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Sylvester forms and Rees algebras

por

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João Pessoa – PB
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sob orientação do

Prof. Dr. Aron Simis

Tese apresentada ao Corpo Docente do Programa em Associação de Pós Graduação em Matemática UFPB/UFCG, como requisito parcial para obtenção do título de Doutor em Matemática.

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Resumo

Este trabalho versa sobre a álgebra de Rees de um ideal quase intersecção completa, de co-comprimento finito, gerado por formas de mesmo grau em um anel de polinômios sobre um corpo. Considera-se duas situações inteiramente diversas: na primeira, as formas são monômios em um número qualquer de variáveis, enquanto na segunda, são formas binárias gerais. O objetivo essencial em ambos os casos é obter a profundidade da álgebra de Rees. É conhecido que tal álgebra é raramente Cohen–Macaulay (isto é, de profundidade máxima). Assim, a questão que permanece é quão distante são do caso Cohen–Macaulay. No caso de monômios prova-se, mediante certa restrição, uma conjectura de Vasconcelos no sentido de que a álgebra de Rees é *quase Cohen–Macaulay*. No outro caso extremo, estabelece-se uma prova de uma conjectura de Simis sobre formas binárias gerais, baseada no trabalho de Huckaba–Marley e em um teorema sobre a filtração de Ratliff–Rush. Além disso, apresenta-se um par de conjecturas mais fortes que implicam a conjectura de Simis, juntamente com uma evidência sólida.

Palavras-chave: álgebra de Rees, número de redução, formas de Sylvester, função de Hilbert, ideais iniciais, quase Cohen-Macaulay, mapping cone.

Abstract

This work is about the Rees algebra of a finite colength almost complete intersection ideal generated by forms of the same degree in a polynomial ring over a field. We deal with two situations which are quite apart from each other: in the first the forms are monomials in an unrestricted number of variables, while the second is for general binary forms. The essential goal in both cases is to obtain the depth of the Rees algebra. It is known that for such ideals the latter is rarely Cohen–Macaulay (i.e., of maximal depth). Thus, the question remains as to how far one is from the Cohen–Macaulay case. In the case of monomials one proves under certain restriction a conjecture of Vasconcelos to the effect that the Rees algebra is almost Cohen–Macaulay. At the other end of the spectrum, one proposes a proof of a conjecture of Simis on general binary forms, based on work of Huckaba–Marley and on a theorem concerning the Ratliff–Rush filtration. Still within this frame, one states a couple of stronger conjectures that imply Simis conjecture, along with some solid evidence.

Keywords: Rees algebra, reduction number, Sylvester forms, Hilbert function, initial ideals, almost Cohen–Macaulayness, mapping cone.

Contents

Introduction

Let $R := k[x_1, \ldots, x_n]$ denote a polynomial ring over a field k. Ideals $I \subset R$ that are almost complete intersections play a critical role in elimination theory of both plane and space parameterizations, while their Rees algebras encapsulate some of the most common tools in both theoretic and applied elimination. Finding a minimal set of generators of the presentation ideal of the Rees algebra – informally referred to as *minimal relations* – is a tall order in commutative algebra. It is tantamount to obtaining minimal syzygies of the powers of I, a problem that can be suitably translated into elimination theory as the method of moving lines, moving surfaces, and so forth (see. e.g., [9]). One idea to reach for minimal relations is to draw them in some sort of recursive way out of others already known. One such recurrence is known as the method of Sylvester forms, where one produces certain square *content matrices* which express the inclusion of two ideals in terms of given sets of generators, where the included ideal is generated by old relations. As an easy consequence of Cramer's rule, the determinant of such a matrix will be a relation. As is well-known, telling whether these relations do not vanish - let alone that they are new minimal relations – is one major problem. The determinants of these content matrices, or a construction that generalizes them, are called *Sylvester forms*. The appearance of Sylvester forms goes back at least to the late sixties in a paper of Wiebe ([42]; see also [10]). They have been largely used in many sources, such as [6–10, 16, 20–22, 38].

Let us succinctly review the main advances in these algebraic methods in the recent history. A good starting point is a couple of conjectures stated in [8], one of which asked whether the minimal relations of a finite colength almost complete intersection in $R = k[x, y]$, generated in degree 4, and with (two) independent syzygies of degree 2, are iterated Sylvester forms – in the terminology of the implicitization school, the case where $\mu = 2$.

The question, originally inspired from some partial affirmative cases by Sederberg, Goldman and Du and, independently, by Jouanolou, both in 1997, soon captured the interest of various authors. In [20] the case of $\mu = 1$ was taken up and it was conjectured that in arbitrary degree the relations are generated by iterated Sylvester forms. This new conjecture was proved in [10] using quite involved homological machinery, including local cohomology and spectral sequences. This case of a finite colength almost complete intersection in arbitrary degree d in 2 variables, with generating syzgies of (standard) degrees 1 and $d-1$, has been further thoroughly examined in [5], [26], [22] and [38]. The methods employed are of varied nature, each enriching the commutative algebra involved in elimination theory.

Among the most interesting conditions on a Rees algebra is the Cohen–Macaulay property, which is knowingly a certain regularity condition on the ideal. Unfortunately, elimination of plane or space parameterizations in high degrees does not commonly lead to a Cohen–Macaulay Rees algebra, as is already the case of binary such parameterizations. Since the property has anyway a difficult translation back into elimination, why should one care about it? Well, as it turns, the property ties beautifully with several other properties and current notions of commutative algebra. In this vein, having a little less than the property itself may be useful. Besides, in the $\mu = 1$ case discussed above it turns out that the Rees algebra is *almost Cohen–Macaulay*, as has been shown in the references mentioned in connection to this case.

This is how the almost Cohen–Macaulay property comes into the picture, namely, as the next best situation from the homological point of view. Looking for this property or its failure is the driving force behind this work.

Now, as usual, looking for some preliminary evidence or some propaedeutics leads one to envisage the case of a monomial parametrization. From the strict angle of elimination theory, where one looks for the implicit equation, this situation is hardly of any interest. On the other hand, quite generally, the relations are binomials. Thus, the interest remains as to whether the minimal (binomial) relations can be obtained by iteration of Sylvester forms and how unique is this procedure. In 2013 W. Vasconcelos formulated the conjecture that the Rees algebra of an Artinian almost complete intersection $I \subset R$ generated by monomials is almost Cohen-Macaulay. For the binary case (i.e., for $n = 2$) a result of M. Rossi and I. Swanson ([36, Proposition 1.9]) implies an affirmative answer to the conjecture, with the machinery of the Ratliff–Rush filtration. Recently, different proofs were established in the binary case of monomials of the same degree as a consequence of work by T. B. Cortadellas and C. D'Andrea ([7]), and independently, of work by A. Simis and S. Tohăneanu ([38]).

Here one tackles the case of a monomial parameterization in arbitrary number of variables firstly with an extra condition on the degrees of the monomials, called *uniformity*. The ternary case has been established in ([38]). In this work we assume an arbitrary number of variables, Under this condition, we answer affirmatively the stated conjecture. In our opinion this contributes a significant step toward the general case, since one has in mind a couple of procedures to reducing the case of general exponents to this one. This is the motivation for Chapter 2 of this presentation. Although sufficiently tighter than the problem of ideals generated by arbitrary forms – a situation still lacking a bona fide conjecture – the general case of monomials of arbitrary degrees and number of variables may require an additional tour de force beyond the facilitation provided by the methods of the present work.

What about the situation where the given monomials are of the same degree throughout? It would look like this is nearly a "geometric" situation and hence more tools at our disposal. One is lead to ask whether the rational map defined by the linear system spanned by these monomials has any perfunctory properties, such as being birational onto the image. A strong asset coming from birationality in the situation of a finite colength almost complete intersection $I \subset k[x_1, \ldots, x_n]$ of forms of the same degree is an exact formula for the value of the Chern number $e_1(I)$ (see [21, Proposition 3.3] and the references thereon).

And in fact, quite generally, one main tool in the binary case of an Artinian almost complete intersections I of forms of the same degree is birationality. This vein has been largely explored in some of the references quoted before.

The other two tools are the Ratliff–Rush filtration theory and the Huckaba–Marley criterion using a minimal reduction of I. While the Ratliff-Rush filtration gives no insight into the conjectured property of the Rees algebra beyond the two variables case, using the criterion of Huckaba–Marley, would probably require as much calculation and besides lead one into no reasonable bound to manage the partial lengths. We add the fact that even when the uniformity assumption degenerates into equigrading, birationality for more than two variables is an issue, and hence computing the first Hilbert coefficient of R/I becomes a hardship.

To use Huckaba–Marley criterion we studied some inequalities involving the Hilbert coefficient $e_1(I)$. A recent source is a paper of L. Ghezzi, S. Goto, J. Hong and W. Vasconcelos ([13]) which gives some inequalites for $e_1(I)$ involving the reduction number. Another interesting source is the survey paper by J. Verma ([41]) of the J. K. Verma, which uses superficial sequences, including the proof of the Huckaba–Marley criterion.

Another important source is the paper by J. Migliore, R. M. Miró-Roig and U. Nagel ([29]). It turns out that a particular case of the *uniformity* hypothesis considered in this thesis is worked out in that reference with a view towards the Hilbert function and Weak Lefschetz Property.

The method in the present work emphasizes the structure of the presentation ideal of $\mathcal{R}_R(I)$ that may benefit from the appeal to Sylvester forms, as we understand them in their modern algebraic formulation. However, additional work became indispensable, emphasizing three pointers: exploiting the natural quasi-homogeneous grading over k of the presentation ideal of $\mathcal{R}_R(I)$, compatible with the usual standard grading of $\mathcal{R}_R(I)$ over R; organizing in a useful algebraic way a certain sequence of iterated Sylvester forms that are Rees generators; a careful computation of certain colon ideals crucial for extracting the homological nature of $\mathcal{R}_R(I)$. As far as we could see, the systematic joint use of these three tools has not been sufficiently applied elsewhere.

We now proceed to a more detailed description of the various parts of the thesis.

Chapter 1 is devoted to the preliminaries used throughout, emphasizing the role of ideal theoretic notions from commutative algebra and a few required numerical invariants thereof. The first two sections are about the Rees algebra of an ideal, the property of Cohen-Macaulayness, the reduction number of a minimal reduction and other related invariants, such as the related type. A section on the notion of Sylvester forms seemed appropriate, as we understand them nowadays and how one uses them. The last section is about two important methods contemplated in this thesis, both referring to an asymptotic behavior of the powers of an ideal – hence naturally related to the Rees algebra. These are the Huckaba-Marley criterion and the Ratliff-Rush filtration theory, substantially applied in steps towards a conjecture in the case of a finite colength almost complete intersection of general forms – so to say, the opposite extreme as regards monomial forms. Although we make strong use of Gröbner basis methods at some point of the work – through the S-polynomials and the Lemma (1.3.3) that connects Sylvester forms and the mapping cone allowing certain homological results – , it seemed inappropriate (or wasteful) including a related section in the preliminaries. Therefore, we chose to expand on this tool in one of the appendices.

Chapter 2 contains the solution to Vasconcelos' conjecture in the *uniform* case of monomials. The core of the chapter is confined to the first two sections, while the technically involved proofs are collected in the third section in order to avoid distraction. In the first section one develops the details of a very precise set of generators of the Rees presentation ideal, drawing upon a weighted grading naturally stemming from the form of the monomial generators of I. Of course, it is well-known that ideals of relations of monomials are generated by binomials. However, for the sake of an efficient generation we need to show that these binomials acquire a special form due to the nature of the given monomials.

One shows that the relation type of I equals the reduction number of I plus 1 and, moreover, state a precise count of the number of the generators in each external (i.e., presentation) degree. Finally, one dedicates a stretch of the section to the identification of these binomial generators as iterated Sylvester forms.

In the subsequent section one states that the above generators can be ordered in a such a way as to describe the Rees presentation ideal $\mathcal{I}$ of I by a finite series of subideals of which any two consecutive ones have a monomial colon ideal. By inducting on the length of this series one is then able to consider mapping cones iteratively culminating with $\mathcal{I}$ itself. As a consequence, the Rees algebra $\mathcal{R}_R(I)$ will be almost Cohen–Macaulay, thus answering affirmatively in this case a conjecture of Vasconcelos stated in [22, Conjecture 4.15]. Furthermore, with a view in ([22, Theorem 3.5]) that describes the regularity of almost Cohen–Macaulay Rees algebra according to the reduction number of I, we present explicitly this invariant for the studied ideal.

The preliminaries of the Chapter 2 require dealing at length with initial ideals and their colon ideals. The calculations along this line of approach though basically straightforward are quite lengthy and seem to be unavoidable. For the purpose of not disturbing the readership smoothness of the main results, we collected those proofs in the subsequent section. Although the details of the proofs can be avoided in a first reading, they constitute the fine tissue legitimating the main results of the work.

Chapter 3 is inspired from a method developed by A. Simis and S. Tohăneanu in ([38]). Although the purpose of these authors was slightly apart, in this thesis we use the procedure to the benefit of reducing the general case of the conjecture to a so-called *b-shaped* parametrization case. This procedure is a homomorphism of the ground ring R induced by mapping a ground variable to one of its powers. Therefater, this is extended to a homomorphism of the relational polynomial ring $R[\mathbf{y}]$ by the identity on $\mathbf{y}$. This simple idea seems to be very efficacious – in a recent, totally akin work, this has been used by P Aluffi ([1]) in order to express the Segre class of a monomial scheme in projective space in terms of log canonical thresholds of associated ideals.

As in in ([38]), we show that the ring homomorphism preserves the essential shape of the Rees relations which allows to deduce certain common homological behavior. Using this reduction procedure to land on a b-shaped situation, and provided an additional hypothesis is satisfied, we then essentially reproduce the arguments of Chapter 2. In order to avoid tedious repetitions, we restated the results, pointing the similarities.

Chapter 4 is dedicated to a conjecture of Simis, further studied by Simis and Tohăneanu. The conjecture states that if $I \subset R = k[x, y]$ is a codimension two ideal generated by 3 general forms of the same degree $d \geq 5$, then its Rees algebra is not almost Cohen-Macaulay. The setup is so to say at the other extreme of the monomial case, hence one may not be surprised by the sharp contrast to the first part of the thesis and all recent akin statements regarding the depth of $\mathcal{R}(I)$ when I is an almost complete intersection (see [20, 22, 36, 38]. Although the conjecture is as yet not solved, we present sufficient evidence for its solution in the affirmative based on various approaches. A curiosity is that these approaches hardly touch directly the structure of the presentation ideal of $\mathcal{R}(I)$ as an R-algebra. In fact, the entire matter is pretty much decided at the level of the second and third powers of the ideal I through the use of two apparently disconnected tools: the Ratliff-Rush filtration and the Huckaba–Marley criterion, with the intermediation of the Hilbert function.

