arguments introduced by [?]. Let $f \in L^2(\mathcal{O} \times (0, T))$ be fixed. Let us define the spaces

$$\mathcal{H}_i = L^2(\mathcal{O}_i \times (0, T)) \;\text{ and }\; \mathcal{H} = \mathcal{H}_1 \times \mathcal{H}_2.$$

The pair (v^1, v^2) is a Nash equilibrium for the functionals J_i if and only if

$$\alpha_i \iint_{\mathcal{O}_{i,d}\times(0,T)} (z - z_{i,d}) \hat{w}^i \, dx \, dt + \mu_i \iint_{\mathcal{O}_i\times(0,T)} v^i \hat{v}^i \, dx \, dt = 0 \;\; \forall \hat{v}_i \in \mathcal{H}_i. \tag{2.12}$$

where $\hat{w}^i$ is the solution to the system

$$\begin{cases} \hat{w}^i_{tt} - \Delta \hat{w}^i + a(x,t)\hat{w}^i = \hat{v}^i 1_{\mathcal{O}_i} & \text{in} \quad Q, \\ \hat{w}^i = 0 & \text{on} \quad \Sigma, \\ \hat{w}^i(\cdot, 0) = 0, \;\; \hat{w}^i_t(\cdot, 0) = 0 & \text{in} \quad \Omega. \end{cases} \tag{2.13}$$

Now, we introduce the operators $L_i : \mathcal{H}_i \mapsto L^2(Q)$, with $L_i(\hat{v}^i) = \hat{w}^i$, where $\hat{w}^i$ is the solution to (3.29). Let u be the solution of to

$$\begin{cases} u_{tt} - \Delta u + a(x,t)u = f 1_{\mathcal{O}} & \text{in} \quad Q, \\ u = 0 & \text{on} \quad \Sigma, \\ u(\cdot, 0) = z^0, \;\; u_t(\cdot, 0) = z^1 & \text{in} \quad \Omega. \end{cases} \tag{2.14}$$

Then, we can write the solution to system (3.12) in the form $z = L_1(v^1) + L_2(v^2) + u$. Consequently, (v^1, v^2) is a Nash equilibrium for J_i if and only if

$$\alpha_i \iint_{\mathcal{O}_{i,d}\times(0,T)} (L_1 v^1 + L_2 v^2 - (z_{i,d} - u))L_i v^i \, dx\, dt + \mu_i \iint_{\mathcal{O}_i\times(0,T)} v^i \hat{v}^i \, dx\, dt = 0 \quad \forall \hat{v}^i \in \mathcal{H}_i,$$

that is to say

$$\iint_{\mathcal{O}_i\times(0,T)} [\alpha_i L_i^*\big((L_1 v^1 + L_2 v^2 - (z_{i,d} - u))1_{\mathcal{O}_{i,d}}\big) + \mu_i v^i]\hat{v}^i \, dx\, dt = 0 \quad \forall \hat{v}^i \in \mathcal{H}_i,$$

where $L_i^* \in \mathcal{L}(L^2(Q), \mathcal{H}_i)$ is the adjoint of L_i. Obviously, this is equivalent to

$$\alpha_i L_i^*((L_1 v^1 + L_2 v^2)1_{\mathcal{O}_{i,d}}) + \mu_i v^i = \alpha_i L_i^*((z_{i,d} - u)1_{\mathcal{O}_{i,d}}), \quad i = 1, 2. \tag{2.15}$$

Let us introduce the operator $\mathbb{L} : \mathcal{H} \mapsto \mathcal{H}$, with

$$\mathbb{L}(v^1, v^2) := (\alpha_1 L_1^*((L_1 v^1 + L_2 v^2)1_{\mathcal{O}_{1,d}}) + \mu_1 v^1, \alpha_2 L_2^*((L_1 v^1 + L_2 v^2)1_{\mathcal{O}_{2,d}}) + \mu_2 v^2).$$

We have that $\mathbb{L} \in \mathcal{L}(\mathcal{H})$. Now, let us assume that the $\mu_i > 0$ satisfy

$$\delta_i = \mu_i - \left(\frac{3\alpha_{3-i} + \alpha_i}{2}\right) ||L_{3-i}||^2 > 0, \quad i = 1, 2.$$

Then

$$(\mathbb{L}(v^1, v^2), (v^1, v^2))_{\mathcal{H}} \geq \delta ||((v^1, v^2))||^2_{\mathcal{H}} \quad \forall (v^1, v^2) \in \mathcal{H}, \tag{2.16}$$

where $\delta = \min\{\delta_1, \delta_2\}$.

Hence, from Lax-Milgram's Theorem, for each $\Phi \in \mathcal{H}'$ there exists exactly one pair (v^1, v^2) satisfying

$$(\mathbb{L}(v^1, v^2), (\hat{v}^1, \hat{v}^2))_{\mathcal{H}} = \langle \Phi, (\hat{v}^1, \hat{v}^2)\rangle_{\mathcal{H}'\times\mathcal{H}} \quad \forall (\hat{v}^1, \hat{v}^2) \in \mathcal{H}. \tag{2.17}$$

In particular, if we take $\Phi \in \mathcal{H}'$ with

$$\langle \Phi, (\hat{v}^1, \hat{v}^2)\rangle_{\mathcal{H}'\times\mathcal{H}} := ((\alpha_1 L_1^*((z_{1,d} - u)1_{\mathcal{O}_{1,d}}), \alpha_2 L_2^*((z_{2,d} - u)1_{\mathcal{O}_{2,d}})), (\hat{v}^1, \hat{v}^2))_{\mathcal{H}},$$

we deduce the existence and uniqueness of a solution to (**??**).

Let us summarize what we have been able to prove:

Proposition 9 *Suppose that the $\mu_i > 0$ satisfy*

$$\mu_i - \left(\frac{3\alpha_{3-i} + \alpha_i}{2}\right) ||L_{3-i}||^2 > 0, \quad i = 1, 2.$$

Then, for each $f \in L^2(\mathcal{O} \times (0,T))$, there exists a unique Nash equilibrium pair $(v^1(f), v^2(f))$ for the J_i.

Remark 2 As a consequence of the proof of Proposition 9 we obtain a constant $C > 0$, such that

$$||z||_{C^0([0,T];H^1(\Omega))} + ||z_t||_{C^0([0,T];L^2(\Omega))} \leq C(1 + ||f||_{L^2(\mathcal{O}\times(0,T))}),$$

where z is the solution of (3.12) corresponding to f, $v^1 = v^1(f)$ and $v^2 = v^2(f)$.

2.2.2 The Optimality system

In this section we obtain an optimality system, which characterizes the Nash equilibrium $(v^1(f), v^2(f))$. Multiplying in (3.29) by a function ϕ^i and integrating by parts, we have

$$\begin{aligned} &\iint_Q (\phi^i_{tt} - \Delta\phi^i + a(x,t)\phi^i)\hat{w}^i \, dx\, dt + \int_\Omega \hat{w}^i_t(x,T)\phi^i(x,T)\, dx \\ &- \int_\Omega \hat{w}^i(x,T)\phi^i_t(x,T)\, dx - \iint_\Sigma \frac{\partial \hat{w}^i}{\partial\nu}\phi^i \, d\sigma\, dt = \iint_{\mathcal{O}_i\times(0,T)} \hat{v}^i\phi^i \, dx\, dt, \end{aligned} \tag{2.18}$$

This leads us to define the adjoint systems

$$\begin{cases} \phi^i_{tt} - \Delta\phi^i + a(x,t)\phi^i = \alpha_i(z - z_{i,d})1_{\mathcal{O}_{i,d}} & \text{in} \quad Q, \\ \phi^i = 0 & \text{on} \quad \Sigma, \\ \phi^i(\cdot,T) = 0, \ \phi^i_t(\cdot,T) = 0 & \text{in} \quad \Omega. \end{cases} \tag{2.19}$$

From (3.44) and (3.45), we have that

$$\iint_{\mathcal{O}_{i,d}\times(0,T)} \alpha_i(z - z_{i,d})\hat{w}^i \, dx\, dt = \iint_{\mathcal{O}_i\times(0,T)} \hat{v}^i\phi^i \, dx\, dt,$$

and, replacing in (3.27), we see that

$$\iint_{\mathcal{O}_i\times(0,T)} (\phi^i + \mu_i v^i)\hat{v}^i \, dx\, dt = 0 \quad \forall \hat{v}^i \in \mathcal{H}_i.$$

As a conclusion, we get the following optimality system for $(v^1(f), v^2(f))$:

$$\begin{cases} z_{tt} - \Delta z + a(x,t)z = f1_{\mathcal{O}} - \displaystyle\sum_{i=1}^{2} \frac{1}{\mu_i}\phi^i 1_{\mathcal{O}_i} & \text{in} \quad Q, \\ \phi^i_{tt} - \Delta\phi^i + a(x,t)\phi^i = \alpha_i(z - z_{i,d})1_{\mathcal{O}_{i,d}} & \text{in} \quad Q, \\ z = 0, \ \phi^i = 0 & \text{on} \quad \Sigma, \\ z(\cdot,0) = z^0, \ z_t(\cdot,0) = z^1, \ \phi^i(\cdot,T) = 0, \ \phi^i_t(\cdot,T) = 0 & \text{in} \quad \Omega, \end{cases} \tag{2.20}$$

$$v^i(f) = -\frac{1}{\mu_i}\phi^i \Big|_{\mathcal{O}_i\times(0,T)}. \tag{2.21}$$

The system (3.15)-(3.16) characterizes the Nash equilibrium $(v^1(f), v^2(f))$ for each $f \in L^2(\mathcal{O}\times(0,T))$. The next objective is to find a leader $f \in L^2(\mathcal{O}\times(0,T))$ such that the solution to system (3.15) satisfies (2.10).

2.2.3 Null controllability

We will prove in this section the null controllability of the system (3.15). The proof is based on an observability inequality for the adjoint to (3.15), which is the following:

$$\begin{cases} \psi_{tt} - \Delta\psi + a(x,t)\psi = \displaystyle\sum_{i=1}^{2} \alpha_i \gamma^i 1_{\mathcal{O}_{i,d}} & \text{in} \quad Q, \\ \gamma^i_{tt} - \Delta\gamma^i + a(x,t)\gamma^i = -\dfrac{1}{\mu_i}\psi 1_{\mathcal{O}_i} & \text{in} \quad Q, \\ \psi = 0, \ \gamma^i = 0 & \text{on} \quad \Sigma, \\ z(\cdot,T) = \psi_0^T, \ \psi_t(\cdot,T) = \psi_1^T, \ \gamma^i(\cdot,0) = 0, \ \gamma^i_t(\cdot,0) = 0 & \text{in} \quad \Omega. \end{cases} \tag{2.22}$$

Note that from the equations in (3.15) and (2.22), one has

$$\sum_{i=1}^{2} \alpha_i \iint_{\mathcal{O}_{i,d}} \gamma^i z \, dx \, dt + \int_\Omega z_t(x,T)\psi_0^T(x)\, dx - \langle \psi(0), z^1 \rangle - \langle \psi_1^T, z(T) \rangle$$
$$+ \int_\Omega z^0 \psi_t(x,0)\, dx = \iint_{\mathcal{O}\times(0,T)} f\psi \, dx \, dt + \sum_{i=1}^{2} \alpha_i \iint_{\mathcal{O}_{i,d}\times(0,T)} (z - z_{i,d})\gamma^i \, dx \, dt,$$

where we have denoted by $\langle \cdot, \cdot \rangle$ the duality pairing $H^{-1}(\Omega) \times H_0^1(\Omega)$. Then

$$\begin{aligned} \iint_{\mathcal{O}\times(0,T)} f\psi \, dx \, dt = \int_\Omega z_t(x,T)\psi_0^T(x)\, dx - \langle \psi_1^T, z(T) \rangle + \int_\Omega z^1(x)\psi(x,0)\, dx \\ -\langle \psi_t(0), z^0 \rangle + \sum_{i=1}^{2} \alpha_i \iint_{\mathcal{O}_{i,d}\times(0,T)} z_{i,d}\gamma^i \, dx \, dt, \end{aligned} \tag{2.23}$$

and consequently, the null controllability of (3.15) holds if and only if, for each (z^0, z^1) given in $H_0^1(\Omega) \times L^2(\Omega)$, there exists $f \in L^2(\mathcal{O} \times (0,T))$ such that

$$\begin{cases} \displaystyle\iint_{\mathcal{O}\times(0,T)} f\psi \, dx \, dt = \int_\Omega z^1(x)\psi(x,0)\, dx - \langle \psi_t(0), z^0 \rangle + \sum_{1}^{2} \alpha_i \iint_{\mathcal{O}_{i,d}\times(0,T)} z_{i,d}\gamma^i \, dx \, dt \\ \forall (\psi_0^T, \psi_1^T) \in L^2(\Omega) \times H^{-1}(\Omega). \end{cases}$$

Given $(z^0, z^1) \in H_0^1(\Omega) \times L^2(\Omega)$ and $\varepsilon > 0$, we introduce $F_\varepsilon : L^2(\Omega) \times H^{-1}(\Omega) \mapsto \mathbb{R}$, as follows

$$\begin{aligned} F_\varepsilon(\psi_0^T, \psi_1^T) := \iint_{\mathcal{O}\times(0,T)} |\psi|^2 \, dx \, dt + \varepsilon ||(\psi_0^T, \psi_1^T)||_{L^2(\Omega)\times H^{-1}(\Omega)} \\ + \langle\langle (\psi(0), \psi_t(0)), (z^0, z^1) \rangle\rangle + \sum_{i=1}^{2} \alpha_i \iint_{\mathcal{O}_{i,d}\times(0,T)} z_{i,d}\gamma^i \, dx \, dt, \end{aligned} \tag{2.24}$$

where we have set by definition

$$\langle\langle (\psi(0), \psi_t(0)), (z^0, z^1) \rangle\rangle := \int_\Omega z^1(x)\psi(x,0)\, dx - \langle \psi_t(0), z^0 \rangle.$$

Note that F_ε is continuous, strictly convex and coercive in $L^2(\Omega) \times H^{-1}(\Omega)$.