We state stronger conjectures that imply the one sought in this part. A neat description of the Hilbert function of the ideal I is discussed. Some of this discussion appeared much earlier in ([12]) within the use of general forms in the sense of forms whose coefficients are algebraically independent elements over the the ground field k. Since the goal here is to actually stay within the original polynomial ring over k, Fröberg's version did not seem all that useful.

A pointer to the interest of considering such general forms appears in the work of J. Migliore and R. M. Miró-roig ([27], [28]) in connection to the Weak Lefschetz property. Unfortunately, in the present case of two variables we could not see how to take advantage of these results.

Chapter 1

Preliminaries

1.1 The Rees algebra

For the basic definitions, R may stand for any ring - later, we will assume that R is a regular local ring or else a standard graded polynomial ring $R = k[x_1, \ldots, x_n]$ over a field.

Let $I \subset R$ be an ideal and t a variable over R. The *Rees algebra* of I, denoted by $\mathcal{R}(I)$, is the subring of the polynomial ring $R[t]$

$$\mathcal{R}(I) = \bigoplus_{r=0}^{\infty} I^r t^r = R + It + \cdots + I^r t^r + \cdots \subseteq R[t].$$

We note that $\mathcal{R}(I)$ is an R-standard graded algebra. Supposing $I = (f_1, \ldots, f_m)$ is finitely generated, $\mathcal{R}(I)$ can be presented by a homogeneous ideal $\mathcal{I}$ in the R-standard graded polynomial ring $R[T_1, \ldots, T_m]$:

$$0 \longrightarrow \mathcal{I} \longrightarrow R[T_1, \ldots, T_m] \stackrel{\psi}{\longrightarrow} \mathcal{R}(I) \longrightarrow 0, \quad T_i \longmapsto f_i t.$$

The ideal $\mathcal{I}$ will be informally called the *Rees ideal* of I.

Set $\mathcal{I} = \mathcal{I}_1 + \mathcal{I}_2 + \cdots$, the uniquely defined decomposition of the Rees ideal in its graded parts. An important invariant used in our work is the *relation type* of I, defined by

$$\text{reltype}(I) = \inf\{n \mid \mathcal{I} = (\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_n)\}.$$

We note the useful fact that $\mathcal{I}_1 = [T_1, \cdots, T_{m+1}] \cdot \varphi$, where φ denotes the matrix of the syzygies of I. The ideal $(\mathcal{I}_1) \subset R[T_1, \ldots, T_m]$ defines the *symmetric algebra* $\mathcal{S}_R(I)$ of I. Thus, one has a natural surjective homomorphism of R-algebras $\mathcal{S}_R(I) \twoheadrightarrow \mathcal{R}(I)$. If moreover the ideal I contains a regular element then the kernel of this surjection is the R-torsion of $\mathcal{S}_R(I)$. From this, one can show the useful relation

$$\mathcal{I} = (\mathcal{I}_1) : I^{\infty},$$

which gives the Rees ideal as the I-saturation of the defining ideal of the symmetric algebra –

or, in an informal perhaps crispier way, as the syzygies stripped out of its I-component.

Let us now for simplicity asuume that $(R, \mathfrak{m})$ is either a Noetherian local ring and its maximal ideal or a standard graded algebra over a field with irrelevant ideal $\mathfrak{m}$. Say, $\dim R = d$.

Let $I \subset \mathfrak{m}$ denote an ideal containing a regular element. The Rees algebra $\mathcal{R}(I)$ has dimension $\dim R + 1 = d + 1$. We will say that $\mathcal{R}(I)$ is *almost Cohen-Macaulay* (respectively, *strictly almost Cohen-Macaulay*) if $\operatorname{depth}(\mathcal{R}(I)) \geq d$ (respectively, $\operatorname{depth}(\mathcal{R}(I)) = d$), where the depth is computed on the maximal graded ideal $(\mathfrak{m}, \mathcal{R}(I)_+)$.

Suppose, moreover, that R is regular (polynomial ring in the standard graded case). Then the extended polynomial ring $R[\mathbf{T}]$ localized in its graded maximal ideal is regular, hence $\mathcal{R}(I)$ admits a finite free resolution as a module over this regular local ring. If R is actually a standard graded polynomial ring and I is a homogeneous ideal then $\mathcal{R}(I)$ admits a finite graded free resolution as a module over $R[\mathbf{T}]$ (no need to localize). In this case, we will make no distinction between these slightly different ways of resolving the Rees algebra into free modules, particularly when measuring the length of such a resolution.

An equally important notion is that of the *associated graded ring* of an ideal $I \subset R$. It is defined as

$$\operatorname{gr}_I(R) := \frac{R}{I} \oplus \frac{I}{I^2} \oplus \frac{I^2}{I^3} \oplus \cdots = \bigoplus_{n \geq 0} I^n/I^{n+1},$$

where the product is defined by $(a + I^i)(b + I^j) = ab + I^{i+j-1}$, $a \in I^{i-1}$ and $b \in I^{j-1}$. Clearly,

$$\operatorname{gr}_I(R) \simeq \frac{\mathcal{R}(I)}{I\mathcal{R}(I)}$$

as R/I-algebras. Note that $\operatorname{gr}_I(R)$ is a standard graded ring over the ground ring R/I. As such, identifying $\mathcal{R}(I)$ with the coordinate ring of the blowup of $\operatorname{Spec}(R)$ along $\operatorname{Spec}(R/I)$, it can be identified with the coordinate ring of the exceptional locus of the blowup. In this thesis we will be solely dealing with the homological properties of $\operatorname{gr}_I(R)$, such as its depth.

1.2 The reduction number

We refer to [39] for the details of this section. Previous sources are in the list of references of this book.

The departing theory is due to D. Northcott and D. Rees in ([32]). They defined *minimal reductions, analytic spread*, proved existence theorem, and connected these ideas with *multiplicity.*

Definition 1.2.1. Let $J \subset I$ be ideals in a ring R. J is said to be a *reduction* of I if there exists an integer $n \geq 0$ such that $I^{n+1} = JI^n$.

Obviously, any ideal is a reduction of itself, but one is interested in "smaller" reductions.

Note that if $JI^n = I^{n+1}$, then for all positive integers m, $I^{m+n} = JI^{m+n-1} = \cdots = J^m I^n$. In particular, if $J \subset I$ is a reduction, there exists an integer n such that for all $m \geq 1$, $I^{m+n} \subset J^m$.

In particular, an ideal share the same radical with all its reductions. Therefore, they share the same set of minimal primes and have the same codimension.

Definition 1.2.2. A reduction J of I is called *minimal* if no ideal strictly contained in J is a reduction of I.

Definition 1.2.3. Let J a reduction of I. The *reduction number of I with respect to J* is the minimum integer n such that $JI^n = I^{n+1}$. It is denoted by $\mathrm{red}_J(I)$. The *reduction number* of I is defined as $\mathrm{red}(I) = \min\{\mathrm{red}_J(I) \mid J \subset I \text{ is a minimal reduction of } I\}$.

The notion of a reduction and the corresponding reduction number can be grasped in terms of Rees algebras, as follows:

Proposition 1.2.4. *Let $J \subset I$ be ideals in a Noetherian ring R. Then J is a reduction of I if and only if $\mathcal{R}(I)$ is module-finite over $\mathcal{R}(J)$, i.e., $\mathcal{R}(J) \hookrightarrow \mathcal{R}(I)$ is a finite morphism of graded algebras. Furthermore, the minimum integer n such that $JI^n = I^{n+1}$ is the largest degree of an element in a minimal homogeneous generating set of the ring $\mathcal{R}(I)$ over the subring $\mathcal{R}(J)$.*

Clearly, this may not be the most efficient way of obtaining the reduction number with respect to a reduction. Therefore, a vast literature has been produced on the subject searching for devices of estimating this important invariant.

In a Noetherian local ring every ideal admits minimal reductions. While the corresponding reduction number is a hard knuckle one can, thanks to a result of Northcott and Rees, explain the minimal number of generators of a minimal reduction in terms of a Krull dimension.

Definition 1.2.5. Let $(R, \mathfrak{m})$ a Notherian local ring and I an ideal of R. The *fiber cone* of I is the ring

$$\mathcal{F}_I(R) = \frac{\mathcal{R}(I)}{\mathfrak{m}\mathcal{R}(I)} \simeq \frac{R}{\mathfrak{m}} \oplus \frac{I}{\mathfrak{m}I} \oplus \frac{I^2}{\mathfrak{m}I^2} \oplus \cdots .$$

The Krull dimension of $\mathcal{F}_I(R)$ is also called the *analytic spread* of I and is denoted $\ell(I)$.

Theorem 1.2.6. (Northcott-Rees). *Let $(R, \mathfrak{m})$ be a Noetherian local ring and let $I \subset R$ be an ideal. Then any reduction of I contains a minimal reduction of I. Moreover, if $R/\mathfrak{m}$ is infinite, then every minimal reduction of I is minimally generated by exactly $\ell(I)$ elements. In particular, every reduction of I contains a reduction generated by $\ell(I)$ elements.*

In this context, the following invariants are related:

Proposition 1.2.7. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and I a ideal. Then*

$$\mathrm{ht}(I) \leq \ell(I) \leq \dim(R).$$

One hardship of looking at reduction numbers is that different minimal reductions of the same ideal in a local ring may have different reduction numbers. Such examples exist in very simple rings, such as polynomial rings in two variables over a field ([23, Example 2.1]). It has thus soon been noted that the problem has to do more with the properties of an individual ideal than with the whole ambient.

The first results along this line of question were obtained by S. Huckaba. He posed that if any two minimal reductions of an ideal $I \subset R$ have the same reduction number, then the ideal is said to have *independent* reduction number. Since one can always define the absolute reduction number $\mathrm{red}(I)$ of I as the minimum of its reduction numbers with respect to all of its minimal reductions, then independence means that the absolute reduction number of I can be computed with respect to any minimal reduction – this is certainly a great computational, if not theoretical, convenience.

Here are the main results of Huckaba in his epoch-making paper of 1986. We will use the original notation of the paper, which is the traditional notation introduced by Nortcott and Rees. Thus, we use the terminology *grade* for the length of a maximal regular sequence inside an ideal. For a standard graded R-algebra $A = \bigoplus_{m\geq 0} A_m$, its irrelevant ideal $A_1 \oplus A_2 \oplus \cdots$ will be denoted A_+.

Theorem 1.2.8. ([23, Theorem 2.1]) *Let $(R, \mathfrak{m})$ be a local ring having an infinite residue field and let I be a ideal of R. Assume $\ell(I) = \mathrm{ht}(I) = \mathrm{grade}(I) = d \geq 1$ and $\mathrm{grade}(\mathrm{gr}_I(R)_+) \geq d-1$. Then $\mathrm{red}(I)$ is independent.*

Theorem 1.2.9. ([23, Theorem 2.3]) *Let $(R, \mathfrak{m})$ be a local ring having an infinite residue field and let I be a ideal of R. Assume $\ell(I) = \mathrm{ht}(I) = \mathrm{grade}(I) = d \geq 1$ and $\mathrm{grade}(\mathrm{gr}_I(R)_+) \geq d-1$. Then $\mathrm{reltype}(I) \leq \mathrm{red}(I) + 1$.*

Theorem 1.2.10. ([23, Theorem 2.4]) *Let $(R, \mathfrak{m})$ be a local ring having an infinite residue field and let I be a ideal of R. Assume I is not principal, $\ell(I) = \mathrm{ht}(I) = \mathrm{grade}(I) = d \geq 1$ and $\mathrm{grade}(\mathrm{gr}_I(R)_+) \geq d-1$. Assume also that I can be generated by d or $d+1$ elements. Then, $\mathrm{reltype}(I) = \mathrm{red}(I) + 1$.*

Note that $d \leq \dim R$. If it happens that $d = \dim R$ and R is Cohen–Macaulay then the first two theorems are applicable for an $\mathfrak{m}$-primary ideal, while the last theorem is applicable for such an ideal which is in addition an almost complete intersection.

We will observe that the ideals considered in Chapter 2 and Chapter 3 satisfy all three theorems.

1.3 Sylvester forms

We refer to ([21]) and ([38]) for the details of this section. Uses of Sylvester forms are spread out in the references [6], [7], [8], [10], [15], [20], [22].

The following definition encapsulates the essence of the classical notion of a Sylvester form, by stressing its nature as the determinant of the content matrix expressing the inclusion of two ideals.

Definition 1.3.1. Let $R = k[x_1, \ldots, x_n]$ the polynomial ring in n variables over a field k. Let $\mathbf{a} = \{a_1, \ldots, a_m\} \subset R$ and let $\mathbf{f} = \{f_1, \ldots, f_m\}$ be a set of polynomials in $S = R[t_1, \ldots, t_k]$. If $f_i \in (\mathbf{a})S$ for all i, we can write

$$\mathbf{f} = \begin{bmatrix} f_1 \\ \vdots \\ f_m \end{bmatrix} = \mathbf{A} \cdot \begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix} = \mathbf{A} \cdot \mathbf{a},$$

where $\mathbf{A}$ is an $m \times m$ matrix with entries in S. We call $(\mathbf{a})$ a R-content of $\mathbf{f}$. We refer to $\det(\mathbf{A})$ as a *Sylvester form* of $\mathbf{f}$ relative to $\mathbf{a}$, in notation

$$\det(\mathbf{f})_{(\mathbf{a})} = \det(\mathbf{A}).$$

Proposition 1.3.2. ([21, Proposition 4.1]) *Let $R = k[x_1, \ldots, x_n], \mathfrak{m} = (x_1, \ldots, x_n)$ and $f_1, \ldots, f_s$ be forms in $S = R[t_1, \ldots, t_m]$. Suppose the R-content of the f_i (the ideal generated by the coefficients in R) is generated by forms $a_1, \ldots, a_q$ of the same degree. Let*

$$\mathbf{f} = \begin{bmatrix} f_1 \\ \vdots \\ f_s \end{bmatrix} = \mathbf{A} \cdot \begin{bmatrix} a_1 \\ \vdots \\ a_q \end{bmatrix} = \mathbf{A} \cdot \mathbf{a}$$

be the corresponding Sylvester decomposition. If $s \leq q$ and the first-order syzygies of $\mathbf{f}$ have coefficients in $\mathfrak{m}$, then $I_s(\mathbf{A}) \neq 0$.

Proof. The condition $I_p(\mathbf{A}) = 0$ means the columns of $\mathbf{A}$ are linearly dependent over $k[t_1, \ldots, t_m]$, thus for some nonzero column vector

$$\mathbf{c} \in k[t_1, \ldots, t_m]^s \quad c \cdot \mathbf{A} = 0.$$

Therefore

$$\mathbf{c} \cdot \mathbf{f} = \mathbf{c} \cdot \mathbf{A} \cdot \mathbf{a} = 0$$

is a syzygy of the f_i whose content in not in $\mathfrak{m}$, against the assumption. □

In this thesis the notion of Sylvester form will be used in the special case when $m = 2$ in the definition 1.3.1, i.e., let $(\alpha, \beta) \subset R = k[x_1, \ldots, x_n]$ be an ideal generated by two nonzero forms and let $f, g \in (\alpha, \beta)R[t_1, \ldots, t_m]$ be given biforms.

The *Sylverter form* of f, g with respect to (α, β) is the determinant of the content 2×2 matrix, denoted $\det(f, g)_{(\alpha,\beta)}$.

Our main use of Sylvester forms is in the case where f, g are biforms in the Rees ideal $\mathcal{I} \subset S$. Under this assumption, by Cramer one has

$$\det(f, g)_{\alpha,\beta} \cdot (\alpha, \beta) \subset (f, g) \subset \mathcal{I}.$$

Since $\mathcal{I}$ is a prime ideal and $\mathcal{I} \cap R = \{0\}$, the determinant belongs to $\mathcal{I}$.

The next result is of fundamental importance to what we will do in the following chapters. This is a basic result of algebraic nature in elimination theory, carrying additional information in the homological side.

The result can be found, in a more encompassing environment, in ([40, Corollary A.140]) and goes back to Northcott.

Lemma 1.3.3. ([38, Lemma 1.2]) *Let S be a commutative ring and let $\{A, B\}$ and $\{C, D\}$ be two regular sequences. Let $a, b, c, d \in S$, be given such that*

$$\begin{bmatrix} C \\ D \end{bmatrix} = \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{\mathcal{M}} \begin{bmatrix} A \\ B \end{bmatrix}.$$

One has:

(a) *If some entry of $\mathcal{M}$ is a nonzero divisor modulo (A, B), then $(C, D) : E = (A, B)$, where $E := \det(\mathcal{M})$.*

(b) *The mapping cone given by the map of complexes*

$$\begin{array}{ccccccccc} 0 & \to & S & \xrightarrow{\begin{bmatrix} -D \\ C \end{bmatrix}} & S^2 & \xrightarrow{\begin{bmatrix} C & D \end{bmatrix}} & S & \to & 0 \\ & & \uparrow \shortparallel & & \uparrow \mathcal{M}^* & & \uparrow \cdot E & & \\ 0 & \to & S & \xrightarrow{\begin{bmatrix} -B \\ A \end{bmatrix}} & S^2 & \xrightarrow{\begin{bmatrix} A & B \end{bmatrix}} & S & \to & 0 \end{array},$$

is a free resolution of $S/(C, D, E)$, where $\mathcal{M}^* := \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$.

□

One knows that $S/(C, D, E)$ is a perfect module of codimension 2, hence the mapping cone above gives a non-minimal free resolution in this case.

An important question in the general setup of Sylvester forms is to decide when such a form is nonzero. In the cases used in this thesis, due to the peculiar data, the shape of the Sylvester form will immediately reveal that it is nonzero.

1.4 The Hilbert function

Let $(R, \mathfrak{m})$ be a Noetherian local ring and let $I \subset R$ stand for an $\mathfrak{m}$-primary ideal. The *Hilbert-Samuel function* of I is $H_I(t) = \lambda(R/I^t)$ for all $t \geq 1$, where λ denotes length. For all large values of t, $H_I(t)$ coincides with the value on n of a polynomial $P_I \in \mathbb{Q}[X]$ of degree $n = \dim R$. This uniquely defined polynomial is called the *Hilbert-Samuel polynomial* of I and is often written in the combinatorial form

$$P_I(X) = \sum_{i=0}^{n} (-1)^i e_i(I) \binom{X+n-i+1}{n-i}$$

where $e_0(I), \ldots, e_n(I)$ are uniquely determined integers. They are called the *Hilbert coefficients* (or the *Chern numbers*) of I. The coefficient $e_0(I)$ is the *multiplicity* of I, important invariant which has a geometric significance. The multiplicity is a well understood invariant – mainly when $I = \mathfrak{m}$, in which case it is called the multiplicity of R. We refer to [37], [35] and [30] for some of the classical results about this invariant.

Not so much $e_1(I)$, which is not entirely understood albeit the existence of a formidable literature on it – see [31], [25] and [33] for basic results when R is Cohen–Macaulay, mainly as how $e_0(I)$ and $e_1(I)$ are related through involving certain lengths.

A largely tractable situation is that of an *equi-homogeneous* ideal I in a standard graded polynomial ring R over a field, by which one means that I is generated by forms of the same degree. This is vastly examined in the references [20, Sections 2 and 3] and [21, Section 3] in connection to the integral closure of I and the rational map defined by the linear system spanned by the generators of I. There is quite a bit of an accomplishment in the case where I is moreover $\mathfrak{m}$-primary and almost complete intersection. In this setup, there is a very explicit formula of $e_1(I)$ since $e_1(I) = e_1(\mathfrak{m}^d)$, where d is the common degree of the forms generating I, namely

$$e_1(I) = \frac{n-1}{2}(d^n - d^{n-1}),$$

with $n = \dim R$.

The usefulness of such an explicit formula cannot be exaggerated as it is crucial for approaching homological properties of the Rees algebra of I via the Hucaba–Marley criterion, to be explained in the next section.

If M is a finitely generated R-module such that $\lambda(M/IM) < \infty$, with I an $\mathfrak{m}$-primary ideal, then one can introduce in a similar fashion the Hilbert function of I on M (the previous definition would then be the Hilbert function of I on R).

We refer to [3, Section 4] for the details on the additive property of the Hilbert function along short exact sequences of modules.

In particular, if R is a standard graded polynomial ring R over a field and $I \subset R$ is a homogeneous ideal, then one can read the Hilbert function $H_I(t)$ off the finite graded free resolution of I over R. This useful result will be drawn upon in Chapter 4.

1.5 The Huckaba-Marley criterion

The basic reference for this part is [24].

With the notation of the previous section, the basic criterion of Huckaba–Marley is expressed in terms of the Chern number $e_1(I)$ and has two parts. One part is a criterion for the associated graded ring $\mathrm{gr}_I(R)$ to be Cohen-Macaulay. This part involves the modules $I^n/J \cap I^n$. The second part is a criterion for the associated graded ring $\mathrm{gr}_I(R)$ to be almost Cohen-Macaulay and is expressed in terms of the modules I^n/JI^{n-1}. Since the latter part involves more directly the inequalities $JI^{n-1} \subset J^n$ as an approximation to $\mathrm{red}_J(I)$, and since in addition almost Cohen–Macaulayness is the property we are interested in, we state only the second part of the criterion:

Theorem 1.5.1. *Let* $(R, \mathfrak{m})$ *be a Cohen-Macaulay local ring of positive dimension and infinite residue field. Let* I *be an* $\mathfrak{m}$*-primary ideal of* R *and* J *a minimal reduction of* I*. Then*

$$\sum_{n \geq 1} \lambda(I^n/JI^{n-1}) \geq e_1(I),$$

with equality if and only if $\mathrm{depth}(\mathrm{gr}_I(R)) \geq \dim R - 1$.

The following observations seem in place:

1. Both sides of the above inequality are (finite) integers. Actually, adding the first part of the criterion, which we have omitted, tells us that the Chern number is squeezed in two sums of lengths of similar shape.

2. It can be shown that, since R is Cohen–Macaulay, then $\mathrm{depth}(\mathrm{gr}_I(R)) \geq \dim R - 1$ is actually equivalent to $\mathrm{depth}(\mathcal{R}_R(I)) \geq \dim R$. Thus, almost Cohen–Macaulayness can be switched between the two rings.

3. In order to show that the above estimate is a strict inequality one has in principle two expected strategies: argue that $\mathrm{red}_J(I)$ is large or prove that the size of the lengths grows by large gaps. Thus, non-almost Cohen–Macaulayness seems to reflect some of these behaviors.

In [38, Section 3.1.3] some hard calculations were made towards showing that the Rees algebra of a certain binary monomial ideal is almost Cohen–Macaulay. These calculations required quite a bit of describing the partial quotients $JI^{n-1} : J^n$ and the resulting behavior of the syzygies of the powers of I. In this thesis we will rather stress another approach toward almost Cohen–Macaulayness, Still, in Chapter 4 we will be interested in exceeding the Chern number, so will need again similar set of calculations.

1.6 The Ratliff–Rush filtration

The basic references for this part are [17], [34] and [36]. Let us mention that there is quite a large literature on the extension of this theory to modules, but we will have no use for it in this work.

The *Ratliff–Rush closure* of an ideal $I \subset R$ in a Noetherian ring R is the ideal

$$\widetilde{I} := \bigcup_{n \geq 1} (I^{n+1} : I^n),$$

where $I^{n+1} : I^n = \{a \in R \mid aI^n \subset I^{n+1}\}$. If I has a regular element, it is shown in ([34]) that $\widetilde{I}$ is the largest ideal for which, for sufficiently large positive integers n, $(\widetilde{I})^n = I^n$.

If I contains a regular element and $I = \widetilde{I}$ it is called *Ratliff–Rush closed*. It is also known that an integrally closed ideal is Ratliff–Rush closed. Although the the Ratliff–Rush closure has a much less stable behavior, it is still very useful in several contexts. One context in which its role is meaningful is related to the depth of the associated graded ring $\mathrm{gr}_I(R)$ of I. We state this as follows:

Theorem 1.6.1. ([36, Remark 1.6]) *Let R be a Noetherian ring and I a proper regular ideal. Then, all powers of I are Ratliff–Rush closed if and only if the graded ideal* $\mathrm{gr}_I(R)_+ = I/I^2 \oplus I^2/I^3 \oplus \cdots$ *contains a nonzerodivisor.*

This will be used in the following weak version: if I is an $\mathfrak{m}$-primary ideal in a Noetherian local ring $(R, \mathfrak{m})$ that is not Ratliff–Rush closed then the Rees algebra $\mathcal{R}_R(I)$ has depth 1.

Chapter 2

Uniform case

We consider the following seetup: $R := k[x_1, \ldots, x_n]$ denotes a polynomial ring over a field k and $I \subset R$ stands for a monomial ideal. Our focus is the following conjecture stated in [22]:

Conjecture 2.0.2. *If I is an almost complete intersection of finite colength its Rees algebra $\mathcal{R}_R(I) = R[It]$ is an almost Cohen–Macaulay ring.*

We will refer to this conjecture as *Vasconcelos conjecture.* In this chapter we deal with a special case of this conjecture. To wit, given integers $0 < b < a$, the monomial ideal $I := (x_1^a, \ldots, x_n^a, (x_1 \cdots x_n)^b) \subset R$ will be called *uniform.*

2.1 Efficient generation

Our main focus is the presentation of the Rees algebra $\mathcal{R}_R(I)$ over a polynomial ring $S := R[y_1, \ldots, y_n, w]$:

$$\mathcal{I} := \ker (S \longrightarrow R[It]),\ y_j \mapsto x_j^a t, w \mapsto (x_1 \cdots x_n)^b t.$$

The presentation ideal $\mathcal{I} \subset S$ is often referred to as the *Rees ideal* of I and $y_1, \ldots, y_n, w$ as the *presentation* or *external* variables. We will moreover let $\mathcal{L} \subset \mathcal{I}$ denote the set of generators coming from the syzygies of I.

A major question is a lower bound for the depth of $\mathcal{R}_R(I)$, where the depth is computed on the maximal graded ideal $(\mathfrak{m}, S_+)$, with $\mathfrak{m} = (x_1, \ldots, x_n)$. Knowingly, $\mathcal{R}_R(I)$ is Cohen–Macaulay when its depth attains the maximum value in the inequality $\operatorname{depth}(\mathcal{R}_R(I)) \leq \dim \mathcal{R}_R(I) = n+1$. One says that $\mathcal{R}_R(I)$ is *almost Cohen–Macaulay* if $\operatorname{depth}(\mathcal{R}_R(I)) \geq n$, a condition equivalent to $\mathcal{R}_R(I)$ having homological dimension $\leq n+1$ over S.

The next results provide us information about the reduction numbers. The following lemma provides information about the general case of the [22, Conjecture 4.15], i.e., when $I = \mathfrak{I} := (x_1^{a_1}, \ldots, x_n^{a_n}, x_1^{b_1} \cdots x_n^{b_n})$ with $0 \leq b_i < a_i$ for every i, and there are at least two different indices i, j for which $b_i \neq 0, b_j \neq 0$.

Lemma 2.1.1. ([38, Lemma 2.3]) *Suppose that $J := (x_1^{a_1}, \ldots, x_n^{a_n})$ is a minimal reduction of $\mathfrak{I}$. Then the reduction number $\operatorname{red}_J(\mathfrak{I})$ is the least integer $d \geq 1$ such that exist $t \geq 2$ distinct indices*

$i_1, \dots, i_t \in \{1, \dots, n\}$ *and corresponding positive integers* $s_{i_1}, \dots, s_{i_t}$ *with* $s_{i_1} + \cdots + s_{i_t} = d+1$ *satisfying the inequalities* $(d+1)b_{i_l} \geq s_{i_l} a_{i_l}$ *for* $l = 1, \dots, t$.