In the sequel, we associate to each $\psi,\ \gamma^1,\ \gamma^2$ the "energy"E_ψ, with

$$E_\psi(t) := \frac{1}{2}\Big[||\psi(\cdot,t)||^2_{L^2(\Omega)} + ||\psi_t(\cdot,t)||^2_{H^{-1}(\Omega)}\Big].$$

The following result furnishes the desired observability inequality:

Proposition 10 *Suppose that* (2.8) *holds. Then there exists* $C > 0$ *such that, for all* $(\psi_0^T, \psi_1^T) \in L^2(\Omega) \times H^{-1}(\Omega)$*, the solution* $(\psi, \gamma^1, \gamma^2)$ *to* (2.22) *satisfies*

$$E_\psi(0) + \sum_{i=1}^{2} \iint_Q e^{R_0^2\lambda}|\gamma^i|^2\,dx\,dt \leq C \iint_{\mathcal{O}\times(0,T)} \psi^2\,dx\,dt, \tag{2.25}$$

where $R_0 := \min\{\sqrt{d(x)}:\ x \in \overline{\Omega}\}$.

Let us assume for the moment that Proposition 10 holds. Since that F_ε is continuous, strictly convex and coercive, it has a unique minimum $(\psi_{\varepsilon,0}^T, \psi_{\varepsilon,1}^T) \in L^2(\Omega) \times H^{-1}(\Omega)$. Moreover, for any $h > 0$ and $(\psi^T, \psi_1^T) \in L^2(\Omega) \times H^{-1}(\Omega)$, one has the following:

- either $(\psi_{\varepsilon,0}^T, \psi_{\varepsilon,1}^T) = 0$, or
- for all $(\psi_0^T, \psi_1^T) \in L^2(\Omega) \times H^{-1}(\Omega)$,

$$\begin{aligned} &\iint_{\mathcal{O}\times(0,T)} \psi_\varepsilon \psi\,dx\,dt + \frac{\varepsilon}{||(\psi_{\varepsilon,0}^T, \psi_{\varepsilon,1}^T)||_{L^2\times H^{-1}}}((\psi_{\varepsilon,0}^T, \psi_{\varepsilon,1}^T), (\psi_0^T, \psi_1^T))_{L^2\times H^{-1}} \\ &\quad + \langle\langle(\psi(0), \psi_t(0)), (z^0, z^1)\rangle\rangle + \sum_{i=1}^{2} \alpha_i \iint_{\mathcal{O}_{i,d}\times(0,T)} \gamma^i z_{i,d}\,dx\,dt = 0 \end{aligned} \tag{2.26}$$

In both cases, taking

$$f_\varepsilon = \psi_\varepsilon\Big|_{\mathcal{O}\times(0,T)}, \tag{2.27}$$

and denoting by $(z_\varepsilon, \phi_\varepsilon^1, \phi_\varepsilon^1)$ the corresponding solution to (3.15), we have

$$\begin{cases} \Big|\langle\langle(z_\varepsilon(\cdot,T), z_{\varepsilon,t}(\cdot,T)), (\psi^T, \psi_1^T)\rangle\rangle\Big| \leq \varepsilon\big|\big|(\psi_0^T, \psi_1^T)\big|\big|_{L^2(\Omega)\times H^{-1}(\Omega)} \\ \forall(\psi_0^T, \psi_1^T) \in L^2(\Omega) \times H^{-1}(\Omega). \end{cases} \tag{2.28}$$

Therefore,

$$\big|\big|(z_\varepsilon(\cdot,T), z_{\varepsilon,t}(\cdot,T))\big|\big|_{H_0^1\times L^2} \leq \varepsilon. \tag{2.29}$$

Furthermore, from (2.25) and (2.27), we get:

$$||f_\varepsilon||_{L^2(\mathcal{O}\times(0,T))} \leq C\left\{||(z^0, z^1)||^2_{H_0^1\times L^2} + \sum_{i=1}^{2} \alpha_i^2 \iint_{\mathcal{O}_{i,d}\times(0,T)} e^{-2\lambda\varphi}|z_{i,d}|^2\,dx\,dt\right\}^{\frac{1}{2}},$$

whence (f_ε) is uniformly bounded in $L^2(\mathcal{O}\times(0,T))$ and possesses a subsequence weakly convergent to some function $\hat{f}\in L^2(\mathcal{O}\times(0,T))$. It is then clear that

$$(z_\varepsilon(\cdot,T),z_{\varepsilon,t}(\cdot,T))\to((\hat{z}(\cdot,T),\hat{z}_t(\cdot,T)))\ \text{ weakly in }\ H_0^1(\Omega)\times L^2(\Omega),$$

where $(\hat{z},\hat{\phi}^1,\hat{\phi}^2)$ is the solution to (3.15) associated to $\hat{f}$. In view of (2.29), we conclude that $\hat{z}$ satisfies (2.10) and the null controllability of (3.15) is satisfied.

Proof of the Proposition 10. The proof relies on the arguments of [19] and [33]. We introduce the function $\varphi: Q\mapsto\mathbb{R}$, with

$$\varphi(x,t):=d(x)-c(t-T/2)^2, \tag{2.30}$$

where $c\in(0,1)$. Let us suppose that (2.8) holds and let us set $\omega:=\mathcal{O}_\delta(\Gamma_+)\cap\Omega\subset\mathcal{O}$. Then, there exists $\lambda_0\geq 1$ such that, for any $\lambda\geq\lambda_0$ and any function $u\in C^0([0,T];L^2(\Omega))$ satisfying

$$u(\cdot,0)=u(\cdot,T)=0,\ (u_{tt}-\Delta u)\in H^{-1}(Q); \tag{2.31}$$

$$(u,\eta_{tt}-\Delta\eta)_{L^2(Q)}=\langle u_{tt}-\Delta u,\eta\rangle_{H^{-1}(Q),H_0^1(Q)}\quad\forall\eta\in H_0^1(Q), \tag{2.32}$$

we have the Carleman inequality

$$\lambda\iint_Q e^{2\lambda\varphi}|u|^2\,dx\,dt\leq C\left(\left|e^{\lambda\varphi}(u_{tt}-\Delta u)\right|^2_{H^{-1}(Q)}+\lambda^2\iint_{\omega\times(0,T)}e^{2\lambda\varphi}|u|^2\,dx\,dt\right), \tag{2.33}$$

where $C>0$ is independent of λ and u. The proof of (2.33) can to be found at [19] or [33].

Recall that we have assumed that $T>2R_1$. Then we can choose c in (2.30) such that

$$\left(\frac{2R_1}{T}\right)^2<c<\frac{2R_1}{T}. \tag{2.34}$$

Now, we can also choose $T_0,\ T_1\in(0,T)$ such that

$$\varphi(x,t)\leq\frac{R_1^2}{2}-c\frac{T^2}{8}<0\quad\forall(x,t)\in\Omega\times((0,T_1)\cup(T_1',T)) \tag{2.35}$$

and

$$\varphi(x,t)\geq\frac{R_0^2}{2},\ \ \forall(x,t)\in\Omega\times(T_0,T_0'). \tag{2.36}$$

where we have set $T_i':=T-T_i$ for $i=0,1$.

Let us consider a function $\xi \in C_0^\infty(0,T)$ so that $\xi(t) \equiv 1$ in (T_1, T_1'). Then $(\tilde{\psi}, \tilde{\gamma}^1, \tilde{\gamma}^1) = (\xi\psi, \xi\gamma^1, \xi\gamma^2)$ is the unique solution of the system

$$\begin{cases} \tilde{\psi}_{tt} - \Delta\tilde{\psi} + a(x,t)\tilde{\psi} = \displaystyle\sum_{i=1}^{2} \alpha_i \xi \gamma^i 1_{\mathcal{O}_{i,d}} + 2\xi_t \psi_t + \xi_{tt}\psi & \text{in} \quad Q, \\ \tilde{\gamma}^i_{tt} - \Delta\tilde{\gamma}^i + a(x,t)\tilde{\gamma}^i = -\dfrac{1}{\mu_i}\xi\psi 1_{\mathcal{O}_i} + 2\xi_t\gamma^i_t + \xi_{tt}\gamma^i & \text{in} \quad Q, \\ \tilde{\psi} = 0, \ \ \tilde{\gamma}^i = 0 & \text{on} \quad \Sigma, \\ \tilde{\psi}(\cdot,T) = \tilde{\psi}(\cdot,0) = \tilde{\psi}_t(\cdot,T) = \tilde{\gamma}^i(\cdot,0) = \tilde{\gamma}^i(\cdot,T) = \tilde{\gamma}^i_t(\cdot,0) = 0 & \text{in} \quad \Omega. \end{cases} \tag{2.37}$$

Since $\tilde{\psi} \in C^0([0,T]; L^2(\Omega))$ satisfies (2.31) and (2.32), for any $\lambda \geq \lambda_0$ we have

$$\begin{aligned} \lambda \iint_Q e^{2\lambda\varphi}|\xi\psi|^2 \, dx \, dt \leq C \Bigg[\Bigg|\Bigg| e^{\lambda\varphi} \left(\sum_{i=1}^{2} \alpha_i \xi \gamma^i 1_{\mathcal{O}_{i,d}} + 2\xi_t\psi_t + \xi_{tt}\psi \right) \Bigg|\Bigg|^2_{H^{-1}(Q)} \\ + \lambda^2 \iint_{\omega\times(0,T)} e^{2\lambda\varphi}|\psi|^2 \, dx \, dt \Bigg]. \end{aligned} \tag{2.38}$$

Then, using the inequality (2.38) and the equations in (2.37), arguing as in [33], we get:

$$\begin{aligned} \lambda \iint_Q e^{2\lambda\varphi}|\psi|^2 \, dx \, dt \leq C \Bigg[\lambda^2 \iint_{\omega\times(0,T)} e^{2\lambda\varphi}|\psi|^2 \, dx \, dt + N \\ + \lambda \sum_{i=1}^{2} \iint_{\mathcal{O}_{i,d}\times(0,T)} e^{2\lambda\varphi}|\gamma^i|^2 \, dx \, dt \Bigg], \end{aligned} \tag{2.39}$$

where $C > 0$ is independent of λ e

$$N := \lambda^2 e^{(R_1^2 - cT^2/4)\lambda} \left(||\psi||^2_{L^2(\Omega\times(0,T_1))} + ||\psi||^2_{L^2(\Omega\times(T_1',T))} \right). \tag{2.40}$$

From (2.39), the following is found:

$$\begin{aligned} &\lambda \iint_Q e^{2\lambda\varphi}|\psi|^2 \, dx \, dt + \lambda \sum_{i=1}^{2} \iint_{\mathcal{O}_{i,d}\times(0,T)} e^{R_0^2\lambda}|\gamma^i|^2 \, dx \, dt \\ &\quad \leq C \Bigg[\lambda^2 \iint_{\omega\times(0,T)} e^{2\lambda\varphi}|\psi|^2 \, dx \, dt + N \\ &\quad + \sum_{i=1}^{2} \iint_{\mathcal{O}_{i,d}\times(0,T)} (\lambda e^{R_0^2\lambda} + e^{2\lambda\varphi})|\gamma^i|^2 \, dx \, dt \Bigg] \end{aligned} \tag{2.41}$$

From (2.35), we deduce that there is a $\lambda_1 \geq \lambda_0$ such that $\lambda^2 e^{(R_1^2 - cT/4)\lambda} < 1$ for all $\lambda \geq \lambda_1$. Consequently, by (2.40) and (2.41),

$$\begin{aligned} &\lambda \iint_Q e^{2\lambda\varphi}|\psi|^2 \, dx \, dt + \sum_{i=1}^{2} \lambda \iint_{\mathcal{O}_{i,d}\times(0,T)} e^{R_0^2\lambda}|\gamma^i|^2 \, dx \, dt \\ &\quad \leq C \Bigg[\lambda^2 \iint_{\omega\times(0,T)} e^{2\lambda\varphi}|\psi|^2 \, dx \, dt + ||\psi||^2_{L^2(\Omega\times(0,T_1))} \\ &\quad + ||\psi||^2_{L^2(\Omega\times(T_1',T))} + \sum_{i=1}^{2} \iint_{\mathcal{O}_{i,d}\times(0,T)} (\lambda e^{R_0^2\lambda} + e^{2\lambda\varphi})|\gamma^i|^2 \, dx \, dt \Bigg], \end{aligned} \tag{2.42}$$

where $C > 0$ is independent of λ.