Proof. Let $\mathbf{x}^{\mathbf{b}} := x_1^{b_1} \cdots x_n^{b_n}$. Since $\mathfrak{I} = (J, \mathbf{x}^{\mathbf{b}})$, then for any $r \geq 1$, one has

$$\mathfrak{I}^{r+1} = (J\mathfrak{I}^r, \mathbf{x}^{(r+1)\mathbf{b}}) = (J^{r+1}, J^r\mathbf{x}^{\mathbf{b}}, \dots, J\mathbf{x}^{r\mathbf{b}}, \mathbf{x}^{(r+1)\mathbf{b}}).$$

Then $\mathrm{red}_J(\mathfrak{I})$ will be the least r such that $\mathbf{x}^{(r+1)\mathbf{b}} \in J\mathfrak{I}^r$. But note that all the generator blocks of $J\mathfrak{I}^r$ are monomials, therefore $\mathbf{x}^{(r+1)\mathbf{b}} \in J\mathfrak{I}^r$ if and only if $\mathbf{x}^{(r+1)\mathbf{b}} \in J^{r+1-s}\mathbf{x}^{s\mathbf{b}}$ for some $s \in \{0, \dots, r\}$. However, this inclusion is only possible if $s = 0$ since otherwise we could cancel a copy of $\mathbf{x}^{\mathbf{b}}$, contradicting that $r+1$ is the least exponent with this property (by definition of reduction number). It follows that $\mathrm{red}_J(\mathfrak{I}) = r$ if and only if $\mathbf{x}^{(r+1)\mathbf{b}} \in J^{r+1}$. Now, since $\mathbf{x}^{(r+1)\mathbf{b}}$ is not a multiple of an $(r+1)$-th power of any $x_i^{a_i}$ (since otherwise $\mathbf{x}^{\mathbf{b}}$ itself would be a multiple of that $x_i^{a_i}$), it must be the case that this membership requires the existence of $t \geq 2$ such pure power $x_{i_1}^{a_{i_1}}, \dots, x_{i_t}^{a_{i_t}}$ and corresponding positive integers $s_{i_1}, \dots, s_{i_t}$ satisfying

$$\mathbf{x}^{(r+1)\mathbf{b}} \in (x_{i_1}^{s_{i_1}a_{i_1}} \cdots x_{i_t}^{s_{i_t}a_{i_t}}),$$

from which our required statement follows.

□

The next result presents the reduction number of I with respect to J a reduction of I. By the theorems 1.2.8 and 1.2.10, note that in this case $\mathrm{red}(I)$ is independent and equal to $\mathrm{reltype}(I) - 1$. In fact, since I is an almost complete intersection of finite colength it follows that $n+1 = \mu(I) \geq \mathrm{ht}(I) = n = \mathrm{grade}(I)$, but $\dim(R) = n$, therefore, by Proposition 1.2.7, we have $\ell(I) = \mathrm{ht}(I) = \mathrm{grade}(I) = n$. Furthermore, the main result of our work, Theorem 2.2.5, we have $\mathrm{depth}(\mathcal{R}_R(I)) \geq n$.

Proposition 2.1.2. ([38, Proposition 2.13]) *For a uniform monomial ideal as above the following hold:*

(a) $J := (x_1^a, \dots, x_n^a)$ *is a minimal reduction of* I *if and only if* $nb \geq a$; *in this case, letting* $1 \leq p \leq n$ *be the smallest integer such that* $pb \geq a$ *(hence* $(p-1)b < a$*), one has* $\mathrm{red}_J(I) = p - 1$.

(b) *If* $nb < a$, *then* $Q := (x_1^a - x_n^a, \dots, x_{n-1}^a - x_n^a, (x_1 \cdots x_n)^b)$ *is a minimal reduction of* I *and* $\mathrm{red}_Q(I) = n - 1$.

Proof.

(a) Suppose that J is a minimal reduction, and let $\mathrm{red}_J(I) = r$. Then, by Lemma 2.1.1, there exist $n \geq t \geq 2$ and $s_{i_1}, \dots, s_{i_t}$ with $s_{i_1} + \cdots + s_{i_t} = r+1$ such that

$$(r+1)b \geq s_{i_j}a, \ \ j = 1, \dots, t.$$

Adding up the inequalities one gets $tb \geq a$ and hence, $nb \geq a$.

Conversely, letting $J := (x_1^a, \ldots, x_n^a)$, since

$$((x_1 \cdots x_n)^b)^p \in (x_1^a \cdots x_n^a),$$

one obtains that $JI^{p-1} = I^p$, and hence $\mathrm{red}_J(I) \leq p-1$. Suppose that $\mathrm{red}_J(I) = p-q$, $q \geq 2$. Then, by Lemma 2.1.1, there exist at least one $1 \leq l \leq t$, such that

$$(p-q+1)b \geq s_{i_l} a.$$

This is a contradiction, since $a > (p-1)b \geq (p-q+1)b$, and $s_{i_l} a \geq a$.

(b) We first claim that $I^n \subset QI^{n-1}$. Thus, let

$$\mathcal{M} = x_1^{i_1 a} \cdots x_n^{i_n a}(x_1 \cdots x_n)^{bj}, \ i_1 + \cdots + i_n + j = n$$

be a typical generator of I^n.
Suppose that for some $1 \leq s \leq n-1, \ i_s \geq 1$. Then

$$x_s^{i_s a} = x_s^{(i_s-1)a} x_s^a = \underbrace{x_s^{(i_s-1)a}(x_s^a - x_n^a)}_{\in Q} + x_s^{(i_s-1)a} x_n^a.$$

We get that $\mathcal{M} = \mathcal{M}' + \mathcal{M}''$, where $\mathcal{M}' \in QI^{n-1}$, and

$$\mathcal{M}'' = x_1^{i_1 a} \cdots x_s^{(i_s-1)a} \cdots x_{n-1}^{i_{n-1}a} x_n^{(i_n+1)a}(x_1 \cdots x_n)^{bj}.$$

Of course, $\mathcal{M} \in QI^{n-1}$ iff $\mathcal{M}'' \in QI^{n-1}$.

Repeating the process we derive that $\mathcal{M} \in QI^{n-1}$ exactly when $\mathcal{N} := x_n^{(n-j)a}(x_1 \cdots x_n)^{bj} \in QI^{n-1}$.
If $j > 0$, then

$$\mathcal{N} = \underbrace{(x_1 \cdots x_n)^b}_{\in Q} \underbrace{(x_n^a)^{(n-j)}((x_1 \cdots x_n)^b)^{(j-1)}}_{\in I^{n-1}}.$$

If $j = 0$, then $\mathcal{N} = x_n^{na}$. Using the generators $x_i^a - x_n^a \in Q, \ 1 \leq i \leq n-1$, we have that $\mathcal{N} \in QI^{n-1}$ if and only if $x_1^a \cdots x_n^a \in QI^{n-1}$. But the latter is always the case because

$$x_1^a \cdots x_n^a = \underbrace{(x_1 \cdots x_n)^b}_{\in Q} \underbrace{(x_1 \cdots x_n)^{a-b}}_{\in I^{n-1}},$$

as $a - b > (n-1)b$.
To complete the proof, we have to show that $I^{n-1} \nsubseteq QI^{n-2}$. Since $x_n^{(n-1)a} \in I^{n-1}$, it is enough to show that $x_n^{(n-1)a} \notin QI^{n-2}$. Suppose the contrary. Then

$$x_n^{(n-1)a} = \sum C_{(i_1,\ldots,i_n)}^{k,j}(x_k^a - x_n^a) x_1^{i_1 a + bj} \cdots x_n^{i_n a + bj} + \sum P_{(i_1,\ldots,i_n)}^j x_1^{i_1 a + b(j+1)} \cdots x_n^{i_n a + b(j+1)},$$

where the sums are taken over all $1 \le k \le n-1$ and $i_1 + \cdots + i_n + j = n-2$.
First, observe that when $j = 0$, $C^{k,0}_{(i_1,\ldots,i_n)}$ are constant polynomials.

The terms which are pure powers of x_n in the right-hand side have $j = i_1 = \cdots = i_{n-1} = 0$, $i_n = n-2$. It follows that

$$x_n^{(n-1)a} = (-\sum_k C^{k,0}_{(0,\ldots,0,n-2)})x_n^{(n-1)a} + + \sum_k C^{k,0}_{(0,\ldots,0,n-2)} x_k^a x_n^{(n-2)a} + \cdots .$$

Hence $-\sum_k C^{k,0}_{(0,\ldots,0,n-2)} = 1$ and the coefficients of all the other monomials must be zero. The monomial $x_k^a x_n^{(n-2)a}$ also can occur only in $(x_k^a - x_n^a)x_k^a x_n^{(n-3)a}$. Therefore we have

$$0 = \sum_k (C^{k,0}_{(0,\ldots,0,n-2)} - C^{k,0}_{(0,\ldots,1,\ldots,0,n-3)})x_k^a x_n^{(n-2)a} + \sum_k C^{k,0}_{(0,\ldots,1,\ldots,0,n-3)} x_k^{2a} x_n^{(n-3)a} + \cdots .$$

The 1 in the multi-index above occurs in position k.
We get that $C^{k,0}_{(0,\ldots,0,n-2)} - C^{k,0}_{(0,\ldots,1,\ldots,0,n-3)} = 0$, for all $1 \le k \le n-1$. If we repeat the process in the end we obtain

$$C^{k,0}_{(0,\ldots,0,n-2)} = C^{k,0}_{(0,\ldots,1,\ldots,0,n-3)} = \cdots = C^{k,0}_{(0,\ldots,n-2,\ldots,0,0)},$$

and

$$0 = \sum_k C^{k,0}_{(0,\ldots,n-2,\ldots,0,0)} x_k^{(n-2)a_k} + \text{ other terms not pure powers of the variables.}$$

This leads to $C^{k,0}_{(0,\ldots,0,n-2)} = C^{k,0}_{(0,\ldots,n-2,\ldots,0,0)} = 0$ for all k. But this contradicts the fact that $\sum_k C^{k,0}_{0,\ldots,0,n-2} = -1$.

□

In particular, for $a \le 2b$ the ideal J is a minimal reduction of I with reduction number 1, hence $\mathcal{R}_R(I)$ is Cohen–Macaulay as is well-known. Since this situation has no interest in our discussion, we will assume $a > 2b$ throughout the work.

In this part we search for a set of binomials of a particular form that minimally generate the Rees ideal $\mathcal{I}$ of I. As we will contend in Theorem 2.1.4, the ring S admits a weighted grading under which an extra behavior will emerge. For now, as a preamble we can prove a basic result that depends solely on the standard grading of S as a polynomial ring over R. It is well-known that ideals of relations of monomials are generated by binomials. In the present case, we show that these binomials acquire a special form due to the nature of the given monomials. This step will be crucial in the subsequent unfolding.

Lemma 2.1.3. *Any binomial in $\mathcal{I}$ belonging to a set of minimal generators thereof is of the form*

$$\mathbf{m}(\mathbf{x})w^\delta - \mathbf{n}(\mathbf{x})y_{i_1}^{\alpha_{i_1}} \cdots y_{i_s}^{\alpha_{i_s}},$$

where $\mathbf{m}(\mathbf{x}), \mathbf{n}(\mathbf{x})$ *are relatively prime monomials in* $\mathbf{x} = x_1, \ldots, x_n$ *and* $1 \leq i_1 < \cdots < i_s \leq n$, $\alpha_{i_j} > 0$.

Proof. One has to show that, for no $1 \leq i \leq n$ do y_i and w divide the same monomial in the expression of a generating binomial.

Assuming the contrary, by change of variables, one has the following two possibilities for a binomial relation:

CASE 1. $y_1^{\alpha_1} \cdots y_t^{\alpha_t} w^\delta - x_1^{d_1} \cdots x_t^{d_t} x_{t+1}^{d_{t+1}} \cdots x_n^{d_n} y_{t+1}^{\alpha_{t+1}} \cdots y_n^{\alpha_n}$, where $\delta > 0$ and $\alpha_1, \ldots, \alpha_t \geq 1$.

Because of the homogeneity of the variables $y_1, \ldots, y_n, w$ and since upon evaluation the degrees of $x_1, \ldots, x_n$ must match on the two sides, we obtain the numerical equalities

$$\begin{aligned}
\alpha_1 + \cdots + \alpha_t + \delta &= \alpha_{t+1} + \cdots + \alpha_n \\
a\alpha_j + \delta b &= d_j, j = 1, \ldots, t \\
\delta b &= a\alpha_k + d_k, k = t+1, \ldots, n.
\end{aligned}$$

From the first of these equalities we can assume that $\alpha_{t+1} \geq 1$ and, from the second one, that $d_1 > a$. Then the binomial can be written as

$$y_1^{\alpha_1} \cdots y_t^{\alpha_t} w^\delta - \underbrace{(K_{1,t+1} + x_{t+1}^a y_1)}_{x_1^a y_{t+1}} x_1^{d_1 - a} \cdots x_t^{d_t} x_{t+1}^{d_{t+1}} \cdots x_n^{d_n} y_{t+1}^{\alpha_{t+1}-1} \cdots y_n^{\alpha_n},$$

where $K_{i,j} = x_i^a y_j - x_j^a y_i,\ i,j \in \{1, \ldots, n\},\ i < j$.

Since $\mathcal{I}$ is a prime ideal, simplifying by y_1 due to minimality, one obtains a binomial in $\mathcal{I}$ of the same shape with y_1 raised to the power $\alpha_1 - 1$. Iterating, we can replace the given generator by another one of the same shape, where the exponent of y_1 vanishes. But this contradicts the assumption that this exponent is nonzero.

CASE 2. $x_1^{d_1} \cdots x_m^{d_m} y_{m+1}^{\alpha_{m+1}} \cdots y_t^{\alpha_t} w^\delta - x_{m+1}^{d_{m+1}} \cdots x_t^{d_t} y_1^{\alpha_1} \cdots y_m^{\alpha_m} x_{t+1}^{d_{t+1}} \cdots x_n^{d_n} y_{t+1}^{\alpha_{t+1}} \cdots y_n^{\alpha_n}$, where $\delta > 0$ and $\alpha_{m+1}, \ldots, \alpha_t \geq 1$.

As before, one has the following equalities between the exponents:

$$\begin{aligned}
d_i + \delta b &= \alpha_i a, i = 1, \ldots, m \\
a\alpha_j + \delta b &= d_j, j = m+1, \ldots, t \\
\delta b &= a\alpha_k + d_k, k = t+1, \ldots, n.
\end{aligned}$$

As $\delta > 0$, the first set of equations gives $\alpha_1, \ldots, \alpha_m \geq 1$. The assumption $\alpha_j \geq 1, j = m+1, \ldots, t$, and the second set of equations give $d_{m+1}, \ldots, d_t > a$. Then the binomial can be written in the form

$$x_1^{d_1}\cdots x_m^{d_m} y_{m+1}^{\alpha_{m+1}}\cdots y_t^{\alpha_t} w^\delta - \underbrace{(K_{m+1,1}+x_1^a y_{m+1})}_{x_{m+1}^a y_1} x_{m+1}^{d_{m+1}-a}\cdots x_t^{d_t} y_1^{\alpha_1 -1}\cdots y_m^{\alpha_m} x_{t+1}^{d_{t+1}}\cdots x_n^{d_n} y_{t+1}^{\alpha_{t+1}}\cdots y_n^{\alpha_n}.$$

By the same token as above, one obtains a binomial in $\mathcal{I}$ of the same shape with y_{m+1} raised to $\alpha_{m+1}-1$. Iterating on α_{m+1} as in the first case gives a contradiction – note that, because α_1 also drops by 1, the first case is around the corner in the inductive process. □

The following notation will be used throughout the rest of the thesis: if $\{i_1,\ldots,i_j\}$ is a subset of $\{1,\ldots,n\}$ we denote by $P(i_1,\ldots,i_j)$ the product of the variables belonging to the subset $\{x_1,\ldots,x_n\}\setminus\{x_{i_1},\ldots,x_{i_j}\}$. A few times around we may deal with a similar situation where we may wish to stress that $\{i_1,\ldots,i_j\}$ is a subset of a smaller subset of $\{1,\ldots,n\}$.

Our first basic result specifies much further the nature of the minimal binomial generators.

Theorem 2.1.4. *Let $I\subset R=k[x_1,\ldots,x_n]$ be a uniform monomial ideal as above. Then the polynomial ring $S:=R[y_1,\ldots,y_n,w]$ admits a grading under which the presentation ideal $\mathcal{I}$ of the Rees algebra of I over it is generated by homogeneous binomials in this grading.*

Moreover:

(a) *If $a\leq nb$, letting $1\leq p\leq n$ be the unique integer such that $(p-1)b<a\leq pb$, then any minimal binomial generator of external degree δ can be written in the form*

$$(x_{i_1}\cdots x_{i_\delta})^{a-\delta b}\, w^\delta - P(i_1,\ldots,i_\delta)^{\delta b}\, y_{i_1}\cdots y_{i_\delta}, \tag{2.1}$$

where $\delta\leq p$, with the convention that if $\delta=p$ then the $\mathbf{x}$-term on the left hand side goes over to the right hand side with exponent $-(a-\delta b)=\delta b-a$.

(b) *If $a>nb$, then any minimal binomial generator of external degree δ can be written in the form*

$$(x_{i_1}\cdots x_{i_\delta})^{a-\delta b}\, w^\delta - P(i_1,\ldots,i_\delta)^{\delta b}\, y_{i_1}\cdots y_{i_\delta}, \tag{2.2}$$

where $\delta\leq n$. (no convention needed in this case since for $\delta=n$, there is no $\mathbf{x}$-term on the right hand side).

Proof. Start with generators of the presentation ideal of the symmetric algebra of I. It is easy to see that the syzygies of I are generated by the Koszul relations of the pure powers $x_1^a,\ldots,x_n^a$ and by the reduced relations of $(x_1\cdots x_n)^b$ with each one of the pure powers. In other words, $\mathcal{L}\subset S=R[y_1,\ldots,y_n,w]$ is generated by the binomials

$$K_{i,j}=x_i^a y_j - x_j^a y_i,\ i,j\in\{1,\ldots,n\},\ i<j,$$

$$L_i = x_i^{a-b}w - P(i)^b y_i,\ i\in\{1,\ldots,n\}.$$

Now, these binomials are homogeneous in S by attributing the following weights to the variables: $\deg(x_i) = 1$ and $\deg(w) = nb - a + 1, \deg(y_j) = 1$ if $a \leq nb$, while $\deg(w) = 1, \deg(y_j) = a - nb + 1$ if $a \geq nb$. Therefore, $\mathcal{L}$ is homogeneous for these weights. Since $\mathcal{I} = \mathcal{L} : I^\infty$ and I is monomial, it follows that $\mathcal{I}$ is generated by binomials which are homogeneous as well under the same weights. Indeed, one has the string of inclusions

$$\mathcal{I} = \mathcal{L} : I^\infty \subset \mathcal{L} : (x_1)^\infty \subset \mathcal{I} : (x_1)^\infty = \mathcal{I},$$

the last equality because $\mathcal{I}$ is a prime ideal. Then by [11, Corollary 1.7 (a)] (or, directly, by [11, Corollary 1.9]), $\mathcal{I}$ is generated by binomials and hence by homogeneous binomials as x_1 is homogeneous of degree 1. (Note that the counterexamples in [11] are non-prime.)

By Lemma 2.1.3, a binomial in $\mathcal{I}$ belonging to a set of minimal generators thereof is of the form

$$\mathbf{m}(\mathbf{x})w^\delta - \mathbf{n}(\mathbf{x})y_{i_1}^{\alpha_{i_1}} \cdots y_{i_s}^{\alpha_{i_s}},$$

with $1 \leq i_1 < \cdots < i_s \leq n$, $\alpha_{i_j} > 0$, and $\mathbf{m}(\mathbf{x}), \mathbf{n}(\mathbf{x})$ suitable monomials in R such that $\gcd\{\mathbf{m}(\mathbf{x}), \mathbf{n}(\mathbf{x})\} = 1$.

In addition, one has the following three basic principles:

• w corresponds to a monomial that involves all variables of R; this implies that the monomial $\mathbf{n}(\mathbf{x})$ must involve the variables indexed by the complementary subset $\{j_{s+1}, \ldots, j_n\} := \{1, \ldots, n\} \setminus \{i_1, \ldots, i_s\}$ and, since $\gcd\{\mathbf{m}(\mathbf{x}), \mathbf{n}(\mathbf{x})\} = 1$, the variables effectively involved in $\mathbf{m}(\mathbf{x})$ must be indexed by a subset of $\{i_1, \ldots, i_s\}$. Therefore, the monomial has the form

$$x_{i_1}^{d_{i_1}} \cdots x_{i_s}^{d_{i_s}} w^\delta - x_{i_{s+1}}^{c_{i_{s+1}}} \cdots x_{i_n}^{c_{i_n}} y_{i_1}^{\alpha_{i_1}} \cdots y_{i_s}^{\alpha_{i_s}}$$

for suitable exponents $d_{i_l} \geq 0$, for $l = 1, \ldots, s$ (some of which may vanish) and c_{i_k}, for $k = s+1, \ldots, n$ (which are positive).

• Weighted homogeneity implies the equalities

$$(nb - a + 1)\delta + \sum_{l=1}^{s} d_{i_l} = \sum_{l=1}^{s} \alpha_{i_l} + \sum_{k=s+1}^{n} c_{i_k} \tag{2.3}$$

if $a \leq nb$, and

$$\delta + \sum_{l=1}^{s} d_{i_l} = (a - nb + 1) \sum_{l=1}^{s} \alpha_{i_l} + \sum_{k=s+1}^{n} c_{i_k} \tag{2.4}$$

if $a \geq nb$.

Moreover, since upon evaluation the powers $x_{i_k}^{c_{i_k}}$ on the right hand side can only cancel against the ones coming from w^δ on the left hand side, we see that $c_{i_k} = \delta b$ for every $k = s+1, \ldots, n$. By the same token, $d_{i_l} = a\alpha_{i_l} - \delta b$ for every $l = 1, \ldots, s$.

• Lastly, since the Rees algebra $\mathcal{R}_R(I)$ is also standard graded over $R = \mathcal{R}_R(I)_0$, we may assume that the binomial is homogeneous with respect to the external variables (however, we warn that $\mathcal{R}_R(I)$ is *standard* bigraded over k if and only if $a = nb$). This means that $\delta = \sum_{l=1}^{s} \alpha_{i_l}$, a formula already found in the above lemma.

So we can assume our binomial to look like

$$x_1^{a\alpha_1-\delta b}\cdots x_s^{a\alpha_s-\delta b}w^\delta-(x_{s+1}\cdots x_n)^{\delta b}y_1^{\alpha_1}\cdots y_s^{\alpha_s}, \alpha_i\geq 1.$$

Case $a\leq nb$: suppose $\delta\geq p+1$. The goal is to show that this binomial can be generated by binomials in $\mathcal{I}$ with w raised to a power $\leq p$. Since $a<(p+1)b$ and $a\alpha_i-\delta b>0$, then $\alpha_i\geq 2$ for all $i=1,\ldots,s$.

If $s\geq p$, consider the polynomial

$$H:=w^p-(x_1\cdots x_p)^{pb-a}(x_{p+1}\cdots x_n)^{pb}y_1\cdots y_p\in\mathcal{I}.$$

If $a=pb$, consider $H:=w^p-y_1\cdots y_p$. By primality of $\mathcal{I}$, using H, our binomial is generated by H and by the following binomial in $\mathcal{I}$

$$x_1^{a(\alpha_1-1)-(\delta-p)b}\cdots x_p^{a(\alpha_p-1)-(\delta-p)b}x_{p+1}^{a\alpha_{p+1}-(\delta-p)b}\cdots x_s^{a\alpha_s-(\delta-p)b}w^{\delta-p}$$

$$-(x_{s+1}\cdots x_n)^{(\delta-p)b}y_1^{\alpha_1-1}\cdots y_p^{\alpha_p-1}y_{p+1}^{\alpha_{p+1}}\cdots y_s^{\alpha_s},$$

where w is raised to $\delta-p$, and in addition the exponents of x_i on the left do not vanish since $a\alpha_i>\delta b$, then $a(\alpha_i-1)-(\delta-p)b>pb-a\geq 0$.

If $s\leq p-1$, consider

$$G:=(x_1\cdots x_s)^{a-sb}w^s-(x_{s+1}\cdots x_n)^{sb}y_1\cdots y_s.$$

Then, by the same token as above, using G, the binomial can be generated by G and by the following binomial in $\mathcal{I}$:

$$x_1^{a(\alpha_1-1)-(\delta-s)b}\cdots x_s^{a(\alpha_s-1)-(\delta-s)b}w^{\delta-s}-(x_{s+1}\cdots x_n)^{(\delta-s)b}y_1^{\alpha_1-1}\cdots y_s^{\alpha_s-1}.$$

Recursively, in both situations above ($s\geq p$ and $s\leq p-1$), our binomial can be generated by binomials in $\mathcal{I}$ of the same shape with w raised to a power $\leq p$.

The concluding blow is given by the following result:

Claim. With the preceding notation, if $\delta\leq p$, then we can assume $\alpha_1=\cdots=\alpha_s=1$, and $s=\delta$.

For the proof, assume $\alpha_1\geq 2$. Then $a\alpha_1-\delta b\geq 2a-\delta b=a-b+a-(\delta-1)b$. Since $p\geq\delta$ and $a>(p-1)b$, then $a-(\delta-1)b>0$. Our binomial can be written as

$$x_1^{a(\alpha_1-1)-(\delta-1)b}x_2^{a\alpha_2-\delta b}\cdots x_s^{a\alpha_s-\delta b}w^{\delta-1}\underbrace{(L_1+(x_2\cdots x_n)^b y_1)}_{x_1^{a-b}w}-(x_{s+1}\cdots x_n)^{\delta b}y_1^{\alpha_1}\cdots y_s^{\alpha_s}.$$

Since $L_1 \in \mathcal{I}$ and we only care for minimal generators, by simplifying by $(x_{s+1}\cdots x_n)^b y_1$ one can assume the binomial to be of the form

$$x_1^{a(\alpha_1-1)-(\delta-1)b} x_2^{a\alpha_2-(\delta-1)b} \cdots x_s^{a\alpha_s-(\delta-1)b} w^{\delta-1} - (x_{s+1}\cdots x_n)^{(\delta-1)b} y_1^{\alpha_1-1} y_2^{\alpha_2}\cdots y_s^{\alpha_s},$$

where both α_1 and δ dropped by 1. Therefore, recursion takes care of the conclusion.

The case where $a > nb$ is more generally shown in Theorem 3.4.3.

This concludes the proof of the claim and also of the theorem. □

2.1.1 Sylvester forms as generators

For the reader's convenience, we recall once more the following notation: if $\{i_1, \ldots, i_j\}$ is a subset of $\{1, \ldots, n\}$ in the natural order of the integers, we denote by $P(i_1, \ldots, i_j)$ the product of the variables in the complementary set $\{x_1, \ldots, x_n\} \setminus \{x_{i_1}, \ldots, x_{i_j}\}$.

The next theorem partly summarizes the results of the preceding part, adding information on the nature of the generators as Sylvester forms.

Theorem 2.1.5. *Let $I \subset R$ be a uniform monomial ideal as above and let r denote its reduction number as established in* Proposition 2.1.2. *Then:*

(a) *$\mathcal{I}$ is generated by*

$$\binom{n}{2} + \sum_{\delta=1}^{r} \binom{n}{\delta} + 1,$$

quasi-homogeneous binomials, where r is the reduction number of I; of these, $\binom{n}{2}$ are the Koszul syzygies of the generators of I and the remaining ones are each a binomial of the form

$$(x_{i_1}\cdots x_{i_\delta})^{a-\delta b}\, w^{\delta} - P(i_1, \ldots, i_\delta)^{\delta b}\, y_{i_1}\cdots y_{i_\delta},$$

where $1 \le \delta \le r+1$ (with the same convention as stated in Theorem 2.1.4 *in the case $a \le nb$).*

(b) *Moreover, each binomial in the previous item is a Sylvester form obtained in an iterative form out of the syzygy forms.*

(c) *The relation type of I is $r+1$.*

Proof. (a) The proof of the generation statement will consist in showing that a quasi-homogeneous generator of $\mathcal{I}$ of arbitrary standard degree in the external variables $y_1, \ldots, y_n, w$ belongs to the ideal generated by the binomials in the statement, with standard external degrees bounded by the reduction number of I. Thus, the result will be a consequence of Theorem 2.1.4 and of Proposition 2.1.2.

From the above degree reduction result and from Theorem 2.1.4 we deduce that, for each $2 \le \delta \le r$, where r is the reduction number of I, $\mathcal{I}$ admits $\binom{n}{\delta}$ generators which are quasi-homogeneous binomials. Generators for $\delta = 1$ are the syzygy binomials, which add up $\binom{n}{2} + n$ generators in standard degree 1.

Finally, we deal with generators in standard degree $r+1$. In the case where $a > nb$, then there is a unique generator in degree n given in Theorem 2.1.4, namely, $(x_1 \cdots x_n)^{a-nb} w^n - y_1 \cdots y_n$. In the case where $a \le nb$ and $p \le n$ is the unique integer such that $(p-1)b < a \le pb$, we obtain $\binom{n}{p}$ generators, one for each choice of an ordered subset $\{i_1, \ldots, i_p\} \subset \{1, \ldots, n\}$:

$$S_{i_1,\ldots,i_p} := w^p - P(i_1, \ldots, i_p)^{pb} \, (x_{i_1} \cdots x_{i_p})^{pb-a} \, y_{i_1} \cdots y_{i_p}.$$

We now show that fixing one of these, the remaining ones belong to the ideal generated by this one and the Koszul relations. To prove this assertion it suffices to fix one subset $\{i_1, \ldots, i_p\}$ and another subset obtained by one transposition. Without loss of generality, we assume the fixed subset is $\{1, \ldots, p\}$ and the other one is $\{1, \ldots, p-1, p+1\}$.