Taking into account (2.36) we have:

$$\iint_Q e^{2\lambda\varphi}|\psi|^2\,dx\,dt \geq e^{R_0^2\lambda}\int_{T_0}^{T_0'}\int_\Omega |\psi|^2\,dx\,dt. \tag{2.43}$$

On the other hand, from the usual energy method (see for example [33], lemmas 3.3 and 3.4), it follows that, for any $S_0 \in (T_0, T/2)$ and $S_0' \in (T/2, T_0')$, we also have

$$\int_{S_0}^{S_0'} E_\psi(t)\,dt \leq C\int_{T_0}^{T_0'}\int_\Omega |\psi|^2\,dx\,dt + \sum_{i=1}^{2}\int_{T_0}^{T_0'}\int_{\mathcal{O}_{i,d}} |\gamma^i|^2\,dx\,dt, \tag{2.44}$$

$$||\psi||^2_{L^2(\Omega\times(0,T_1))} + ||\psi||^2_{L^2(\Omega\times(T_1',T))} \leq CE_\psi(0) + C\sum_{i=1}^{2}\iint_{\mathcal{O}_{i,d}\times(0,T)} |\gamma^i|^2\,dx\,dt \tag{2.45}$$

and also

$$CE_\psi(0) \leq \int_{S_0}^{S_0'} E_\psi(t)\,dt + \sum_{i=1}^{2}\int_0^{S_0'}\int_{\mathcal{O}_{i,d}} |\gamma^i|^2\,dx\,dt. \tag{2.46}$$

Then, combining (2.42)–(2.46) we arrive at the inequality

$$\begin{aligned} C_1\,\lambda E_\psi(0)\,e^{R_0^2\lambda + C_1} + \sum_{i=1}^{2}\iint_Q e^{R_0^2\lambda}|\gamma^i|^2\,dx\,dt &\leq C\lambda^2\iint_{\omega\times(0,T)} e^{2\lambda\varphi}|\psi|^2\,dx\,dt \\ &+ C_2\,e^{C_2}E_\psi(0) + \frac{C}{\min\{\mu_1,\mu_2\}}e^{2\lambda R_1}E_\psi(0), \end{aligned}$$

where C, C_1 and C_2 are independent of λ. Thus, choosing λ and μ_i such that

$$C_1\lambda > C\,C_2, \quad \lambda R_0^2 + C_1 > C_2, \quad \frac{Ce^{2\lambda R_1}}{\min\{\mu_1,\mu_2\}} < 1,$$

we get (2.25). ■

2.3 The semilinear case

In this section we consider the semilinear case, with $F \in W^{1,\infty}(\mathbb{R})$ in (3.1). The main is to prove the Theorem 8. We will follow arguments similar to those in [3] and [6].

First, let us deduce the optimality system, which in this case characterizes the Nash quasi-equilibrium pair. Thus, given $f \in L^2(\mathcal{O}\times(0,T))$, let $(v^1(f), v^2(f)) \in \mathcal{H}$ be a Nash quasi-equilibrium corresponding to the functionals J_i. Then, we have by definition

$$J_i'(f, v^1, v^2)\cdot\hat{v}^i = 0 \quad \forall \hat{v}^i \in \mathcal{H}_i.$$

Note that this is equivalent to

$$\alpha_i \iint_{\mathcal{O}_{i,d}\times(0,T)} (y - y_{i,d})\hat{y}^i \, dx\, dt + \mu_i \iint_{\mathcal{O}_i\times(0,T)} v^i \hat{v}^i \, dx\, dt = 0 \quad \forall \hat{v}^i \in \mathcal{H}_i, \tag{2.47}$$

where $\hat{y}^i$ is the solution to the system

$$\begin{cases} \hat{y}^i_{tt} - \Delta \hat{y}^i + a(x,t)\hat{y}^i = F'(y)\hat{y}^i + \hat{v}^i 1_{\mathcal{O}_i} & \text{in} \quad Q, \\ \hat{y}^i = 0, & \text{on} \quad \Sigma, \\ \hat{y}^i(\cdot,0),\ \hat{y}^i_t(\cdot,0) = 0, & \text{in} \quad \Omega. \end{cases} \tag{2.48}$$

Multiplying in (2.48) by a function ϕ^i, integrating in Q with respect to x and t and using integration by parts, we see that

$$\begin{aligned} &\iint_Q (\phi^i_{tt} - \Delta\phi^i + a(x,t)\phi^i)\hat{y}^i \, dx\, dt - \iint_\Sigma \phi^i \frac{\partial \hat{y}^i}{\partial \nu} \, d\sigma\, dt + \left\langle \hat{y}^i_t(T), \phi^i(T) \right\rangle \\ &\quad - \left(\hat{y}^i(T), \phi^i(T)\right) = \iint_Q F'(y)\, \hat{y}^i \phi^i \, dx\, dt + \iint_{\mathcal{O}_i\times(0,T)} \hat{v}^i \phi^i \, dx\, dt. \end{aligned} \tag{2.49}$$

Then, we define the following adjoint systems to (2.48)

$$\begin{cases} \phi^i_{tt} - \Delta\phi^i + a(x,t)\phi^i = F'(y)\phi^i + \alpha_i(y - y_{i,d})1_{\mathcal{O}_{i,d}} & \text{in} \quad Q, \\ \phi^i = 0, & \text{on} \quad \Sigma, \\ \phi^i(\cdot,T),\ \phi^i_t(\cdot,T) = 0, & \text{in} \quad \Omega. \end{cases} \tag{2.50}$$

From (2.48) - (2.50), we have

$$\alpha_i \iint_{\mathcal{O}_{i,d}\times(0,T)} (y - y_{i,d})\hat{y}^i \, dx\, dt = \iint_{\mathcal{O}_i\times(0,T)} \hat{v}^i \phi^i \, dx\, dt \quad \forall \hat{v}^i \in \mathcal{H}_i. \tag{2.51}$$

Hence, replacing (2.51) in (2.47), it follows that

$$\iint_{\mathcal{O}_i\times(0,T)} (\phi^i + \mu_i v^i)\hat{v}^i \, dx\, dt = 0 \quad \forall \hat{v}^i \in \mathcal{H}_i.$$

This leads to the optimality system

$$\begin{cases} y_{tt} - \Delta y + a(x,t)y = F(y) + f1_{\mathcal{O}} - \sum_{i=1}^{2} \frac{1}{\mu_i}\phi^i 1_{\mathcal{O}_i} & \text{in} \quad Q, \\ \phi^i_{tt} - \Delta\phi^i + a(x,t)\phi^i = F'(y)\phi^i + \alpha_i(y - y_{i,d})1_{\mathcal{O}_{i,d}} & \text{in} \quad Q, \\ y = 0,\ \phi^i = 0 & \text{on} \quad \Sigma, \\ y(\cdot,0) = y^0,\ y_t(\cdot,0) = y^1,\ \phi^i(\cdot,T) = 0,\ \phi^i_t(\cdot,T) = 0 & \text{in} \quad \Omega, \end{cases} \tag{2.52}$$

$$v^i = -\frac{1}{\mu_i}\phi^i\Big|_{\mathcal{O}_i\times(0,T)}. \tag{2.53}$$

Now, given $(\bar{y}^0, \bar{y}^1) \in H_0^1(\Omega) \times L^2(\Omega)$, let $\bar{y}$ be a free trajectory of the system (3.1), that is, the solution to (2.5). Let us consider the change of variables $z = y - \bar{y}$. From (2.52) and (2.5), we see that (z, ϕ^1, ϕ^2) is the solution to

$$\begin{cases} z_{tt} - \Delta z + a(x,t)z = G(x,t;z)z + f 1_{\mathcal{O}} - \displaystyle\sum_{i=1}^{2} \frac{1}{\mu_i} \phi^i 1_{\mathcal{O}_i} & \text{in} \quad Q, \\ \phi^i_{tt} - \Delta \phi^i + a(x,t)\phi^i = F'(\bar{y} + z)\phi^i + \alpha_i (z - z_{i,d}) 1_{\mathcal{O}_{i,d}} & \text{in} \quad Q, \\ z = 0, \ \phi^i = 0 & \text{on} \quad \Sigma, \\ z(\cdot,0) = z^0, \ z_t(\cdot,0) = z^1, \ \phi^i(\cdot,T) = 0, \ \phi^i_t(\cdot,T) = 0 & \text{in} \quad \Omega, \end{cases} \tag{2.54}$$

where, $z^0 = y^0 - \bar{y}^0$, $z^1 = y^1 - \bar{y}^1$, $z_{i,d} = y_{i,d} - \bar{y}$ and

$$G(x,t;z) = \int_0^1 F'(\bar{y} + \tau z)\, d\tau.$$

With this change of variable, it becomes clear that the exact controllability property for (2.52) is equivalent to the null controllability property for (2.54). We will prove the latter by a fixed point method.

For each $z \in L^2(Q)$ and each $f \in L^2(\mathcal{O} \times (0,T))$, let us consider the linear system

$$\begin{cases} w_{tt} - \Delta w + a(x,t)w = G(x,t;z)w + f 1_{\mathcal{O}} - \displaystyle\sum_{i=1}^{2} \frac{1}{\mu_i} \phi^i 1_{\mathcal{O}_i} & \text{in} \quad Q, \\ \phi^i_{tt} - \Delta \phi^i + a(x,t)\phi^i = F'(\bar{y} + z)\phi^i + \alpha_i (w - z_{i,d}) 1_{\mathcal{O}_{i,d}} & \text{in} \quad Q, \\ w = 0, \ \phi^i = 0 & \text{on} \quad \Sigma, \\ w(\cdot,0) = z^0, \ w_t(\cdot,0) = z^1, \ \phi^i(\cdot,T) = 0, \ \phi^i_t(\cdot,T) = 0 & \text{in} \quad \Omega. \end{cases} \tag{2.55}$$

Since $F \in W^{1,\infty}(\mathbb{R})$, there exists $M > 0$ such that

$$|G(x,t;s)| + |F'(s)| \le M \quad \forall (x,t;s) \in Q \times \mathbb{R}. \tag{2.56}$$

Furthermore, there exists $C > 0$ such that

$$||w||_{L^\infty(0,T;H_0^1(\Omega))} + ||w_t||_{L^\infty(0,T;L^2(\Omega))} \le (1 + ||f||_{L^2(\mathcal{O}\times(0,T))}). \tag{2.57}$$

For any fixed $z \in L^2(Q)$, let us denote by $(w_z, \phi^1_z, \phi^2_z)$ the solution to (2.55) associated to z. Then, by multiplying in $(2.55)_1$ and $(2.55)_2$ by functions ψ_z and γ^i_z (respectively) and integrating by parts in Q, we get:

$$\begin{aligned}\iint_Q (\psi_{z,tt} - \Delta\psi_z + a(x,t)\psi_z) w_z \,dx\,dt + \int_\Omega w_{z,t}(x,T)\psi_z(x,T)\,dx \\ - \int_\Omega z^1(x)\psi_z(x,0)\,dx - \langle w_z(T), \psi_{z,t}(T)\rangle + \langle z^0, \psi_{z,t}(0)\rangle \\ = \iint_\Sigma \psi_z \frac{\partial w_z}{\partial \nu}\,d\sigma\,dt + \iint_Q G(x,t;z)\, w_z\psi_z\,dx\,dt \\ - \sum_{i=1}^{2} \frac{1}{\mu_i} \iint_{\mathcal{O}_i\times(0,T)} \phi_z^i \psi_z\,dx\,dt + \iint_{\mathcal{O}\times(0,T)} f\psi_z\,dx\,dt\end{aligned} \tag{2.58}$$

and

$$\begin{aligned}\iint_Q (\psi_{z,tt} - \Delta\psi_z + a(x,t)\psi_z) w_z\,dx\,dt + \langle \phi_z^i(0), \gamma_{z,t}^i(0)\rangle \\ - \int_\Omega \phi_z^i(x,0)\gamma_z^i(x,0)\,dx - \iint_\Sigma \gamma_z^i \frac{\partial \phi_z^i}{\partial \nu}\,d\sigma\,dt \\ = \iint_Q F'(\bar{y}+z)\phi_z^i\gamma_z^i\,dx\,dt + \alpha_i \iint_{\mathcal{O}_{i,d}\times(0,T)} (w_z - z_{i,d})\gamma_z^i\,dx\,dt.\end{aligned} \tag{2.59}$$

In view of (2.58) and (2.59), we introduce the following adjoint system

$$\begin{cases} \psi_{z,tt} - \Delta\psi_z + a(x,t)\psi_z = G(x,t;z)\psi_z + \displaystyle\sum_{i=1}^{2} \alpha_i \gamma_z^i 1_{\mathcal{O}_{i,d}} & \text{in} \quad Q, \\ \gamma_{z,tt}^i - \Delta\gamma_z^i + a(x,t)\gamma_z^i = F'(\bar{y}+z)\gamma_z^i - \frac{1}{\mu_i}\psi_z 1_{\mathcal{O}_i} & \text{in} \quad Q, \\ \psi_z = 0, \ \gamma_z^i = 0 & \text{on} \quad \Sigma, \\ \psi_z(\cdot,T) = \psi_0^T, \ \psi_{z,t}(\cdot,T) = \psi_1^T, \ \gamma_z^i(\cdot,0) = 0, \ \gamma i_{z,t}(\cdot,0) = 0 & \text{in} \quad \Omega. \end{cases} \tag{2.60}$$

Then,

$$-\frac{1}{\mu_i} \iint_{\mathcal{O}_i\times(0,T)} \phi_z^i\psi_z\,dx\,dt = \alpha_i \iint_{\mathcal{O}_{i,d}\times(0,T)} (w_z - z_{i,d})\gamma_z^i\,dx\,dt,$$

whence

$$-\sum_{i=1}^{2} \frac{1}{\mu_i} \iint_{\mathcal{O}_i\times(0,T)} \phi_z^i\psi_z\,dx\,dt = \sum_{i=1}^{2} \alpha_i \iint_{\mathcal{O}_{i,d}\times(0,T)} (w_z - z_{i,d})\gamma_z^i\,dx\,dt \tag{2.61}$$

and, by (2.58) and (2.60), we see that

$$\begin{aligned}\int_\Omega w_{z,t}(x,T)\psi_0^T(x)\,dx - \int_\Omega z^1(x)\psi_z(x,0)\,dx - \langle w_z(T), \psi_1^T\rangle \\ + \langle z^0, \psi_{z,t}(0)\rangle + \sum_{i=1}^{2} \alpha_i \iint_{\mathcal{O}_{i,d}\times(0,T)} w_z\gamma_z^i\,dx\,dt \\ = -\sum_{i=1}^{2} \frac{1}{\mu_i} \iint_{\mathcal{O}_i\times(0,T)} \phi_z^i\psi_z\,dx\,dt + \iint_{\mathcal{O}\times(0,T)} f\psi_z\,dx\,dt.\end{aligned} \tag{2.62}$$