Claim: With the above notation and the previous notation for the Koszul relations, one has

$$S_{i_1,\ldots,i_{p-1},i_{p+1}} = S_{i_1,\ldots,i_p} + M(\mathbf{x}) \, y_2 \cdots y_p \, K_{1,p+1} - M(\mathbf{x}) \, y_2 \cdots y_{p-1} y_{p+1} \, K_{1,p},$$

where $M(\mathbf{x}) = (x_1 \cdots x_p x_{p+1})^{pb-a} (x_{p+2} \cdots x_n)^{pb}$.

The proof is a straightforward calculation by developing the right hand side.

As a consequence, also for the case $(p-1)b < a \le pb$ there is a unique minimal generator in standard degree p. Summing up, in both cases, we get

$$\binom{n}{2} + \sum_{\delta=1}^{r} \binom{n}{\delta} + 1$$

minimal quasi-homogeneous binomial generators.

(b) We next show that the generators of the first part are indeed Sylvester forms obtained iteratively.

Recall once more the form of the generators of $\mathcal{L} \subset S = R[y_1, \ldots, y_n, w]$: the Koszul relations

$$K_{i,j} = x_i^a y_j - x_j^a y_i, \; i, j \in \{1, \ldots, n\}, \; i < j \tag{2.5}$$

and the reduced (Taylor) relations

$$L_i = x_i^{a-b} w - P(i)^b y_i, \; i \in \{1, \ldots, n\}. \tag{2.6}$$

We start by availing ourselves of Sylvester forms of degree 2. For this, take any two distinct indices $l, i \in \{1, \ldots, n\}$, say, $l < i$. We form the Sylvester content matrix of $\{L_l, L_i\}$ with respect to the complete intersection $\{x_l^b, x_i^b\}$:

$$\begin{bmatrix} L_l \\ L_i \end{bmatrix} = \begin{bmatrix} x_l^{a-b} w - P(l)^b y_l \\ x_i^{a-b} w - P(i)^b y_i \end{bmatrix} = \underbrace{\begin{pmatrix} x_l^{a-2b} w & -P(l,i)^b y_l \\ -P(l,i)^b y_i & x_i^{a-2b} w \end{pmatrix}}_{M_2^{l,i}} \begin{bmatrix} x_l^b \\ x_i^b \end{bmatrix}.$$

Set $H_2^{l,i} = \det(M_2^{l,i}) = (x_l x_i)^{a-2b} w^2 - P(l,i)^{2b} y_l y_i$. Note that, since we are assuming that $a > 2b$, we obtain this way $\binom{n}{2}$ distinct forms of external degree 2.

We now induct on the degree. Thus, suppose that for $j \in \{1, \ldots, n\}$ with $a > jb$, one has found $\binom{n}{j}$ Sylvester forms, of external degree j, each of the shape

$$H_j^{i_1,\ldots,i_j} = (x_{i_1} x_{i_2} \cdots x_{i_j})^{a-jb} w^j - P(i_1, \ldots, i_j)^{jb} y_{i_1} \cdots y_{i_j}, \tag{2.7}$$

with $i_1, \ldots, i_j \in \{1, \ldots, n\}$ and $i_1 < \cdots < i_j$. Then for every $l \in \{1, \ldots, n\} \setminus \{i_1, \ldots, i_j\}$, we obtain a Sylvester content matrix of $L_l, H_j^{i_1,\ldots,i_j}$ with respect to the complete intersection $(x_l^{jb}, (x_{i_1} \cdots x_{i_j})^b)$:

$$\begin{aligned} \begin{bmatrix} L_l \\ H_j^{i_1,\ldots,i_j} \end{bmatrix} &= \begin{bmatrix} x_l^{a-b} w - P(l)^b y_l \\ (x_{i_1} \cdots x_{i_j})^{a-jb} w^j - P(i_1, \ldots, i_j)^{jb} y_{i_1} \cdots y_{i_j} \end{bmatrix} \\ &= \underbrace{\begin{pmatrix} x_l^{a-(j+1)b} w & -P(i_1, \ldots, i_j, l)^b y_l \\ -P(i_1, \ldots, i_j, l)^{jb} y_{i_1} \cdots y_{i_j} & (x_{i_1} \cdots x_{i_j})^{a-(j+1)b} w^j \end{pmatrix}}_{M_{j+1}^{i_1,\ldots,l,\ldots,i_j}} \begin{bmatrix} x_l^{jb} \\ (x_{i_1} \cdots x_{i_j})^b \end{bmatrix}. \end{aligned}$$

This yields a new Sylvester form of external degree $j+1$: $H_{j+1}^{i_1,\ldots,\ldots,i_j,l} = \det(M_{j+1}^{i_1,\ldots,\ldots,i_j,l})$

$$= (x_{i_1} \cdots x_{i_j} x_l)^{a-(j+1)b} w^{j+1} - P(i_1, \ldots, i_j, l))^{(j+1)b} y_{i_1} \cdots y_{i_j} \cdots y_l.$$

(Here, we assume that $\{i_1, \ldots, i_j, l\}$ is written in increasing order.) This way we have produced $\binom{n}{j+1}$ distinct Sylvester forms of external degree $j+1$.

To conclude the inductive procedure, we divide the proof into the two basic cases:

(i) $a \le nb$.

In this case, let $1 \le p \le n$ be the smallest integer such that $(p-1)b < a \le pb$. By the previous argument, since $a > (p-1)b$ then a Sylvester form of standard degree $(p-1)$ over R has the shape

$$H_{p-1}^{i_1,\ldots,i_{p-1}} = (x_{i_1} \cdots x_{i_{p-1}})^{a-(p-1)b} w^{p-1} - P(i_1, \ldots, i_{p-1})^{(p-1)b} y_{i_1} \cdots y_{i_{p-1}},$$

with $\{i_1, \ldots, i_{p-1}\}$ an ordered subset of $\{1, \ldots, n\}$. Take the Sylvester form of $\{L_l, H_{p-1}^{i_1,\ldots,i_{p-1}}\}$ with respect to $\{x_l^{a-b}, (x_{i_1} \cdots x_{i_{p-1}})^{a-(p-1)b}\}$, since $a \le pb$:

$$\begin{bmatrix} L_l \\ H_{p-1}^{i_1,\ldots,i_{p-1}} \end{bmatrix} = M_p^{i_1,\ldots,l,\ldots,i_{p-1}} \cdot \begin{bmatrix} x_l^{a-b} \\ (x_{i_1} \cdots x_{i_{p-1}})^{a-(p-1)b} \end{bmatrix},$$

where $M_p^{i_1,\ldots,l,\ldots,i_{p-1}}$ denotes the content matrix

$$\begin{pmatrix} w & -P(i_1, \ldots, i_{p-1}, l)^b (x_{i_1} \cdots x_{i_{p-1}})^{pb-a} y_l \\ -P(i_1, \ldots, i_{p-1}, l)^{(p-1)b} x_l^{pb-a} y_{i_1} \cdots y_{i_{p-1}} & w^{p-1} \end{pmatrix}.$$

Thus,

$$\begin{aligned} H_p^{i_1,\ldots,l,\ldots,i_{p-1}} &= \det(M_p^{i_1,\ldots,l,\ldots,i_{p-1}}) \\ &= w^p - P(i_1,\ldots,i_{p-1},l)^{pb}(x_{i_1}\ldots x_l \ldots x_{i_{p-1}})^{pb-a} y_{i_1}\cdots y_l \cdots y_{i_{p-1}}. \end{aligned}$$

(ii) $a > nb$.

By the previous argument, since $a > nb$ then a Sylvester form of standard degree n over R has the shape

$$H_n^{1,\ldots,n} = (x_1 \cdots x_n)^{a-nb} w^n - y_1 \cdots y_n.$$

(c) This follows immediately from the details of the generation as described in (a). □

Remark 2.1.6. Note the sharp difference between cases (i) and (ii) at the end of the proof above: if $p = n$ then there is a unique binomial Sylvester form with a term a pure power of w (namely, w^n), while for $p < n$ there are various such binomials having the pure term w^p – although only one emerges as part of a minimal set of generators, as explained in the proof of the previous theorem.

2.2 Combinatorial structure of the Rees ideal

We keep the notation of the previous part. Recall that, given an integer $2 \le j \le p-1$, where $p-1 \le n-1$ is the reduction number of the ideal $I \subset S = k[x_1,\ldots,x_n]$, and an increasing sequence of integers $i_1 < \cdots < i_j$ in $\{1,\ldots,n\}$, we had a well-defined Sylvester form $H_j^{i_1,\ldots,i_j}$ in the set of generators of the Rees ideal of $\mathcal{R}_R(I)$. This polynomial is weighted homogeneous in all concerned variables and homogeneous of degree j in the presentation variables $y_1,\ldots,y_n,w$. We will order the set of these forms in the following way: first, if two of these forms $H_j^{i_1,\ldots,i_j}$ and $H_j^{k_1,\ldots,k_j}$ have the same presentation degree j then we set $H_j^{i_1,\ldots,i_j}$ before $H_j^{k_1,\ldots,k_j}$ provided $i_r < k_r$, where r is the first index from the left such that $i_r \neq k_r$; second, we decree that the last form $H_j^{n-j+1,\ldots,n}$ of degree j in this ordering precedes the first form $H_{j+1}^{1,2,\ldots,j+1}$ of the next presentation degree $j+1$.

The presentation ideal of the symmetric algebra of I is denoted $\mathcal{L}$ as before. It is generated by the Koszul relations $K_{i,j}$, $1 \le i < j \le n$ and the reduced Taylor relations L_i, $1 \le i \le n$, as in (2.5) and (2.6).

We will need the following easy properties of the colon ideal in the proof of the next proposition:

Lemma 2.2.1. *Let $J \subset R$ be an ideal in a ring and $f \in R$. Then:*

(a) $(J : f)f = J \cap (f)$.

(b) *Suppose that R is a polynomial ring over a field and $<$ is a monomial order. Then* $\mathrm{in}_<(J : f) \subset \mathrm{in}_<(J) : \mathrm{in}_<(f)$; *if in addition* $\mathrm{in}_<(J) : \mathrm{in}_<(f) \subset J : f$ *then the equality* $\mathrm{in}_<(J) : \mathrm{in}_<(f) = J : f$ *holds.*

Proof. (a) This is straightforward from the definition of the colon ideal.

(b) The inclusion $\mathrm{in}_<(J:f)\subset \mathrm{in}_<(J):\mathrm{in}_<(f)$ follows immediately from the definition of the initial ideal.

Now let $F\in J:f$. Then, by the above inclusion and the assumption, one has $\mathrm{in}_<(F)\in J:f$, hence $G:=F-\mathrm{in}_<(F)\in J:f$. By induction on the number of nonzero terms of a polynomial in R, we have $G\in \mathrm{in}_<(J):\mathrm{in}_<(f)$. It follows that $F\in \mathrm{in}_<(J):\mathrm{in}_<(f)$. □

2.2.1 Initial ideals

In the following propositions we discuss the preliminaries on Gröbner basis and initial ideals related to the ordered sequence of Sylvester forms.

Proposition 2.2.2. *Let $2\le j\le p-1$, where $p-1\le n-1$ is the reduction number of the ideal $I\subset S=k[x_1,\ldots,x_n]$, and let $i_1<\cdots<i_j$ be an ordered subset of $\{1,\ldots,n\}$. The set*

$$\Sigma(i_1,\ldots,i_j):=\{K_{i,k}\,(1\le i<k\le n), L_i\,(1\le i\le n), H_2^{1,2},\ldots,H_j^{i_1,\ldots,i_j}\}$$

is a Gröbner basis of the ideal $\mathcal{H}(i_1,\ldots,i_j):=(\mathcal{L},H_2^{1,2},\ldots,H_j^{i_1,\ldots,i_j})$ in the lexicographic order on $w>x_n>\cdots>x_1>>\cdots$. In particular, the initial ideal of $\mathcal{H}(i_1,\ldots,i_j)$ is generated by

$$\{x_i^a y_k\,(1\le i<k\le n),\ x_i^{a-b}w\,(1\le i\le n),\ (x_1x_2)^{a-2b}w^2,\ldots,(x_{i_1}\cdots x_{i_j})^{a-jb}w^j\},\tag{2.8}$$

where j and $\{x_{i_1},\ldots,x_{i_j}\}$ flow as in the statement.

Since the proof is a case-by-case scrutiny of S-pairs, we postpone it to Section 2.3.

Proposition 2.2.3. *With the above setting, let $H_{j'}^{k_1,\cdots,k_{j'}}\in S$ denote the first Sylvester form succeeding the Sylvester from $H_j^{i_1,\cdots,i_j}\in S$ in the prescribed ordering of these forms.*

(a) *If $j=j'$, one has*

$$\mathrm{in}(\mathcal{H}(i_1,\ldots,i_j)):\mathrm{in}(H_j^{k_1,\ldots,k_j})=\Big(x_{k_1}^{(j-1)b},\ldots,x_{k_j}^{(j-1)b},x_u^{a-jb},x_{k_j+1}^{a-b},\ldots,x_n^{a-b},(x_{r_1}\cdots x_{r_s})^{(j-s)b},$$
$$(x_{q_1}\cdots x_{q_r})^{(j-s)b}(x_{d_1}\cdots x_{d_{s-r}})^{a-sb}\Big)\,S,$$

for all $k_i<u<k_{i+1}, i=1,\ldots,j-1$ and all choices of indices $s\in\{1,\ldots,j-1\}$, $r\in\{0,\ldots,s-1\}$, of ordered subsets $\{r_1<\cdots<r_s\}\subset\{k_1,\ldots,k_j\}$, $\{q_1<\cdots<q_r\}\subset\{k_1,\ldots,k_j\}$ and of an ordered set $d_1<\cdots<d_{s-r}$ with $k_j<d_1$.

(b) *If $j'=j+1$ (and hence $\{k_1,\cdots,k_{j'}\}=\{1,\ldots,j+1\}$), one has*

$$\mathrm{in}(\mathcal{H}(i_1,\ldots,i_j)):\mathrm{in}(H_{j+1}^{1,\ldots,j+1})=\Big(x_1^{jb},\ldots,x_{j+1}^{jb},x_{j+2}^{a-b},\ldots,x_n^{a-b},(x_{r_1}\cdots x_{r_s})^{(j+1-s)b},$$
$$(x_{q_1}\cdots x_{q_r})^{(j+1-s)b}(x_{d_1}\cdots x_{d_{s-r}})^{a-sb}\Big)\,S,$$

for all choices of indices $s\in\{1,\ldots,j\}$, $r\in\{0,\ldots,s-1\}$, of ordered subsets $\{r_1<\cdots<r_s\}\subset\{1,\ldots,j+1\}$, $\{q_1<\cdots<q_r\}\subset\{1,\ldots,j+1\}$ and of an ordered set $d_1<\cdots<d_{s-r}$ with $j+1<d_1$.