Finally, replacing (2.61) in (2.62), the following identity is found:

$$\begin{aligned}\iint_{\mathcal{O}\times(0,T)} f\psi_z\,dx\,dt = \int_\Omega w_{z,t}(x,T)\psi_0^T(x)\,dx - \int_\Omega z^1(x)\psi_z(x,0)\,dx \\ - \langle w_z(T), \psi_1^T\rangle + \langle z^0, \psi_{z,t}(0)\rangle + \sum_{i=1}^{2} \alpha_i \iint_{\mathcal{O}_{i,d}\times(0,T)} z_{i,d}\gamma_z^i\,dx\,dt.\end{aligned}$$

As a consequence, we have the null controllability of (2.55) if and only if, for each $(z^0, z^1) \in H_0^1(\Omega) \times L^2(\Omega)$, there exists $f \in L^2(\mathcal{O} \times (0,T))$ such that

$$\begin{cases} \displaystyle\iint_{\mathcal{O}\times(0,T)} f\psi_z \, dx\, dt = \langle z^0, \psi_{z,t}(0)\rangle + \sum_{i=1}^{2} \alpha_i \iint_{\mathcal{O}_{i,d}\times(0,T)} z_{i,d}\gamma_z^i \, dx\, dt \\ \qquad\qquad\qquad\qquad - \displaystyle\int_\Omega z^1(x)\psi_z(x,0)\, dx \quad \forall (\psi_0^T, \psi_1^T) \in L^2(\Omega) \times H^{-1}(\Omega). \end{cases}$$

As in the previous section, let us define the functional $F_{z,\varepsilon}$, with

$$\begin{aligned} F_{z,\varepsilon}(\psi_0^T, \psi_1^T) := &\iint_{\mathcal{O}\times(0,T)} |\psi_z|^2 \, dx\, dt + \varepsilon ||(\psi^T, \psi_1^T)|| \\ &+ \langle (\psi_z(0), \psi_{z,t}(0)), (z^0, z^1)\rangle + \sum_{i=1}^{2} \alpha_i \iint_{\mathcal{O}_{i,d}\times(0,T)} z_{i,d}\gamma_z^i \, dx\, dt. \end{aligned}$$

As in the linear case, we get the following observability inequality

$$E_{\psi_z}(0) + \sum_{i=1}^{2} \iint_Q e^{R_0^2\lambda} |\gamma_z^i|^2 dxdt \le C \iint_{\mathcal{O}\times(0,T)} |\psi_z|^2 dxdt, \tag{2.63}$$

where $C > 0$ is a constant independent of z and E_{ψ_z} denotes the $L^2 \times H^{-1}$ energy associated to ψ_z. Arguing as in the linear case, the inequality (2.63) allows us to prove the existence of minimizer $(\psi_{0,\varepsilon}^T, \psi_{1,\varepsilon}^T)$ of $F_{z,\varepsilon}$ in $L^2(\Omega) \times H^{-1}(\Omega)$. Thus, taking limits as $\varepsilon \to 0$, we get a unique control $f_z \in L^2(\mathcal{O} \times (0,T))$ such that the associated solution $(w_z, \phi_z^1, \phi_z^2)$ to system (2.55), satisfy

$$w_z(\cdot, T) = 0 \;\text{ in }\; \Omega, \tag{2.64}$$

and

$$||f_z||_{L^2(\mathcal{O}\times(0,T))} \le C, \tag{2.65}$$

where C is independent of z. The remainder of the proof is completely standard and well known. It suffies to introduce the mapping $\Lambda : L^2(Q) \to L^2(Q)$, with $\Lambda(z) := w_z$, where w_z is, together with ϕ_z^i, the solution to (2.55) satisfying (2.64) that we have just constructed. From the properties of f_z and the usual energy estimates, it is clear that Λ satisfies the assumptions of Schauder's Theorem and, consequently, possesses at least one fixed point. This ends the proof of Theorem 8.

2.3.1 Equilibrium and Quasi-equilibrium

In this section we will prove Proposition 8 that says that under some conditions, the definitions of Nash equilibrium and Nash quasi-equilibbrium are equivalent. Similar questions are analyzed in [3] and [6] for parabolic systems.

Suppose that $F \in W^{2,\infty}(\mathbb{R})$. Let $f \in L^2(\mathcal{O} \times (0,T))$ be given and let $(v^1, v^2) \in \mathcal{H}$ be the Nash quasi-equilibrium associated to f. For all $\hat{v}^1 \in \mathcal{H}_1$ and $s \in \mathbb{R}$, let y^s be the solution to

$$\begin{cases} y^s_{tt} - \Delta y^s + a(x,t)y^s = F(y^s) + f1_{\mathcal{O}} + (v^1 + s\hat{v}^1)1_{\mathcal{O}_1} + v^2 1_{\mathcal{O}_2} & \text{in} \quad Q, \\ y^s = 0 & \text{on} \quad \Sigma, \\ y^s(\cdot,0) = y^0, \ y^s_t(\cdot,0) = y^1 & \text{in} \quad \Omega. \end{cases} \tag{2.66}$$

Let us consider the notation $y^s|_{s=0} = y$. Now, for all $\tilde{v}^1 \in \mathcal{H}_1$, we have:

$$\begin{aligned} &D_1 J_1(f; v^1 + s\hat{v}^1, v^2) \cdot \tilde{v}^1 - D_1 J_1(f; v^1, v^2) \cdot \tilde{v}^1 \\ &\qquad = \alpha_1 \iint_{\mathcal{O}_{1,d}\times(0,T)} (y^s - y_{1,d}) q^s \, dx\, dt - \alpha_1 \iint_{\mathcal{O}_{1,d}\times(0,T)} (y - y_{1,d}) q \, dx\, dt \\ &\qquad + s\mu_1 \iint_{\mathcal{O}_1\times(0,T)} \hat{v}^1 \tilde{v}^1 \, dx\, dt, \end{aligned} \tag{2.67}$$

where q^s is the solution to

$$\begin{cases} q^s_{tt} - \Delta q^s + a(x,t)q^s = F(y^s)q^s + \tilde{v}^1 1_{\mathcal{O}_1} & \text{in} \quad Q, \\ q^s = 0 & \text{on} \quad \Sigma, \\ q^s(\cdot,0) = 0, \ q^s_t(\cdot,0) = 0 & \text{in} \quad \Omega, \end{cases} \tag{2.68}$$

and we denote used the notation $q := q^s|_{s=0}$. Multiplying in (2.68) by a function ϕ^s and integrating by parts in Q, we have

$$\begin{aligned} &\iint_Q (\phi^s_{tt} - \Delta\phi^s + a(x,t)\phi^s) q^s \, dx\, dt - \iint_\Sigma \phi^s \frac{\partial q^s}{\partial \nu} \, d\sigma\, dt \\ &\qquad + \langle q^s_t(T), \phi^s(T)\rangle - \int_\Omega q^s(x,T)\phi^s(x,T)\, dx \\ &\qquad = \iint_Q F'(y^s) q^s \phi^s \, dx\, dt + \iint_{\mathcal{O}_i\times(0,T)} \tilde{v}^1 \phi^s \, dx\, dt. \end{aligned} \tag{2.69}$$

Let us introduce the adjoint

$$\begin{cases} \phi^s_{tt} - \Delta\phi^s + a(x,t)\phi^s = F(y^s)\phi^s + \alpha_1 (y^s - y_{1,d}) 1_{\mathcal{O}_{1,d}} & \text{in} \quad Q, \\ \phi^s = 0 & \text{on} \quad \Sigma, \\ \phi^s(\cdot,T) = 0, \ \phi^s_t(\cdot,T) = 0 & \text{in} \quad \Omega. \end{cases} \tag{2.70}$$

By (2.69) and (2.70), we have

$$\alpha_1 \iint_{\mathcal{O}_{i,d}\times(0,T)} (y^s - y_{1,d}) q^s \, dx\, dt = \iint_{\mathcal{O}_i\times(0,T)} \tilde{v}^1 \phi^s \, dx\, dt.$$

Replacing in (2.67), we get

$$\begin{aligned} D_1J_1(f;v^1+s\hat{v}^1,v^2)\cdot\tilde{v}^1 - D_1J_1(f;v^1,v^2) = \iint_{\mathcal{O}_1\times(0,T)} \tilde{v}^1(\phi^s-\phi)\,dx\,dt \\ +s\mu_1\iint_{\mathcal{O}_1\times(0,T)} \hat{v}^1\tilde{v}^1\,dx\,dt \quad \forall\hat{v}^1,\tilde{v}^1\in\mathcal{H}_1. \end{aligned} \tag{2.71}$$

Note that

$$(\phi^s-\phi)_{tt}-\Delta(\phi^s-\phi)+a(x,t)(\phi^s-\phi)=[F'(y^s)-F(y)]\phi^s+F'(y)(\phi^s-\phi)+\alpha_1(y^s-y)1_{\mathcal{O}_{1,d}},$$

and the following limits exist

$$h=\lim_{s\to0}\frac{1}{s}(y^s-y),\quad \eta=\lim_{s\to0}\frac{1}{s}(\phi^s-\phi),$$

moreover, (h,η) is the solution to the system

$$\begin{cases} h_{tt}-\Delta h+a(x,t)h=F'(y)h+\hat{v}^1 1_{\mathcal{O}_1} & \text{in} \quad Q,\\ \eta_{tt}-\Delta\eta+a(x,t)\eta=F''(y)h\phi+F'(y)\eta+\alpha_1 h1_{\mathcal{O}_{1,d}} & \text{in} \quad Q,\\ h=0,\ \eta=0 & \text{on} \quad \Sigma,\\ h(\cdot,0)=0,\ h_t(\cdot,0)=0,\ \eta(\cdot,T)=0,\ \eta_t(\cdot,T)=0 & \text{in} \quad \Omega. \end{cases} \tag{2.72}$$

Thus, by (2.71) and (2.72), we deduce that

$$\begin{cases} D_1^2J_1(f;v^1,v^2)\cdot(\hat{v}^1,\tilde{v}^1)=\displaystyle\iint_{\mathcal{O}_1\times(0,T)}\tilde{v}^1\eta\,dx\,dt+\mu_1\iint_{\mathcal{O}_1\times(0,T)}\hat{v}^1\tilde{v}^1\,dx\,dt\\ \forall\hat{v}^1,\tilde{v}^1\in\mathcal{H}_1. \end{cases}$$

In particular,

$$\begin{cases} D_1^2J_1(f;v^1,v^2)\cdot(\hat{v}^1,\hat{v}^1)=\displaystyle\iint_{\mathcal{O}_1\times(0,T)}\hat{v}^1\eta\,dx\,dt+\mu_1\iint_{\mathcal{O}_1\times(0,T)}|\hat{v}^1|^2\,dx\,dt\\ \forall\hat{v}^1\in\mathcal{H}_1. \end{cases} \tag{2.73}$$

Let us see that for some $C_1>0$ independent of $f,\eta,h,\hat{v}^1$, one has

$$\left|\iint_{\mathcal{O}_1\times(0,T)}\hat{v}^1\eta\,dx\,dt\right|\le C_1\left(1+||f||_{L^2(\mathcal{O}\times(0,T))}\right)||\hat{v}^1||_{\mathcal{H}_1}\quad\forall\hat{v}^1\in\mathcal{H}_1. \tag{2.74}$$

Indeed, since $F'\in W^{1,\infty}(\mathbb{R})$, from the energy estimates, we first see that

$$\int_\Omega|h_t|^2\,dx+\int_\Omega|\nabla h|^2\,dx\le C||\hat{v}^1||^2_{\mathcal{H}_1},\quad t\in[0,T]\ \text{a. e.} \tag{2.75}$$

Also,

$$\iint_Q\hat{v}^1\eta\,dx\,dt=\iint_Q F''(y)\phi h^2\,dx\,dt+\alpha_1\iint_{\mathcal{O}_{i,d}\times(0,T)}h^2\,dx\,dt. \tag{2.76}$$

Let us find r and s such that

$$\phi \in L^r(0,T;L^s(\Omega)) \ \text{ and } \ h \in L^{r'}(0,T;L^{s'}(\Omega)),$$

where r' and s' are conjugate of r and s, respectively. In fact, as $h \in C^0([0,T];H_0^1(\Omega))$ and $h_t \in C^0([0,T];L^2(\Omega))$, we have in particular, by Sobolev embedding

$$h \in L^\infty(0,T;L^{\frac{2n}{n-2}}),$$

that is, $r' = +\infty$ and $s' = n/(n-2)$. Thus $r = 1$ and $s = n/2$.

Now, since

$$y, y_{1,d} \in C^0([0,T];H_0^1(\Omega)) \cap C^1([0,T];L^2(\Omega)),$$

we have

$$\begin{cases} \phi \in C^1([0,T];H_0^1(\Omega)) \cap C^0([0,T],H^2(\Omega)\cap H_0^1(\Omega)), \\ \phi_t \in C^1([0,T];L^2(\Omega)) \cap C^0([0,T],H_0^1(\Omega)). \end{cases}$$

As a consequence, $\phi \in L^1(0,T;L^a(\Omega))$, where $a = 2n/(n-4)$. Thus, we have that $\phi \in L^1(0,T;L^s(\Omega))$ if, and only if,

$$\frac{n}{2} \leq \frac{2n}{n-4},$$

that is, if and only if $n \leq 8$.

Then, if $n \leq 8$ by (2.66), (2.70), (2.75) and (2.76), we obtain

$$\begin{aligned} \left| \iint_{\mathcal{O}_1\times(0,T)} \hat{v}^1 \eta \, dx\, dt \right| &\leq C \left(||h||^2_{\mathcal{O}_{1,d}\times(0,T)} + ||F''||_\infty ||h||^2_{L^\infty(0,T;L^{2s'}(\Omega))} ||\phi||_{L^1(0,T;L^s(\Omega))} \right) \\ &\leq C(1 + ||\phi||_{L^1(0,T;L^s(\Omega))})||\hat{v}^1||_{\mathcal{H}_1} \\ &\leq C(1 + ||y||_{L^2(Q)})||\hat{v}^1||_{\mathcal{H}_1} \\ &\leq C_1(1 + ||f||_{L^2(\mathcal{O}\times(0,T))})||\hat{v}^1||_{\mathcal{H}_1}, \end{aligned}$$

which proves (2.74).

Thus, by (2.73) and (2.74), we have

$$D_1^2 J_1(f;v^1,v^2)\cdot(\hat{v}^1,\hat{v}^1) \geq [\mu_1 - C_1(1 + ||f||_{L^2(\mathcal{O}\times(0,T))})]||\hat{v}^1||^2_{\mathcal{H}_1} \quad \forall \hat{v}^1 \in \mathcal{H}_1.$$

Analogously, we get a constant $C_2 > 0$, such that

$$D_2^2 J_2(f;v^1,v^2)\cdot(\hat{v}^2,\hat{v}^2) \geq [\mu_2 - C_2(1 + ||f||_{L^2(\mathcal{O}\times(0,T))})]||\hat{v}^2||^2_{\mathcal{H}_2} \quad \forall \hat{v}^2 \in \mathcal{H}_2.$$

Finally, noting that C_1 and C_2 are independent of μ_i, for μ_i large enough we have that $\mu_i - C_i(1 + ||f||_{L^2(\mathcal{O}\times(0,T))})$ is a positive constant. In other words, (v^1,v^2) minimizes simultaneously the functionals J_i and (v^1,v^2) is a Nash equilibrium.

2.4 Final comments

In this section we indicate several problems related to hierarchic control and exact controllability that can be analyzed.

2.4.1 Boundary controllability

In [50], the author analyzes the hierarchic control of a hyperbolic equation with one leader and one follower and he solves the problem of approximate controllability. The boundary exact controllability problem with at least two followers is still not completely understood.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with boundary Γ of class C^2 and let $\Gamma_0,\ \Gamma_1,\ \Gamma_2 \subset \Gamma$ be nonempty subsets. Let us consider the linear system

$$\begin{cases} y_{tt} - \Delta y + a(x,t)y = 0 & \text{in} \quad Q, \\ y = f 1_{\Gamma_0} + v^1 1_{\Gamma_1} + v^2 1_{\Gamma_2} & \text{on} \quad \Sigma, \\ y(\cdot,0) = y^0,\ y_t(\cdot,0) = y^1 & \text{in} \quad \Omega, \end{cases} \tag{2.77}$$

the cost functionals

$$J_i(f, v^1, v^2) := \frac{\alpha_i}{2} \iint_{\Gamma_{i,d}\times(0,T)} |y - y_{i,d}|^2 \, dx\, dt + \frac{\mu_i}{2} \iint_{\Gamma_i\times(0,T)} |v^i|^2 \, dx\, dt, \tag{2.78}$$

where the $\Gamma_{i,d} \subset \Gamma$ are nonempty $y_{i,d} \in L^2(\Gamma_{i,d} \times (0,T))$ are given and $\alpha_i,\ \mu_i > 0$ and the main functional

$$J(f) := \frac{1}{2} \iint_{\Gamma_0\times(0,T)} |f|^2 \, dx\, dt, \tag{2.79}$$

Following similar arguments to those in Section 2.2, we can prove the existence and uniqueness of a Nash equilibrium for each f. We can also deduce an optimality system similar to (3.15). However, at present we do not know whether an approppriate observability inequality holds for the solutions of the corresponding adjoint. The hierarchic exact controllability of (2.77)–(2.78) is therefore an open question.

2.4.2 On the nonlinearity

In this work we have considered nonlinear terms locally Lischitz-continuous. We could think of other assumptions on F. For example, in [33] the authors consider nonlinear $F \in C^1(\mathbb{R})$ satisfying the following condition

$$\limsup_{s\to+\infty} \frac{F(s)}{|s| \cdot \log^{\frac{1}{2}} |s|} = 0.$$

Following the arguments in Section 2.3, we can obtain an optimality system similar to (1.90). However, in this case, will show the function F', on which we have no information. The analysis of hierarchic control in the context of the exact controllability for hyperbolic equations with nonlinear terms of this type is an open question.

Capítulo 3

Hierarchic control for Burgers equation via Stackelberg–Nash strategy

Hierarchic control for Burgers equation via Stackelberg–Nash strategy

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Abstract. We present some results on the null controllability of the Burgers equation. We analyze the hierarchic control through Stackelberg–Nash strategy, where we consider one leader and two followers. To each leader we associate a Nash equilibrium corresponding to a bi-objective optimal control problem; then we look for a leader that solves the null controllability problem. We prove linear case and we use a fixed point method to solve the semilinear problem.

3.1 Introducton

The Burgers' equation is the simplest of evolution PDE's that describe a fluid flux. This equation was introduced in the paper [11] by Burgers. The study of this equation is important to investgate proprieties of systems governed by more complex equations in higher dimensions. Por example, the analysis of the controllability for Burgers' equation has been used to try understand questions on the controllability of Navier–Stokes system.

3.1.1 Statement of the problem

Let $T > 0$ be and consider the nonempty open sets $\mathcal{O}$, $\mathcal{O}_1$, $\mathcal{O}_2 \subset (0,1)$, with $0 \notin \overline{\mathcal{O}}$. We introduce the Burgers' system:

$$\begin{cases} y_t - y_{xx} + yy_x = f1_{\mathcal{O}} + v^1 1_{\mathcal{O}_1} + v^2 1_{\mathcal{O}_2} & \text{in} \quad (0,1)\times(0,T), \\ y(0,t) = y(1,t) = 0 & \text{on} \quad (0,T), \\ y(x,0) = y^0(x) & \text{in} \quad (0,1), \end{cases} \tag{3.1}$$

where the controls are f, v^1, v^2, the state is y and with the notation 1_A we denote the characteristic function of the set A.

The controllability of systems as in (3.1) has been extensively analyzed in the last years. In [34], it is shown that, in general, a stationary solution of system as (3.1) with large L^2-norm cannot be reached (not even approximately) at any time T. Similar questions are analyzed in [15].

The local null conrollability for Burgers equation is investigate in [26], where was obtained explicit sharp estimates of the minimal time of controllability $T(r)$ of initial data with L^2-norm less or igual r. Now in [7] is obtained results on the local null controllability of the Burgers-alpha model.

In a more recent work [53] the author, considering two controls forces, proves that one can drive the solution of the corresponding Burgers equation from any initial state y^0 in $L^2(0,1)$ to null state in any time $T > 0$.

When we have a problem multi-objective, for example, beyond to obtain that solves the controllability problem we want that the associate solution not go away too of a given function. For to study problem of that type generaly we use the hierarchic control introduced by Lions in [50], which cosists in the process of leader (independent control) and follower (dependent on leader). In [50] was considered one leader and one follower and applied the Stackelberg optimization, [61], to solve a bi-objective optimal approximate controllability problem for a hyperbolic equation. In [51], similar results are obtained for a parabolic system.

The hierarchic control using the Stackelberg–Nash strategy was introduced in [16], where was associated the strategies of cooperative optimization of Stackelberg and of non-cooperative optimization of Nash. The authors has been considered this strategy to solve a approximate controllability problem for a parabolic system with a leader and a number N of followers. Another references on this strategy, but in the framework of the exact controllability for trajectories of the parabolic system are [1], [3] and [6].

The objective this paper is analyze the null controllability of the system (3.1) in the context of the hierarchic control applying the Stackelberg–Nash strategy with leader f and followers v^1 and v^2.

In sequel, to describe the methodology, we consider the cost functionals for the followers:

$$J_i(f; v^1, v^2) := \frac{\alpha_i}{2} \iint_{\mathcal{O}_{i,d}\times(0,T)} |y - y_{i,d}|^2 \, dx \, dt + \frac{\mu_i}{2} \iint_{\mathcal{O}_i\times(0,T)} |v^i|^2 \, dx \, dt, \tag{3.2}$$

and main functional

$$J(f) := \frac{1}{2} \iint_{\mathcal{O}\times(0,T)} |f|^2 \, dx \, dt,$$

where $\mathcal{O}_{i,d} \subset (0,1)$ is a open nonempty set, $\alpha_i, \mu_i > 0$ are constants and $y_{i,d} = y_{i,d}(x,t)$ is a given function in $L^2(\mathcal{O}_{i,d} \times (0,T))$.

For each leader control f choosing we look for an pair (v^1, v^2) that minimize, simultaneously, the functionals J_i, that is,

$$J_1(f; v^1, v^2) = \min_{\hat{v}^1} J_1(f; \hat{v}^1, v^2) \quad \text{and} \quad J_2(f; v^1, v^2) = \min_{\hat{v}^2} J_2(f; v^1, \hat{v}^2). \tag{3.3}$$

An pair $(v^1, v^2) = (v^1(f), v^2(f))$ that satisfies (3.3) is called Nash equilibrium for the functionals J_i.

Note that when J_i is convex, $i = 1, 2$, (3.3) is equivalent to

$$J_i'(f; v^1, v^2) \cdot \hat{v}^i = 0 \quad \forall\, \hat{v}^i \in L^2(\mathcal{O}_i \times (0, T)), \tag{3.4}$$

but, in general, as (3.1) is nonlinear the equivalence between (3.3) and (3.4) is not true. Then, we must to prove that an pair (v^1, v^2) satisfying (3.4) is unique and we have

$$\left\langle J''(f; v^1, v^2), \hat{v}^i \right\rangle > 0 \quad \forall\, \hat{v}^i \in L^2(\mathcal{O}_i \times (0, T)). \tag{3.5}$$

Thus, (v^1, v^2) is a Nash equilibrium if and only if satisfies (3.4) and (3.5).

After proved the existence and uniqueness of Nash equilibrium for each leader $f \in L^2(\mathcal{O} \times (0, T))$, we look for to prove the null controllability for the system (3.1). More precisely, we prove that there exists $f \in L^2(\mathcal{O} \times (0, T))$ such that

$$J(f) = \min_{\hat{f}} J(\hat{f}) \tag{3.6}$$

and

$$y(\cdot, T) = 0 \quad \text{in} \quad (0, 1). \tag{3.7}$$

In (3.1), we can think in the state $y(x, t)$ as a one dimensional velocity of a fluid and $y^0(x)$ as a initial datum. Then, roughly speaking, the problem that we will analyze has the following interpretation: we want to act with minimal control in a set of paticles of the fluid of such a way that the velocity of the all fluid, which is $y^0(x)$ initialy, vanish at time $T > 0$, but we also want to act with minimal control, during the all time, for that determined set of particles of fluid the velocity $y(x, t)$ does not go away too much from the velocity $y_{i,d}$.

3.1.2 Main results

In the first reult we have conditons to guarantee the existence and uniqueness of the Nash equilibrium.

Theorem 7 *Suppose μ_i large enough. Then, for each $f \in L^2(\mathcal{O} \times (0,T))$ given, there exists a Nash Equilibrium pair $(v^1(f), v^2(f))$ for the functionals J_i.*

The local null controllability with the initial data in $H_0^1(0,1)$ given by the result:

Theorem 8 *Suppose $y^0 \in H_0^1(0,1)$ with $||y^0||_{H_0^1(0,1)} \leq r$ for some $r > 0$ and $\mathcal{O}_{i,d} \cap \mathcal{O} \neq \emptyset$, $i = 1, 2$. Assume that one of the following conditions holds:*

$$\mathcal{O}_{1,d} = \mathcal{O}_{2,d} := \mathcal{O}_d, \tag{3.8}$$

or

$$\mathcal{O}_{1,d} \cap \mathcal{O} \neq \mathcal{O}_{2,d} \cap \mathcal{O}. \tag{3.9}$$

If the constants $\mu_i > 0$ $(i = 1, 2)$ are large enough, and there exists a positive function $\rho = \rho(t)$, which decay exponentialy to 0 when $t \to T^-$, such that, functions $y_{i,d} \in L^2(\mathcal{O}_{i,d} \times (0,T))$ $(i = 1, 2)$ satisfies

$$\iint_{\mathcal{O}_{i,d}\times(0,T)} \rho^{-2}\, y_{i,d}^2 \, dx\, dt < +\infty, \quad i = 1, 2, \tag{3.10}$$

then, there exist a control $f \in L^2(\mathcal{O} \times (0,T))$ and a associated Nash equilibrium $(v^1(f), v^2(f))$ such that has one (3.6) *and the corresponding solution to* (3.1) *satisfies* (3.7).

Since that we want to drive the solution of (3.1) to null state in the time $T > 0$ without that the state y go away of the functions $y_{i,d}$, the hypothesis (3.10) is natural.

In view of the result obtained in [26] for $y^0 \in L^2(0,1)$ with $||y^0||_{L^2(0,1)} \leq r$, $r > 0$, we must have a minimal time for null controllability $T(r) > 0$ with

$$C_0 \log(1/r)^{-1} \leq T(r) \leq C_1 \log(1/r)^{-1},$$

for suitables constants C_0, $C_1 > 0$. Now we present a result on local null controllability with initial data in $L^2(0,1)$.

Theorem 9 *Suppose $y^0 \in L^2(0,1)$ with $||y^0||_{L^2(0,1)} \leq r$ for some $r > 0$ and $T \geq T(r)$. Assuming as in the Theorem 7 that the conditions* (3.8), (3.9) *holds, μ_i is large enough and there exists a function ρ such that the hypothesis* (3.10), *then there exists a control $f \in L^2(\mathcal{O} \times (0,T))$ and a associated Nash equilibrium $(v^1(f), v^2(f))$ such that the corresponding solution to* (3.1) *satisfies* (3.7).