(*In both cases, we adopt the convention that* $x_{q_0} = 1$.)

For both items, we will apply Lemma 2.2.1 (i), by which one is to compute a minimal set of generators of the intersection of the two initial ideals on the left hand side, then divide each generator by the initial term of $H_{j'}^{k_1,\dots,k_{j'}}$. To get a minimal set of generators of the intersection we use a well-known principle (Proposition C.2 and Proposition C.3), by which this set is the set of the least common multiples of $\mathrm{in}(H_{j'}^{k_1,\dots,k_{j'}})$ and each minimal generator of $\mathrm{in}(\mathcal{H}(i_1,\dots,i_j))$.

The details of the proof are given in Section 2.3.

The next result is slightly surprising.

Proposition 2.2.4. *With the previously established notation, one has*

$$\mathcal{H}(i_1,\dots,i_j) : H_{j'}^{k_1,\dots,k_{j'}} = \mathrm{in}(\mathcal{H}(i_1,\dots,i_j)) : \mathrm{in}(H_{j'}^{k_1,\dots,k_{j'}}).$$

In particular, the colon ideal on the left hand side is a monomial ideal.

The proof hinges on the explicit form of the generators given in the previous proposition. The computation is again a case-by-case calculation and quite often it requires some ingenuity as to how the generator looks and how the result of the calculation ought to look like. Since at this point it will give no additional conceptual contribution to the rhythm of the exposition, we once more postpone the details to Section 2.3.

2.2.2 Almost Cohen–Macaulayness

In this part we deal with the depth of the Rees algebra of the ideal $I \subset S$, which wraps-up the main goal of the thesis.

In the notation of the preceding sections, the main result is as follows.

Theorem 2.2.5. *Let* $I \subset R = k[x_1,\dots,x_n]$ *denote a uniform monomial ideal as in* Section 2.1.1. *Then* $S/\mathcal{H}(i_1,\dots,i_j)$ *has depth at least* n *for every tuple* $i_1 < \cdots < i_j$. *In particular, the Rees algebra* $\mathcal{R}_R(I)$ *of* I *is an almost Cohen–Macaulay ring.*

Proof. We basically follow the idea of [38, Theorem 3.14 (b)]. Namely, produce a sequence of mapping cones, each a free resolution of the sequential ideal

$$\mathcal{H}(i_1,\dots,i_j) := (\mathcal{L}, H_2^{1,2},\dots,H_j^{i_1,\dots,i_j})$$

discussed above, ending with a free resolution of $\mathcal{R}_R(I)$; at each step the mapping cone has length at most $n+1$. Therefore, the depth of $\mathcal{R}_R(I)$ will turn out to be at least $2n+1-(n+1) = n$, as desired.

In a precise way, we now argue that for each tuple $i_1 < \cdots < i_j$, starting from the first tuple $1 < 2$, a free S-resolution of $S/\mathcal{H}(k_1,\dots,k_{j'})$ is the mapping cone of the map of complexes from a resolution of $S/(\mathcal{H}(i_1,\dots,i_j) : H_{j'}^{k_1,\dots,k_{j'}})$ to a resolution of $S/\mathcal{H}(i_1,\dots,i_j)$ induced by multiplication by $H_{j'}^{k_1,\ldots,k_{j'}}$ on S, where $k_1 < \cdots < k_{j'}$ is the first tuple succeeding $i_1 < \cdots < i_j$ in the ordering explained before.

To see this, we induct on the number of generators $\mathcal{H}(i_1, \ldots, i_j)$.

Now, by Proposition 2.2.4 and Proposition 2.2.3, the generators of the colon ideal $\mathcal{H}(i_1, \ldots, i_j) : H_{j'}^{k_1,\ldots,k_{j'}}$ are elements of R containing powers of all variables. Therefore, these monomials generate an R_+-primary ideal of R, and hence a free S-resolution of $S/(\mathcal{H}(i_1, \ldots, i_j) : H_{j'}^{k_1,\ldots,k_{j'}})$ is obtained by flat base change $R \subset S$ from a minimal free R-resolution of length n.

In the first step one has $\mathcal{H}(1,2) = (\mathcal{L}, H_2^{1,2})$. Since the ideal $I \subset R$ is an almost complete intersection of finite length, $S/\mathcal{L}$ is Cohen–Macaulay by Theorem D.15. As the codimension of the Rees algebra of I on S is n, the codimension of $S/\mathcal{L}$ is at least n. But since $\mathcal{L} \subset R_+ = (x_1, \ldots, x_n)S$ then the codimension is n.

We consider the map of complexes induced by multiplication by $H_2^{1,2}$ on S:

$$\begin{array}{ccccccccccc} 0 & \to & S^{\beta_n} & \longrightarrow & \cdots & \longrightarrow & S^{\beta_2} & \longrightarrow & S^{\beta_1} & \longrightarrow & S & \to & 0 \\ & & \uparrow & & & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \to & S^{\alpha_n} & \longrightarrow & \cdots & \longrightarrow & S^{\alpha_2} & \longrightarrow & S^{\alpha_1} & \longrightarrow & S & \to & 0 \end{array},$$

where the upper complex is a free resolution of $S/\mathcal{L}$ and the lower one is the free S-resolution of $S/(\mathcal{L} : H_2^{1,2})$ extended from the free R-resolution by flat base change $R \subset S$. (Note that $\beta_1 = \binom{n+1}{2}$ is the minimal number of generators of $\mathcal{L}$, but all the remaining Betti number of both resolutions are harder to guess.)

The mapping cone is a free S-resolution of $S/\mathcal{H}(1,2)$ (not minimal as there will be cancellation in general). By definition, this S-resolution has length at most $n+1$.

The general step of the induction is entirely similar, by taking the mapping cone of the map of complexes induced by multiplication by $H_{j'}^{k_1,\ldots,k_{j'}}$:

$$\begin{array}{ccccccccccccc} 0 & \to & S^{\beta_{n+1}} & \longrightarrow & S^{\beta_n} & \longrightarrow & \cdots & \longrightarrow & S^{\beta_2} & \longrightarrow & S^{\beta_1} & \longrightarrow & S & \to & 0 \\ & & \uparrow & & \uparrow & & & & \uparrow & & \uparrow & & & & \\ & & 0 & \longrightarrow & S^{\alpha_n} & \longrightarrow & \cdots & \longrightarrow & S^{\alpha_2} & \longrightarrow & S^{\alpha_1} & \longrightarrow & S & \to & 0 \end{array},$$

where the upper complex is a (not necessarily minimal) free resolution of $S/\mathcal{H}(i_1, \ldots, i_j)$ and the lower one is the S-resolution of $S/(\mathcal{H}(i_1, \ldots, i_j) : H_{j'}^{k_1,\ldots,k_{j'}})$ extended by flat base change from a minimal free R-resolution. Here we have used for simplicity the same notation for the Betti number as above, but of course they are different.

Because the lower complex has length at most the length of the upper complex, the mapping cone is again a free S-resolution of length at most $n+1$.

By Theorem 2.1.5 and the previous discussion of this section, the presentation ideal $\mathcal{I}$ of the Rees algebra on S is the sequential ideal $\mathcal{H}(1, \ldots, p)$, where $p-1$ is the reduction number of I. Therefore the above gives that $\mathcal{R}_R(I)$ has an S-resolution of length at most $n+1$, as was to be shown. □

Since the reduction number of I is independent, by the Proposition 2.1.2 and by the ([22, Theorem 3.5]), we can describe the regularity of the Rees algebra of I.

Corollary 2.2.6. *Let $I \subset R = k[x_1, \ldots, x_n]$ denote a uniform monomial ideal as above. Then,*

(a) *If $nb \geq a$, in this case, letting $1 \leq p \leq n$ be the smallest integer such that $pb \geq a$ (hence $(p-1)b < a$), then $\operatorname{reg}(\mathcal{R}_R(I)) = p$.*

(b) *If $nb < a$, then $\operatorname{reg}(\mathcal{R}_R(I)) = n$.*

2.3 Proofs

2.3.1 Proof of Proposition 2.2.2

The proof will compute all $S-$pairs of elements in the set $\Sigma = \Sigma(i_1, \ldots, i_j)$. As usual, pairs F, G such $\gcd(\text{in}(F), \text{in}(G)) = 1$ will be overlooked.

Case 1. $S(K_{i,k}, K_{i',k'})$.
In this case, $\text{in}(K_{i,k}) = x_k^a y_i$ and $\text{in}(K_{i',k'}) = x_{k'}^a y_{i'}$.

Case 1.1. Let $i < k < i' < k'$. No action here since $\text{in}(K_{i,k})$ and $\text{in}(K_{i',k'})$ are relatively prime.

Case 1.2. Let $i < i' < k < k'$. No action here since $\text{in}(K_{i,k})$ and $\text{in}(K_{i',k'})$ are relatively prime.

Case 1.3. Let $i' < i < k < k'$. No action here since $\text{in}(K_{i,k})$ and $\text{in}(K_{i',k'})$ are relatively prime.

Case 1.4. Let $k = k'$. Then

$$S(K_{i,k}, K_{i',k}) = \frac{x_k^a y_i y_{i'}}{-x_k^a y_i} K_{i,k} - \frac{x_k^a y_i y_{i'}}{-x_k^a y_{i'}} K_{i',k} = -y_k(x_i^a y_{i'} - x_{i'}^a y_i) \equiv 0 \mod \Sigma.$$

Case 1.5. Let $i < k = i' < k'$. No action here since $\text{in}(K_{i,k})$ and $\text{in}(K_{k,k'})$ are relatively prime.

Case 1.6. Let $i' < k' = i < k$. No action here since $\text{in}(K_{i,k})$ and $\text{in}(K_{k,i})$ are relatively prime.

Case 1.7. Let $i = i'$. Then

$$S(K_{i,k}, K_{i,k'}) = \frac{x_k^a x_{k'}^a y_i}{-x_k^a y_i} K_{i,k} - \frac{x_k^a x_{k'}^a y_i}{-x_{k'}^a y_i} K_{i,k'} = -x_i^a(x_{k'}^a y_k - x_k^a y_{k'}) \equiv 0 \mod \Sigma.$$

Case 2. $S(L_j, L_{j'})$, with $j < j'$.
In this case, $\text{in}(L_j) = x_j^{a-b} w$ and $\text{in}(L_{j'}) = x_{j'}^{a-b} w$.
Then

$$S(L_j, L_{j'}) = \frac{x_j^{a-b} x_{j'}^{a-b} w}{x_j^{a-b} w} L_j - \frac{x_j^{a-b} x_{j'}^{a-b} w}{x_{j'}^{a-b} w} L_{j'} = P(j, j')^b K_{j,j'} \equiv 0 \mod \Sigma.$$

Case 3. $S(L_j, K_{i,k})$.
In this case, $\text{in}(L_j) = x_j^{a-b} w$ and $\text{in}(K_{i,k}) = x_k^a y_i$.

Case 3.1. Let $j < i < k$. No action here since $\text{in}(L_j)$ and $\text{in}(K_{i,k})$ are relatively prime.

Case 3.2. Let $i < j < k$. No action here since $\text{in}(L_j)$ and $\text{in}(K_{i,k})$ are relatively prime.

Case 3.3. Let $i < k < j$. No action here since $\text{in}(L_j)$ and $\text{in}(K_{i,k})$ are relatively prime.

Case 3.4. Let $j = i < k$. No action here since $\mathrm{in}(L_i)$ and $\mathrm{in}(K_{i,k})$ are relatively prime.

Case 3.5. Let $i < j = k$. Then

$$S(L_k, K_{i,k}) = \frac{x_k^a w y_i}{x_k^{a-b} w} L_k - \frac{x_k^a w y_i}{-x_k^a y_i} K_{i,k} = x_i^b y_k L_i \equiv 0 \mod \Sigma.$$

Case 4. $S(K_{u,k}, H_j^{i_1,\ldots,i_j})$.

In this case, $\mathrm{in}(K_{u,k}) = x_k^a y_u$ and $\mathrm{in}(H_j^{i_1,\ldots,i_j}) = (x_{i_1} \cdots x_{i_j})^{a-jb} w^j$.

Case 4.1. Let $u < k$, $u, k \notin \{i_1, \ldots, i_j\}$. No action here since $\mathrm{in}(K_{u,k})$ and $\mathrm{in}(H_j^{i_1,\ldots,i_j})$ are relatively prime.

Case 4.2. Let $u < k$, $u \in \{i_1, \ldots, i_j\}$ and $k \notin \{i_1, \ldots, i_j\}$. No action here since $\mathrm{in}(K_{u,k})$ and $\mathrm{in}(H_j^{i_1,\ldots,i_j})$ are relatively prime.

Case 4.3. Let $u < k$, $u \notin \{i_1, \ldots, i_j\}$, and $k \in \{i_1, \ldots, i_j\}$. Then

$$S(K_{u,k}, H_j^{i_1,\ldots,i_j}) = \frac{x_k^a y_u (x_{i_1} \cdots \widehat{x_k} \cdots x_{i_j})^{a-jb} w^j}{-x_k^a y_u} K_{u,k} - \frac{x_k^a y_u (x_{i_1} \cdots \widehat{x_k} \cdots x_{i_j})^{a-jb} w^j}{(x_{i_1} \cdots x_k \cdots x_{i_j})^{a-jb} w^j} H_j^{i_1,\ldots,i_j}$$

$$= (-x_u^{jb} y_k) H_j^{I'} \equiv 0 \mod \Sigma, \text{ where } I' = (\{i_1, \ldots, i_j\} \setminus \{k\}) \cup \{u\}.$$

Case 4.4. Let $u < k$, $u, k \in \{i_1, \ldots, i_j\}$. Then

$$S(K_{u,k}, H_j^{i_1,\ldots,i_j}) = \frac{x_k^a y_u (x_{i_1} \cdots \widehat{x_k} \cdots x_{i_j})^{a-jb} w^j}{-x_k^a y_u} K_{u,k} - \frac{x_k^a y_u (x_{i_1} \cdots \widehat{x_k} \cdots x_{i_j})^{a-jb} w^j}{(x_{i_1} \cdots x_k \cdots x_{i_j})^{a-jb} w^j} H_j^{i_1,\ldots,i_j}$$

$$= -y_k [x_u^a (x_{i_1} \cdots \widehat{x_k} \cdots x_{i_j})^{a-jb} w^j - x_k^{jb} y_u P(i_1, \ldots, i_j)^{jb} y_{i_1} \cdots y_u \cdots \widehat{y_k} \cdots y_{i_j}].$$

Since $x_u^{a-b} w = L_u + P(u)^b y_u$, it obtains

$$S(K_{u,k}, H_j^{i_1,\ldots,i_j}) \equiv -P(i_1, \ldots, \widehat{k}, \ldots, i_j)^b y_u y_k H_{j-1}^{I''} \equiv 0 \mod \Sigma,$$

where $I'' = \{i_1, \ldots, i_j\} \setminus \{k\}$.