The rest this paper is organized as follows. In Section 3.2, we prove the Theorem 7. In Section 3.3, we concern in the prove of the Theorems 8 and 9. Finally, in Section 3.4, we comment on some open problems.

3.2 Existence and uniqueness of Nash equilibrium

In this section we will prove the Theorem 7, that is, for each $f \in L^2(\mathcal{O} \times (0,T))$ we will obtain a unique Nash equilibrium pair $(v^1(f), v^2(f))$ for the functionals J_i. By defintion this mean that $(v^1(f), v^2(f))$ is the solution to minimal problem (3.3), which is equivalent to satisfy (3.4) and (3.5), as we has mentioned above. Firstly, we will characterize the pair satisfying (3.4) by a optimality system. Then, after to prove the well posedness that system, we will verify that the pair satisfies (3.5).

Suppose that there exists an pair (v^1, v^2) satisfying (3.4). Then, for $i = 1, 2$, we have

$$\alpha_i \iint_{\mathcal{O}_{i,d}\times(0,T)} (y - y_{i,d}) w^i \, dx\, dt + \mu_i \iint_{\mathcal{O}_i\times(0,T)} v^i \hat{v}^i \, dx\, dt = 0 \quad \forall\, \hat{v}^i \in L^2(\mathcal{O}_i \times (0,T)), \tag{3.11}$$

where w^i is the Gateaux derivative of y in relation to v^i in the direction of $\hat{v}^i$. Thus, w^i is solution to

$$\begin{cases} w^i_t - w^i_{xx} + y w^i_x + w^i y_x = v^i 1_{\mathcal{O}_i} & \text{in} \quad (0,1)\times(0,T), \\ y(0,t) = y(1,t) = 0 & \text{on} \quad (0,T), \\ y(x,0) = 0 & \text{in} \quad (0,1). \end{cases} \tag{3.12}$$

We introduce the adjoint to (3.12)

$$\begin{cases} -\phi^i_t - \phi^i_{xx} - y\phi^i_x = \alpha_i (y - y_{i,d}) 1_{\mathcal{O}_{i,d}} & \text{in} \quad (0,1)\times(0,T), \\ \phi(0,t) = \phi(1,t) = 0 & \text{on} \quad (0,T), \\ \phi(x,T) = 0 & \text{in} \quad (0,1), \end{cases} \tag{3.13}$$

where the right-hand side in $(3.27)_1$ is motivated from (3.11). Using the PDE's in (3.12) and (3.27) we deduce

$$\alpha_i \iint_{\mathcal{O}_{i,d}\times(0,T)} (y - y_{i,d}) w^i \, dx\, dt = \mu_i \iint_{\mathcal{O}_i\times(0,T)} \phi^i \hat{v}^i \, dx\, dt \quad \forall \hat{v}^i \in L^2(\mathcal{O}_i \times (0,T)),$$

then, replacing in (3.11), we see that

$$\iint_{\mathcal{O}_i\times(0,T)} (\mu_i v^i + \phi^i) \hat{v}^i \, dx\, dt = 0 \quad \forall \hat{v}^i \in L^2(\mathcal{O}_i \times (0,T)). \tag{3.14}$$

Thus, we get the following optimality system

$$\begin{cases} y_t - y_{xx} + y y_x = f 1_{\mathcal{O}} - \displaystyle\sum_{i=1}^{2} \frac{1}{\mu_i} \phi^i 1_{\mathcal{O}_i}, & \text{in} \quad (0,1)\times(0,T), \\ -\phi^i_t - \phi^i_{xx} - y\phi^i_x = \alpha_i (y - y_{i,d}) 1_{\mathcal{O}_{i,d}}, & \text{in} \quad (0,1)\times(0,T), \\ y(0,t) = y(1,t) = \phi^i(0,t) = \phi^i(1,t) = 0, & \text{on} \quad (0,T), \\ y(x,0) = y^0(x),\ \phi^i(x,T) = 0 & \text{in} \quad (0,1), \end{cases} \tag{3.15}$$

$$v^i = -\frac{1}{\mu_i}\phi^i\Big|_{\mathcal{O}_i\times(0,T)}. \tag{3.16}$$

We will prove that the system (3.15) is well posed. Fix $\tilde{y} \in L^2(0,T;H_0^1(0,1))$ and consider the system

$$\begin{cases} y_t - y_{xx} + yy_x = f1_{\mathcal{O}} - \sum_{i=1}^{2}\frac{1}{\mu_i}\phi^i 1_{\mathcal{O}_i}, & \text{in} \quad (0,1)\times(0,T), \\ -\phi_t^i - \phi_{xx}^i - \tilde{y}\phi_x^i = \alpha_i(y - y_{i,d})1_{\mathcal{O}_{i,d}}, & \text{in} \quad (0,1)\times(0,T), \\ y(0,t) = y(1,t) = \phi^i(0,t) = \phi^i(1,t) = 0, & \text{on} \quad (0,T), \\ y(x,0) = y^0(x),\ \phi^i(x,T) = 0 & \text{in} \quad (0,1), \end{cases} \tag{3.17}$$

Note that in (3.17) the equation are not conjugate, we can obtain a unique solution

$$\left(y,\, \phi^1,\, \phi^2\right) \in \left[L^\infty(0,T;L^2(0,1)) \cap L^2(0,T;H_0^1(0,1))\right]^3.$$

Multiplying in $(3.17)_1$ by y and integrating in $(0,1)$, using integration by parts, we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}|y|^2_{L^2(0,1)} + |y_x|^2_{L^2(0,1)} \leq |y|^2_{L^4(0,1)} + \frac{1}{4}|y_x|^2_{L^2(0,1)} + \frac{1}{2}\left(|f|^2_{L^2(\mathcal{O})} + |y|^2_{L^2(0,1)}\right) \\ + \sum_{i=1}^{2}\frac{1}{\mu_i^2}|\phi^i|^2_{L^2(\mathcal{O}_i)} + \frac{1}{2}|y|^2_{L^2(0,1)}, \end{aligned} \tag{3.18}$$

on the other hand

$$|y|^2_{L^4(0,1)} \leq C\,|y_x|_{L^2(0,1)} \cdot |y|_{L^2(0,1)} \leq \frac{1}{4}\,|y_x|^2_{L^2(0,1)} + C^2\,|y|^2_{L^2(0,1)},$$

then, replacing in (3.18)

$$\frac{d}{dt}|y|^2_{L^2(0,1)} + |y_x|^2_{L^2(0,1)} \leq 2\left(C^2+1\right)|y|^2_{L^2(0,1)} + |f|^2_{L^2(\mathcal{O})} + \sum_{i=1}^{2}\frac{1}{\mu_i^2}|\phi^i|^2_{L^2(\mathcal{O}_i)}. \tag{3.19}$$

Integrating in $(0,t)$, $t \in (0,T]$ we obtain

$$\begin{aligned} |y(t)|^2_{L^2(0,1)} + \int_0^t |y_x(s)|^2_{L^2(0,1)}\,ds \leq |y^0|^2_{L^2(0,1)} + 2\left(C^2+1\right)\int_0^t |y(s)|^2_{L^2(0,1)}\,ds \\ + ||f||^2_{L^2(\mathcal{O}\times(0,T))} + \sum_{i=1}^{2}\frac{1}{\mu_i^2}||\phi^i||^2_{L^2(\mathcal{O}_i\times(0,T))}, \end{aligned}$$

thus, by Gronwall's lemma we get

$$||y||^2_{L^\infty(0,T;L^2(0,1))} + ||y||^2_{L^2(0,T;H_0^1(0,1))} \leq C_1 + C\sum_{i=1}^{2}\frac{1}{\mu_i^2}||\phi^i||^2_{L^2(\mathcal{O}_i\times(0,T))}, \tag{3.20}$$

where $C_1 \geq \left(|y^0|^2_{L^2(0,1)} + ||f||^2_{L^2(\mathcal{O}\times(0,T))}\right)$.

Now multiplying in $(3.17)_2$ by ϕ^i and integrating in $(0,1)$, using integration by parts, we have

$$-\frac{1}{2}\frac{d}{dt}|\phi^i|^2_{L^2(0,1)} + |\phi^i_x|^2_{L^2(0,1)} - \int_0^1 \tilde{y}\phi^i_x\phi^i\,dx = \alpha_i \int_{\mathcal{O}_{i,d}} (\tilde{y} - y_{i,d})\,\phi^i dx, \tag{3.21}$$

but,

$$\int_0^1 \tilde{y}\phi^i_x\phi^i\,dx \frac{1}{4}|\phi^i_x|^2_{L^2(0,1)} + \frac{C}{2}|\tilde{y}_x|^2_{L^2(0,1)} \cdot |\phi^i|^2_{L^2(0,1)}$$

and

$$\alpha_i \int_{\mathcal{O}_{i,d}} (\tilde{y} - y_{i,d})\,\phi^i\,dx \leq C\alpha_i^2 \left(\int_0^1 |\tilde{y}|^2 dx + \int_{\mathcal{O}_{i,d}} |y_{i,d}|^2 dx \right) + \frac{1}{4}\int_0^1 |\phi^i_x|^2 dx,$$

then, replacing in (3.21) and integrating in (t,T), $t \in [0,T)$, we get

$$\begin{aligned} |\phi^i(t)|^2_{L^2(0,1)} + \int_t^T |\phi^i_x(s)|^2_{L^2(0,1)}\,ds \leq\ & C\left(||\tilde{y}||^2_{L^2(0,1)\times(0,T)} + ||y_{i,d}||^2_{L^2(\mathcal{O}_{i,d}\times(0,T))} \right) \\ & + \int_t^T |y_x(s)|^2_{L^2(0,1)} \cdot |\phi^i(s)|^2_{L^2(0,1)}\,ds, \end{aligned}$$

thus, by Gronwall's lemma, we obtain

$$\begin{aligned} & ||\phi^i||^2_{L^\infty(0,T;L^2(0,1))} + ||\phi^i||^2_{L^2(0,T;H^1_0(0,1))} \\ & \leq C\widetilde{C}\left(||\tilde{y}||^2_{L^2((0,1)\times(0,T))} + ||y_{i,d}||^2_{L^2(\mathcal{O}_{i,d}\times(0,T))} \right) \end{aligned} \tag{3.22}$$

where $\widetilde{C} = e^{||\tilde{y}||^2_{L^2(0,T;H^1_0(0,1))}}$. From (3.20) and (3.22) we deduce

$$||y||^2_{L^\infty(0,T;L^2(0,1))} + ||y||^2_{L^2(0,T;H^1_0(0,1))} \leq C_1 + \frac{\widetilde{C}}{\min\{\mu_1,\mu_2\}}\left(C_2 + C||\tilde{y}||^2_{L^2((0,1)\times(0,T))} \right), \tag{3.23}$$

where $C_2 \geq C||y_{i,d}||^2_{L^2(\mathcal{O}_{i,d}\times(0,T))}$, $i = 1,\, 2$.

Now we consider $r > \sqrt{C_1}$ and the set

$$K := \{z \in L^2(0,T;H^1_0(0,1));\ ||z||_{L^2(0,T;H^1_0(0,1))} \leq r\}.$$

Then, taking

$$\min\{\mu_1,\,\mu_2\} \geq \sqrt{\frac{(C_2 + C\,r^2)\,e^{r^2}}{r^2 - C_1}},$$

we can define the operator $\Lambda : K \to K$, with $\Lambda(\tilde{y}) := y$, where $(y,\,\phi^1,\,\phi^2)$ is solution to (3.17). Note that Λ is a compact operator, in fact, K is a bounded subset of $L^2(0,T;H^1_0(0,1))$ which is compactly immerse in $L^2(0,T;L^2(0,1)) = L^2((0,1)\times(0,T))$ and y depends continuously on ϕ^i, then, Λ is the composition od the operators

$$\begin{array}{ccccc} K & \xrightarrow{F_1} & L^2((0,1)\times(0,T)) & \xrightarrow{F_2} & K, \\ \tilde{y} & \mapsto & \phi^i & \mapsto & y, \end{array}$$

where F_1 is a compact operator and F_2 is a continuous operator.

Hence, by Schauder's Theorem, there exists a fixed point $y \in K$ to Λ, that is, $\Lambda(y) = y$. Thus, $(y,\, \phi^1,\, \phi^2)$ is solution to (3.15). To prove the uniqueness of solution, suppose that $(y^1,\, \phi^{1,1},\, \phi^{1,2})$ and $(y^2,\, \phi^{2,1},\, \phi^{2,2})$ are solutions to (3.15). Then, considering $y = y^1 - y^2$ and $\phi^i = \phi^{1,i} - \phi^{2,i}$, we have that $(y,\, \phi^1,\, \phi^2)$ is solution to

$$\begin{cases} y_t - y_{xx} + \dfrac{1}{2}\left(yy^1 + yy^2\right)_x = -\displaystyle\sum_{i=1}^{2} \frac{1}{\mu_i}\phi^i 1_{\mathcal{O}_i}, & \text{in} \quad (0,1)\times(0,T), \\ -\phi^i_t - \phi^i_{xx} - y\phi^i_x = \alpha_i\, y\, 1_{\mathcal{O}_{i,d}}, & \text{in} \quad (0,1)\times(0,T), \\ y(0,t) = y(1,t) = \phi(0,t) = \phi(1,t) = 0, & \text{on} \quad (0,T), \\ y(x,0) = 0,\ \phi(x,T) = 0 & \text{in} \quad (0,1), \end{cases} \tag{3.24}$$

Multiplying in $(3.24)_1$ by y and integrating in $(0,1)$, applying integration by parts, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}|y(t)|^2_{L^2(0,1)} + |y_x(t)|^2_{L^2(0,1)} - \frac{1}{2}\int_0^1 y\left(y^1 + y^2\right) y_x\, dx \\ &\le \sum_{i=1}^{2} \frac{1}{2\mu_i^2}\int_{\mathcal{O}_i} |\phi^i|^2\, dx + \frac{1}{2}\int_0^1 |y|^2\, dx \\ &\quad + \frac{1}{4}\int_0^1 |y_x|^2\, dx + \frac{1}{4}\int_0^1 |y^1 + y^2|^2 |y|^2\, dx, \end{aligned}$$

Since $y^1,\, y^2 \in L^2(0,T;H^1_0(0,1))$, follows that:

$$t \mapsto \left|y^1(\cdot,t) + y^2(\cdot,t)\right|^2_{L^\infty(0,1)},$$

belongs to $L^1(0,T)$, then, integrating in $(0,t)$, with $t \in (0,T)$, we get:

$$\begin{aligned} &\frac{1}{2}|y(t)|^2_{L^2(0,1)} + \frac{3}{4}\int_0^t |y_x(s)|^2_{L^2(0,1)} \\ &\le \sum_{i=1}^{2} \frac{1}{2\mu_i^2}\, ||\phi^i||^2_{L^2(\mathcal{O}_i\times(0,T))} + \frac{1}{2}\int_0^t |y(s)|^2_{L^2(0,1)}\, ds \\ &\quad + \frac{1}{2}\int_0^t \left|y^1(s) + y^2(s)\right|^2_{L^\infty(0,1)} |y(s)|^2_{L^2(0,1)}\, ds \end{aligned} \tag{3.25}$$

Analogously to we did to get (3.22), we obtain:

$$||\phi^i||^2_{L^2(\mathcal{O}_i\times(0,T))} \le C\left(||y||^2_{L^\infty(0,T;L^2(0,1))} + ||y||^2_{L^2(0,T;H^1_0(0,1))}\right). \tag{3.26}$$

Hence, replacing (3.26) in (3.25) and using the Gronwall's lemma, we get

$$||y||^2_{L^\infty(0,T;L^2(0,1))} + ||y||^2_{L^2(0,T;H^1_0(0,1))} \le \frac{C}{\min\{\mu_1^2,\, \mu_2^2\}}\left(||y||^2_{L^\infty(0,T;L^2(0,1))} + ||y||^2_{L^2(0,T;H^1_0(0,1))}\right),$$

thus, considering $\min\{\mu_1^2,\, \mu_2^2\} > C$, we conclude that $y \equiv 0$ and, as consequence, $\phi^i \equiv 0$, $i = 1, 2$. This prove the uniqueness of solution to (3.15).

As the system (3.15) is well posed, there exists a unique pair $(v^1,\, v^2) \in L^2(\mathcal{O}_1 \times (0,T)) \times L^2(\mathcal{O}_2 \times (0,T))$ to satisfies (3.11). We will prove that the pair $(v^1,\, v^2)$ is a Nash equilibrium, for this, we will prove that this pair satisfies (3.5). That is, already we obtain the critical point $(v^1,\, v^2)$ for the functionals J_i, now using the second derivative test we will prove that it is a minimum point to J_i.

In fact, given $\tilde{v}^1 \in L^2(\mathcal{O}_1 \times (0,T))$ and $s \in \mathbb{R}$, we denote by y^s the solution of the following system

$$\begin{cases} y_t^s - y_{xx}^s + y^s y_x^s = f 1_{\mathcal{O}} + \left(v^1 + \tilde{v}^1\right) 1_{\mathcal{O}_1} + v^2 1_{\mathcal{O}_2}, & \text{in} \quad (0,1) \times (0,T), \\ y(0,t) = y(1,t) = 0, & \text{on} \quad (0,T), \\ y(x,0) = 0, & \text{in} \quad (0,1), \end{cases} \tag{3.27}$$

and let us use the notation $y^s|_{s=0} = y$. For each $\hat{v}^1 \in L^2(\mathcal{O} \times (0,T))$ we have

$$\begin{aligned} D_1 J_1(f; v^1 + s\tilde{v}^1, v^2) \cdot \hat{v}^1 - D_1 J_1(f; v^1, v^2) := \alpha_1 \iint_{\mathcal{O}_{i,d} \times (0,T)} (y^s - y_{i,d})\, w^{1,s}\, dx\, dt \\ - \alpha_1 \iint_{\mathcal{O}_{1,d} \times (0,T)} (y - y_{1,d})\, w^1\, dx\, dt + s\, \mu_1 \iint_{\mathcal{O}_{1,d} \times (0,T)} \tilde{v}^1 \hat{v}^1\, dx\, dt, \end{aligned} \tag{3.28}$$

where $w^{1,s}$ is the derivative of y^s in relation to $v^1 + s\tilde{v}^1$ in the direction of $\hat{v}^1$. Then, $w^{1,s}$ is the solution to

$$\begin{cases} w_t^{1,s} - w_{xx}^{1,s} + w^{1,s} y_x + y w_x^{1,s} = \hat{v}^1 1_{\mathcal{O}_1}, & \text{in} \quad (0,1) \times (0,T), \\ w^{1,s}(0,t) = w^{1,s}(1,t) = 0, & \text{on} \quad (0,T), \\ w^{1,s}(x,0) = 0, & \text{in} \quad (0,1), \end{cases} \tag{3.29}$$

and let us use the notation $w^{1,s}|_{s=0} = w^1$. We have the following adjoint system to (3.29)

$$\begin{cases} -\phi_t^{1,s} - \phi_{xx}^{1,s} - y\phi_x^{1,s} = \alpha_1\, (y^s - y_{1,d})\, 1_{\mathcal{O}_{1,d}}, & \text{in} \quad (0,1) \times (0,T), \\ \phi^{1,s}(0,t) = \phi^{1,s}(1,t) = 0, & \text{on} \quad (0,T), \\ \phi^{1,s}(x,T) = 0, & \text{in} \quad (0,1), \end{cases} \tag{3.30}$$

where the right-hand side in $(3.30)_1$ s motivated by (3.11). From (3.31), (3.29) and (3.30), we get

$$\begin{aligned} D_1 J_1(f; v^1 + s\tilde{v}^1, v^2) \cdot \hat{v}^1 - D_1 J_1(f; v^1, v^2) := \iint_{\mathcal{O}_1 \times (0,T)} \left(\phi^{1,s} - \phi^1\right) \hat{v}^1\, dx\, dt \\ + s\, \mu_1 \iint_{\mathcal{O}_1 \times (0,T)} \tilde{v}^1 \hat{v}^1\, dx\, dt, \end{aligned} \tag{3.31}$$

then, taking the limit as $s \to 0$, we obtain

$$\begin{cases} D_1^2(f; v^1, v^2) \cdot (\tilde{v}^1, \hat{v}^1) = \displaystyle\iint_{\mathcal{O}_1 \times (0,T)} \eta \hat{v}^1 \, dx \, dt + \mu_1 \iint_{\mathcal{O}_1 \times (0,T)} \tilde{v}^1 \hat{v}^1 \, dx \, dt \\ \forall \tilde{v}^1, \hat{v}^1 \in L^2(\mathcal{O} \times (0,T)), \end{cases}$$

in particular,

$$\begin{cases} D_1^2(f; v^1, v^2) \cdot (\tilde{v}^1, \tilde{v}^1) = \displaystyle\iint_{\mathcal{O}_1 \times (0,T)} \eta \, \tilde{v}^1 \, dx \, dt + \mu_1 \iint_{\mathcal{O}_1 \times (0,T)} |\tilde{v}^1|^2 \, dx \, dt \\ \forall \tilde{v}^1 \in L^2(\mathcal{O} \times (0,T)), \end{cases} \tag{3.32}$$

where

$$\eta := \lim_{s \to 0} \frac{1}{s} \left(\phi^{1,s} - \phi^1 \right).$$

Note that also exists the limit

$$h := \lim_{s \to 0} \frac{1}{s} \left(y^s - y \right),$$

and the pair $(h, \, \eta)$ is solution to

$$\begin{cases} h_t - h_{xx} + y h_x + h y_x = \tilde{v}^1 1_{\mathcal{O}_1}, & \text{in} \quad (0,1) \times (0,T), \\ -\eta_t - \eta_{xx} - y \eta_x = \alpha_1 h 1_{\mathcal{O}_{1,d}}, & \text{in} \quad (0,1) \times (0,T), \\ h(0,t) = h(1,t) = \eta(0,t) = \eta(1,t) = 0, & \text{on} \quad (0,T), \\ h(x,0) = 0, \; \eta(x,T) = 0 & \text{in} \quad (0,1). \end{cases} \tag{3.33}$$

Multiplying in $(3.33)_1$ by η and integrating in $(0,1) \times (0,T)$, using integration by parts, we get

$$\iint_{\mathcal{O}_1 \times (0,T)} \eta \, \tilde{v}^1 \, dx \, dt = \alpha_1 \iint_{\mathcal{O}_{1,d} \times (0,T)} |h|^2 \, dx \, dt,$$

then, replacing in (3.32), we obtain

$$\begin{cases} D_1^2(f; v^1, v^2) \cdot (\tilde{v}^1, \tilde{v}^1) = \alpha_1 \displaystyle\iint_{\mathcal{O}_{1,d} \times (0,T)} |h|^2 \, dx \, dt, + \mu_1 \iint_{\mathcal{O}_1 \times (0,T)} |\tilde{v}^1|^2 \, dx \, dt \\ \forall \tilde{v}^1 \in L^2(\mathcal{O}_1 \times (0,T)). \end{cases} \tag{3.34}$$

Hence,

$$D_1^2(f; v^1, v^2) \cdot (\tilde{v}^1, \tilde{v}^1) \geq C ||\tilde{v}^1||^2_{L^2(\mathcal{O}_1 \times (0,T))} \quad \forall \tilde{v}^1 \in L^2(\mathcal{O}_1 \times (0,T)),$$

and analogously,

$$D_1^2(f; v^1, v^2) \cdot (\tilde{v}^1, \tilde{v}^1) \geq C ||\tilde{v}^1||^2_{L^2(\mathcal{O}_1 \times (0,T))} \quad \forall \tilde{v}^1 \in L^2(\mathcal{O}_1 \times (0,T)).$$

Thus, the pair $(v^1, \, v^2)$ is a Nash equilibrium to functionals J_i.

3.3 Null controllability

In this section we will prove the Theorems 7 and 8. In other words, we will obtain the local null controllability to system (3.15) considering initial data in $H_0^1(0,1)$ and $L^2(0,1)$, respectively.

Initially, for each $\bar{y} \in L^\infty((0,1)\times(0,T))$, let us consider the linearized system corresponding (3.15)

$$\begin{cases} y_t - y_{xx} + \bar{y}(x,t)y_x = f1_{\mathcal{O}} - \displaystyle\sum_{i=1}^{2} \frac{1}{\mu_i}\phi^i 1_{\mathcal{O}_i}, & \text{in} \quad (0,1)\times(0,T), \\ -\phi^i_t - \phi^i_{xx} - \bar{y}(x,t)\phi^i_x = \alpha_i(y - y_{i,d})1_{\mathcal{O}_{i,d}}, & \text{in} \quad (0,1)\times(0,T), \\ y(0,t) = y(1,t) = \phi^i(0,t) = \phi^i(1,t) = 0, & \text{on} \quad (0,T), \\ z(x,0) = y^0(x), \ \phi^i(x,T) = 0 & \text{in} \quad (0,1). \end{cases} \tag{3.35}$$

We have the following adjoint system for (3.35)

$$\begin{cases} -\psi_t - \psi_{xx} - (\bar{y}(x,t)\psi)_x = \displaystyle\sum_{i=1}^{2} \alpha_i\gamma^i 1_{\mathcal{O}_{i,d}}, & \text{in} \quad (0,1)\times(0,T), \\ \gamma^i_t - \gamma^i_{xx} + (\bar{y}(x,t)\gamma^i)_x = -\dfrac{1}{\mu_i}\psi 1_{\mathcal{O}_i}, & \text{in} \quad (0,1)\times(0,T), \\ \psi(0,t) = \psi(1,t) = \gamma^i(0,t) = \gamma^i(1,t) = 0, & \text{on} \quad (0,T), \\ \psi(x,T) = \psi^T(x), \ \gamma^i(x,0) = 0 & \text{in} \quad (0,1). \end{cases} \tag{3.36}$$

From (3.35) and (3.36), we get

$$\iint_{\mathcal{O}\times(0,T)} f\,\psi\,dx\,dt = \sum_{i=1}^{2}\alpha_i \iint_{\mathcal{O}_{i,d}\times(0,T)} \gamma^i y_{i,d}\,dx\,dt + \big(y(\cdot,T),\psi^T\big)_{L^2(0,1)} - \big(y^0,\psi(\cdot,0)\big)_{L^2(0,1)}, \tag{3.37}$$

then, we have null controllability if and only if, for each $y^0 \in L^2(0,1)$ given, there exists a control $f \in L^2(\mathcal{O}\times(0,T))$, such that

$$\iint_{\mathcal{O}\times(0,T)} f\psi\,dx\,dt = \sum_{i=1}^{2}\alpha_i \iint_{\mathcal{O}_{i,d}\times(0,T)} \gamma^i y_{i,d}\,dx\,dt - \big(y^0,\psi(\cdot,0)\big)_{L^2(0,1)}.$$

Now, for each $\varepsilon > 0$, we define the functional

$$F_\varepsilon(\psi^T) := \frac{1}{2}\iint_{\mathcal{O}\times(0,T)} |\psi|^2\,dx\,dt + \varepsilon|\psi^T|_{L^2(0,1)} + \big(y^0,\psi(\cdot,0)\big)_{L^2(0,1)} - \sum_{i=1}^{2}\alpha_i \iint_{\mathcal{O}_{i,d}\times(0,T)} \gamma^i y_{i,d}\,dx\,dt.$$

Adapting the proofs of observability found in [1] and [6] to Carleman estimates found in [41], we have that the following result for observability inequality:

Proposition 11 *Suppose that* $\mathcal{O}_{i,d} \cap \mathcal{O} \neq \emptyset$, $i = 1, 2$. *Assume that one of the following conditions holds:*

$$\mathcal{O}_{1,d} = \mathcal{O}_{2,d} = \mathcal{O}_d, \tag{3.38}$$

or

$$\mathcal{O}_{1,d} \cap \mathcal{O} \neq \mathcal{O}_{2,d} \cap \mathcal{O}. \tag{3.39}$$

Then, in both cases, for all $\psi^T \in L^2(\Omega)$, *there are* $C = C(\Omega, \mathcal{O}, \mathcal{O}_{i,d}, \alpha_i, \mu_i, T, ||\bar{y}||_{L^\infty(Q)}) > 0$ *and a weight function* $\rho = \rho(t)$, *which decay exponentially to* 0 *when* $t \to T^-$, *such that for the solution* $(\psi,\ \gamma^1,\ \gamma^2)$ *of* (3.36), *we have*

$$\int_0^1 |\psi(x,0)|^2\, dx + \sum_{i=1}^{2} \iint_{(0,1)\times(0,T)} \rho^2 |\gamma^i|^2\, dx\, dt \leq C \iint_{\mathcal{O}\times(0,T)} |\psi|^2\, dx\, dt. \tag{3.40}$$

By Proposition 11, F_ε is continuous, strictly convex and coercive in $L^2(0,1)$. Then it has a unique minimum $\psi_\varepsilon^T \in L^2(0,1)$. Then, either $\psi_\varepsilon^T = 0$ or for all $\psi^T \in L^2(0,1)$, we have

$$\begin{aligned} \iint_{\mathcal{O}\times(0,T)} \psi_\varepsilon \psi\, dx\, dt + \varepsilon \left(\frac{\psi_\varepsilon^T}{||\psi_\varepsilon^T||}, \psi^T \right)_{L^2(0,1)} + \left(y^0(x), \psi(x,0)\right)_{L^2(0,1)} \\ - \sum_{i=1}^{2} \alpha_i \iint_{\mathcal{O}_{i,d}\times(0,T)} y_{i,d} \gamma^i\, dx\, dt = 0, \end{aligned} \tag{3.41}$$

where $(\psi_\varepsilon, \gamma_\varepsilon^1, \gamma_\varepsilon^2)$ is the solution of (3.36) corresponding to ψ_ε^T. Thus, considering $f = \psi_\varepsilon|_{\mathcal{O}\times(0,T)}$, by (3.37) and (3.41), we get

$$\int_\Omega \left(y_\varepsilon(x,T) + \varepsilon \frac{\psi_\varepsilon^T}{||\psi_\varepsilon^T||} \right) \psi^T(x)\, dx = 0 \quad \forall \psi^T \in L^2(0,1).$$

This way, follows that

$$||y_\varepsilon(\cdot, T)|| \leq \varepsilon. \tag{3.42}$$

Moreover, from (3.40) and (3.41), we deduce

$$||f_\varepsilon||_{L^2(\mathcal{O}\times(0,T))} \leq C \left(\sum_{i=1}^{2} \iint_{\mathcal{O}_{i,d}\times(0,T)} \rho^{-2} |y_{i,d}|^2\, dx\, dt + ||y^0||^2_{L^2(0,1)} \right)^{\frac{1}{2}},$$

where we has used that (3.10) holds.

So, f_ε is uniformly bounded in $L^2(\mathcal{O}\times(0,T))$, then, it possesses a subsequence weakly convergent for some $f \in L^2(\mathcal{O}\times(0,T))$. From regularity of solution of the system (3.35), taking the limit as $\varepsilon \to 0$, by inequality (3.42) and Banach-Steinhaus' Theorem, we conclude that correponding solution $(y,\ \phi^1,\ \phi^2)$ to (3.35) satisfies the null controllability condition $y(\cdot, T) = 0$.

Now we will use this result and fixed point method to prove the Theorem 8.

Proof of Theorem 8. From the linear case, for each $\bar{y} \in L^\infty((0,1)\times(0,T))$ fixed, given $y^0 \in H_0^1(0,1)$ with $||y^0||_{H_0^1(0,1)} \leq r$, for some $r > 0$, if the constants μ_i are large enough we get a minimal control $f = f_{\bar{y}} \in L^2(\mathcal{O}\times(0,T))$ such that the corresponding solution $(y,\, \phi^1,\, \phi^2) = (y_{\bar{y}},\, \phi^1_{\bar{y}},\, \phi^2_{\bar{y}})$ to (3.35) satisfies $y(\cdot, T) = 0$. Moreover, we have that

$$||f_{\bar{y}}||_{L^2(\mathcal{O}\times(0,T))} \leq C\,r \quad \forall\, \bar{y} \in L^\infty((0,1)\times(0,T))$$

and $y \in L^2(0,T;H^2(0,1)) \cap L^\infty(0,T;H_0^1(0,1))$ with

$$||y||_{L^\infty(0,T;H_0^1(0,1))} + ||y||_{L^2(0,T;H^2(0,1))} \leq C\,r\,e^{C||\bar{y}||_{L^\infty((0,1)\times(0,T))}},$$

where $C > 0$. So, in partcular $y = y_{\bar{y}} \in L^\infty((0,1)\times(0,T))$ and we can define the map

$$\Lambda : L^\infty((0,1)\times(0,T)) \to L^\infty((0,1)\times(0,T)), \quad \Lambda(\bar{y}) := y_{\bar{y}}.$$

Let $K > 0$ be a constant, we introduce the set

$$Y := \{\bar{y} \in L^\infty((0,1)\times(0,T)) : ||\bar{y}||_{L^\infty((0,1)\times(0,T))} \leq K\},$$

then, for all $\bar{y} \in Y$, if

$$r \leq K/C\,e^{CK} \tag{3.43}$$

we obtain $y \in Y$. That is, $\Lambda(Y) \subset Y$.

Note that Λ is continuous and since that $L^2(0,T;H^2(0,1)) \cap L^\infty(0,T;H_0^1(0,1)) \hookrightarrow Y$ with compact embedding, Λ maps the space Y into compact subset. Hence, by Shauder's fixed point theorem, there exists $y \in L^2(0,T;H^2(0,1)) \cap L^\infty(0,T;H_0^1(0,1))$ such that $\Lambda(y) = y$. Thus, there exists a minimal control $f \in L^2(\mathcal{O}\times(0,T))$ such that the corresponding solution $(y,\, \phi^1,\, \phi^2)$ to (3.15), with $y^0 \in H_0^1(0,1)$, satisfies $y(\cdot, T) = 0$. This proves the Theorem 8. ■

In sequel we will prove the Theorem 9 using the result of the Theorem 8.

Proof of Theorem 9. We assume that $y^0 \in L^2(0,1)$, with $||y^0||_{L^2(0,1)} \leq r$. Then, considering the trajectory $\tilde{y}$ of the system (3.1), that is, the solution to

$$\begin{cases} \tilde{y}_t - \tilde{y}_{xx} + \tilde{y}\tilde{y}_x = 0 & \text{in} \quad (0,1)\times(0,T), \\ \tilde{y}(0,t) = \tilde{y}(1,t) = 0 & \text{on} \quad (0,T), \\ \tilde{y}(x,0) = y^0(x) & \text{in} \quad (0,1), \end{cases} \tag{3.44}$$

from parabolic regularity, we have $\tilde{y}(\cdot,t) \in H_0^1(0,1)$ for any $t > 0$ and there exists constants τ and M such that

$$||\tilde{y}(\cdot,t)||_{L^\infty(0,1)} \le Mt^{-\frac{1}{4}}||y^0||_{L^2(0,1)} \quad \forall t \in (0,\tau).$$

Then, considering $t_0 = M^4$ and taking $\bar{y}^0 = \tilde{y}(\cdot,t_0)$, we have that

$$\bar{y}^0 \in H_0^1(0,1) \quad \text{and} \quad ||\bar{y}^0||_{L^\infty(0,1)} \le r.$$

Now we consider the system

$$\begin{cases} y_t - y_{xx} + yy_x = f1_{\mathcal{O}} + v^1 1_{\mathcal{O}_1} + v^2 1_{\mathcal{O}_2} & \text{in} \quad (0,1) \times (t_0, t_0+t_1), \\ y(0,t) = y(1,t) = 0 & \text{on} \quad (t_0, t_0+t_1), \\ y(x,0) = \bar{y}^0(x) & \text{in} \quad (0,1), \end{cases} \tag{3.45}$$

where $t_1 = C/\log(1/r)$, where $C > 0$ is a suitable constant such that allow as to obtain a estimate as in (3.43) . Hence, considering the functionals

$$\bar{J}_i(f;v^1,v^2) := \frac{\alpha_i}{2}\iint_{\mathcal{O}_{i,d}\times(t_0,t_0+t_1)} |y - y_{i,d}|^2\,dx\,dt + \frac{\mu_i}{2}\iint_{\mathcal{O}_i\times(t_0,t_0+t_1)} |v^i|^2\,dx\,dt, \tag{3.46}$$

and main functional

$$\bar{J}(f) := \frac{1}{2}\iint_{\mathcal{O}\times(t_0,t_0+t_1)} |f|^2\,dx\,dt,$$

we can apply the Stackelbrg–Nash strategy is this case, where we obtain

$$\bar{f} \in L^2(\mathcal{O}\times(t_0,t_0+t_1))$$

a associate Nash equilibrium

$$(\bar{v}^1,\bar{v}^2) \in L^2(\mathcal{O}_1\times(t_0,t_0+t_1)) \times L^2(\mathcal{O}_2\times(t_0,t_0+t_1))$$

such that the corresponding solution to (3.45) satisfies $y(\cdot,t_0+t_1) = 0$.

Thus, we define the control

$$f(x,t) := \begin{cases} 0, & \text{if} \quad (x,t) \in \mathcal{O}\times(0,t_0); \\ \bar{f}(x,t), & \text{if} \quad (x,t) \in \mathcal{O}\times(t_0,t_0+t_1), \end{cases} \tag{3.47}$$

we have $f \in L^2(\mathcal{O}\times(0,t_0+t_1))$ and obtain a Nash equilibrium $(v^1(f),v^2(f)) \in L^2(\mathcal{O}_1 \times (0,t_0+t_1)) \times L^2(\mathcal{O}_2\times(0,t_0+t_1))$ such that the corresponding solution to (3.15) satisfies $y(\cdot,t_0+t_1) = 0$. Hence, the Theorem 9 is proved. ■

The control f defined in (3.47) is not optimal. But as mentioned it before, by [26] we not have null contrallability for any small time $T > 0$, then is rasonable to consider the control null in a suitable interval.

3.4 Final comments

3.4.1 Stackelberg–Nash strategy with controls in the boundary

A natural and interesting question is the hierarchic control applying Stackelberg–Nash stategy for null controllability of the burgers equation with at last a control, leader or follower, in the boundary. For example, we can consider systems in the form

$$\begin{cases} y_t - y_{xx} + yy_x = v^1 1_{\mathcal{O}_1} + v^2 1_{\mathcal{O}_2} & \text{in} \quad (0,1)\times(0,T), \\ y(0,t) = 0, \quad y(1,t) = f(t) & \text{on} \quad (0,T), \\ y(x,0) = y^0(x) & \text{in} \quad (0,1), \end{cases} \tag{3.48}$$

or

$$\begin{cases} y_t - y_{xx} + yy_x = f 1_{\mathcal{O}} & \text{in} \quad (0,1)\times(0,T), \\ y(0,t) = v^1(t), \quad y(1,t) = v^2(t) & \text{on} \quad (0,T), \\ y(x,0) = y^0(x) & \text{in} \quad (0,1), \end{cases} \tag{3.49}$$

where f is the leader and v^1, v^2 are the followers.

Then, we apply the Stackeberg–Nash strategy using cost functionals similar to (3.2). This way for each leader f fixed, we obtain a Nash equilibrium arguing as in the Section 3.2, but in both cases, the difficult is combine suitables Carleman inequalities to obtain the observability. This situations will be analyzed in a forthcoming paper.

3.4.2 The Navier–Stokes system

The system (3.1) is a one dimensional simplification of the control system associated to the Navier–Stokes system. Let us consider a bounded domain Ω of $\mathbb{R}^N$ ($N = 2$ or $N = 3$), which boundary Γ is regular anough. Let us consider the Navier–Stokes system

$$\begin{cases} y_t - \Delta y + (y\cdot\nabla)y + \nabla p = f 1_{\mathcal{O}} + v^1 1_{\mathcal{O}_1} + v^2 1_{\mathcal{O}_2} & \text{in} \quad Q, \\ \nabla\cdot y = 0 & \text{in} \quad Q, \\ y = 0 & \text{on} \quad \Sigma, \\ y(\cdot,0) = y^0 & \text{in} \quad \Omega, \end{cases} \tag{3.50}$$

where $T > 0$ is given, $Q = \Omega\times(0,T)$, $\Sigma = \Gamma\times(0,T)$ and the sets $\mathcal{O}$, $\mathcal{O}_1$, $\mathcal{O}_2 \subset \Omega$ are nonempty open.

We consider y^0 in the space

$$H := \{z \in L^2(\Omega)^N : \nabla\cdot z = 0 \quad \text{in} \quad \Omega, \quad z\cdot\nu \quad \text{on} \quad \Gamma\},$$

where $\nu(x)$ denotes the outward unit normal to Ω at the point $x \in \Gamma$.

The controllability to system (3.50) has been extensively studied these last years, see for example, [27] and [34], where we can find results on the exact controllability with distributed and boundary controls, respectively.

In the context of the hierarchic control, in [2] the authors has been analized the system (3.50), applying the Stackelberg–Nash strategy, where they already obtained positives results on existence and uniqueness of Nash equilibrium for functionals similar to J_i defined in (3.2). But the null controllability for the (3.50) is still a open question.

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