Case 5. $S(L_u, H_j^{i_1,\ldots,i_j})$.

In this case, $\mathrm{in}(L_u) = x_u^{a-b} w$ and $\mathrm{in}(H_j^{i_1,\ldots,i_j}) = (x_{i_1} \cdots x_{i_j})^{a-jb} w^j$.

Case 5.1. Let $u \notin \{i_1, \ldots, i_j\}$. Then

$$\begin{aligned} S(L_u, H_j^{i_1,\ldots,i_j}) &= -P(i_1, \ldots, i_j, u)^b [(x_{i_1} \cdots x_{i_j})^{a-(j-1)b} w^{j-1} y_u \\ &- x_u^{a+(j-1)b} P(i_1, \ldots, i_j, u)^{(j-1)b} y_{i_1}, \ldots, y_{i_j}]. \end{aligned}$$

Pick any subset $I' \subset \{i_1, \ldots, i_j\}$, with $|I'| = j - 1$ and reduce modulo $H_{j-1}^{I'}$ the monomial with w occurring within the square brackets. The result is a binomial not involving w. By the same argument as before, we conclude that this pair reduces to 0 modulo Σ.

Case 5.2. Let $u \in \{i_1, \ldots, i_j\}$. Then

$$S(L_u, H_j^{i_1,\ldots,i_j}) = -P(i_1, \ldots, i_j)^b y_u H_{j-1}^{i_1,\ldots,\widehat{u},\ldots,i_j} \equiv 0 \mod \Sigma.$$

Case 6. Consider $H_m^{i_1,\ldots,i_m}$ and $H_{m'}^{j_1,\ldots,j_{m'}}$, with the respective external degrees $m \le m'$. Denote $\mathfrak{I} := \{i_1, \ldots, i_m\}$, $\mathfrak{I}' := \{j_1, \ldots, j_{m'}\}$ and let $\mathfrak{I} \cap \mathfrak{I}' = \{k_1, \ldots, k_s\}$, for some $s \in \{0, \ldots, m\}$.

Under the given order, the two leading terms of the two binomials are $\mathrm{in}(H_m^{i_1,\ldots,i_m}) = (x_{i_1} \cdots x_{i_m})^{a-mb} w^m$ and $\mathrm{in}(H_{m'}^{j_1,\ldots,j_{m'}}) = (x_{j_1} \cdots x_{j_{m'}})^{a-m'b} w^{m'}$, so their least common multiple is $w^{m'}(x_{j_1} \cdots \widehat{x_{i_1}} \cdots \widehat{x_{i_m}} \cdots x_{j_{m'}})^{a-m'b}(x_{i_1} \cdots x_{i_m})^{a-mb}$. Therefore

$$\begin{aligned} S(H_m^{i_1,\ldots,i_m}, H_{m'}^{j_1,\ldots,j_{m'}}) = & -P(i_1, \ldots, j_1, \ldots, i_m, \ldots, j_{m'})^{mb} y_{k_1} \cdots y_{k_s} \\ & \cdot \Big[w^{m'-m}(x_{j_1} \cdots \widehat{x_{i_1}} \cdots \widehat{x_{i_m}} \cdots x_{j_{m'}})^{a-m'b+mb} y_{i_1} \cdots \widehat{y_{j_1}} \cdots \widehat{y_{j_{m'}}} \cdots y_{i_m} \\ & -(x_{i_1} \cdots \widehat{x_{j_1}} \cdots \widehat{x_{j_{m'}}} \cdots x_{i_m})^{a-mb+m'b}(x_{k_1} \cdots x_{k_s})^{(m'-m)b} \\ & \cdot P(i_1, \ldots, j_1, \ldots, i_m, \ldots, j_{m'})^{(m'-m)b} y_{j_1} \cdots \widehat{y_{i_1}} \cdots \widehat{y_{i_m}} \cdots y_{j_{m'}} \Big]. \end{aligned}$$

If $m' = m$, then the binomial inside the square brackets does not involve w and therefore reduces to 0 modulo Σ by previous cases.

If $m' > m$, then $|\mathfrak{I}'| = m' > m = |\mathfrak{I}|$, and therefore $|\mathfrak{I}' \setminus \mathfrak{I}| \ge m' - m \ge 1$. Let $\widetilde{\mathfrak{I}} \subset \mathfrak{I}' \setminus \mathfrak{I}$ with $|\widetilde{\mathfrak{I}}| = m' - m$. Reducing the binomial inside the square brackets modulo $H_{m'-m}^{\widetilde{\mathfrak{I}}} \in \Sigma$, will result in the cancellation of the monomial involving w, hence we are back to the previous situation. □

2.3.2 Proof of Proposition 2.2.3

To apply Lemma 2.2.1 (i) we set ourselves to compute a minimal set of generators of the intersection of the two initial ideals on the left hand side, then divide each generator by the initial term of $H_{j'}^{k_1,\ldots,k_{j'}}$. A minimal set of generators of the intersection turns out to be the set of the least common multiples of $\mathrm{in}(H_{j'}^{k_1,\ldots,k_{j'}})$ and each minimal generator of $\mathrm{in}(\mathcal{H}(i_1, \ldots, i_j))$.

We separate the two cases, according as to whether $j' = j$ or $j' = j + 1$.

(a) Same degree: $j = j'$

One has $\mathrm{in}(H_j^{k_1,\ldots,k_j}) = (x_{k_1} \cdots x_{k_j})^{a-jb} w^j$. Drawing upon (2.8), according to the external degree of a monomial, we have

Degree 1:

- $x_d^{a-b} w$, $d \notin \{k_1, \ldots, k_j\}$ and $d < k_j$ (coming from $L_d \in \mathcal{L}$)

$$\frac{\mathrm{lcm}(x_d^{a-b} w, (x_{k_1} \cdots x_{k_j})^{a-jb} w^j)}{(x_{k_1} \cdots x_{k_j})^{a-jb} w^j} = x_d^{a-b}.$$

As $j \le p - 1$, and $a > (p-1)b$, then $a - jb > 0$. But then $x_d^{a-b} = x_d^{(j-1)b} x_d^{a-jb}$, and x_d^{a-jb} is among the generators listed in the right hand side monomial ideal.

- $x_d^{a-b}w,\ d \notin \{k_1, \ldots, k_j\}$ and $d > k_j$ (coming from $L_d \in \mathcal{L}$)

$$\frac{\operatorname{lcm}(x_d^{a-b}w, (x_{k_1} \cdots x_{k_j})^{a-jb}w^j)}{(x_{k_1} \cdots x_{k_j})^{a-jb}w^j} = x_d^{a-b}.$$

As above, x_d^{a-jb} is among the generators listed in the right hand side monomial ideal.

- $x_d^{a-b}w,\ d \in \{k_1, \ldots, k_j\}$ (coming from $L_d \in \mathcal{L}$)

$$\frac{\operatorname{lcm}(x_d^{a-b}w, (x_{k_1} \cdots x_{k_j})^{a-jb}w^j)}{(x_{k_1} \cdots x_{k_j})^{a-jb}w^j} = x_d^{(j-1)b},$$

which is among the generators listed in the right hand side monomial ideal.

- $x_d^a y_v,\ d \notin \{k_1, \ldots, k_j\}$ and $d < k_j$ (coming from $K_{d,v} \in \mathcal{L}$)

$$\frac{\operatorname{lcm}(x_d^a y_v, (x_{k_1} \cdots x_{k_j})^{a-jb}w^j)}{(x_{k_1} \cdots x_{k_j})^{a-jb}w^j} = x_d^a y_v$$

Note that $x_d^a y_v = (x_d^{jb} y_v)x_d^{a-jb}$, while x_d^{a-jb} is among the generators listed in the right hand side monomial ideal.

- $x_d^a y_v,\ d \notin \{k_1, \ldots, k_j\}$ and $d > k_j$ (coming from $K_{d,v} \in \mathcal{L}$)

$$\frac{\operatorname{lcm}(x_d^a y_v, (x_{k_1} \cdots x_{k_j})^{a-jb}w^j)}{(x_{k_1} \cdots x_{k_j})^{a-jb}w^j} = x_d^a y_v$$

One has $x_d^a y_v = (x_d^b y_v)x_d^{a-b}$, while x_d^{a-b} is among the generators listed in the right hand side monomial ideal.

- $x_d^a y_v,\ d \in \{k_1, \ldots, k_j\}$ (coming from $K_{d,v} \in \mathcal{L}$)

$$\frac{\operatorname{lcm}(x_d^a y_v, (x_{k_1} \cdots x_{k_j})^{a-jb}w^j)}{(x_{k_1} \cdots x_{k_j})^{a-jb}w^j} = x_d^{jb} y_v$$

Note that $x_d^{jb} y_v = (x_d^{(j-1)b} y_v)x_d^b$, so once more we get a generator listed in the right hand side monomial ideal.

Degree s $(2 \leq s \leq j-1)$:

- $(x_{d_1} \cdots x_{d_r} x_{q_1} \cdots x_{q_{s-r}})^{a-sb}w^s,\ \{d_1 < \ldots < d_r\} \cap \{k_1, \ldots, k_j\} = \emptyset$, $d_1 < k_j$ and $\{q_1 < \cdots < q_{s-r}\} \subset \{k_1, \ldots, k_j\}$.

$$\begin{aligned}&\frac{\operatorname{lcm}((x_{d_1} \cdots x_{d_r} x_{q_1} \cdots x_{q_{s-r}})^{a-sb}w^s, (x_{k_1} \cdots x_{k_j})^{a-jb}w^j)}{(x_{k_1} \cdots x_{k_j})^{a-jb}w^j} \\ = \ & (x_{d_1} \cdots x_{d_r})^{a-sb}(x_{q_1} \cdots x_{q_{s-r}})^{(j-s)b}\end{aligned}$$

Note that $x_{d_1}^{a-sb}$ is a factor thereof factoring further as $x_{d_1}^{a-sb} = x_{d_1}^{(j-s)b} x_{d_1}^{a-jb}$, while $x_{d_1}^{a-jb}$ is among the generators listed in the right hand side monomial ideal since $d_1 \notin \{k_1, \ldots, k_j\}$ and $d_1 < k_j$.

- $(x_{q_1} \cdots x_{q_s})^{a-sb} w^s,\ \{q_1 < \cdots < q_s\} \subset \{k_1, \ldots, k_j\}$.

$$\frac{\operatorname{lcm}((x_{q_1} \cdots x_{q_s})^{a-sb} w^s, (x_{k_1} \cdots x_{k_j})^{a-jb} w^j)}{(x_{k_1} \cdots x_{k_j})^{a-jb} w^j} = (x_{q_1} \cdots x_{q_s})^{(j-s)b}$$

- $(x_{q_1} \cdots x_{q_r} x_{d_1} \cdots x_{d_{s-r}})^{a-sb} w^s,\ \{q_1 < \cdots < q_r\} \subset \{k_1, \ldots, k_j\},\ d_1 < \ldots < d_{s-r}$ with $k_j < d_1$.

$$\begin{aligned} &\frac{\operatorname{lcm}((x_{q_1} \cdots x_{q_r} x_{d_1} \cdots x_{d_{s-r}})^{a-sb} w^s, (x_{k_1} \cdots x_{k_j})^{a-jb} w^j)}{(x_{k_1} \cdots x_{k_j})^{a-jb} w^j} \\ = \ & (x_{q_1} \cdots x_{q_r})^{(j-s)b} (x_{d_1} \cdots x_{d_{s-r}})^{a-sb} \end{aligned}$$

- $(x_{d_1} \cdots x_{d_s})^{a-sb} w^s,\ d_1 < \ldots < d_s$ with $k_j < d_1$.

$$\frac{\operatorname{lcm}((x_{d_1} \cdots x_{d_s})^{a-sb} w^s, (x_{k_1} \cdots x_{k_j})^{a-jb} w^j)}{(x_{k_1} \cdots x_{k_j})^{a-jb} w^j} = (x_{d_1} \cdots x_{d_s})^{a-sb}.$$

In all three cases above the resulting monomial is among the generators listed in the right hand side monomial ideal.

Degree j :

- $(x_{k_1} \cdots \widehat{x_{k_c}} \ldots x_{k_j})^{a-jb} {x_d}^{a-jb} w^j, d \notin \{k_1, \ldots, k_j\},\ d < k_j,\ c \in \{1, \ldots, j\}$.

$$\frac{\operatorname{lcm}((x_{k_1} \cdots \widehat{x_{k_c}} \ldots x_{k_j})^{a-jb} {x_d}^{a-jb} w^j, (x_{k_1} \cdots x_{k_j})^{a-jb} w^j)}{(x_{k_1} \cdots x_{k_j})^{a-jb} w^j} = x_d^{a-jb}.$$

Again we conclude as before.

(b) Degree jump: $j' = j + 1$

We now consider the case where the degree goes up, that is, one is dealing with

$$\operatorname{in}(H_{j+1}^{1,\ldots,j+1}) = (x_1 \cdots x_{j+1})^{a-(j+1)b} w^{j+1}.$$

We go through similar calculations as before. In each case below the resulting monomial is among the generators listed in the right hand side monomial ideal.

Degree 1:

- $x_d^{a-b} w,\ d \notin \{1, \ldots, j+1\}$.

$$\frac{\operatorname{lcm}(x_d^{a-b} w, (x_1 \cdots x_{j+1})^{a-(j+1)b} w^{j+1})}{(x_1 \cdots x_{j+1})^{a-(j+1)b} w^{j+1}} = x_d^{a-b}$$

- $x_d^{a-b}w,\ d \in \{1,\ldots,j+1\}$.

$$\frac{\operatorname{lcm}(x_d^{a-b}w,(x_1\cdots x_{j+1})^{a-(j+1)b}w^{j+1})}{(x_1\cdots x_{j+1})^{a-(j+1)b}w^{j+1}} = x_d^{jb}$$

- $x_d^a y_v,\ d \notin \{1,\ldots,j+1\}$.

$$\frac{\operatorname{lcm}(x_d^a y_v,(x_1\cdots x_{j+1})^{a-(j+1)b}w^{j+1})}{(x_1\cdots x_{j+1})^{a-(j+1)b}w^{j+1}} = x_d^a y_v$$

  Note that $x_d^a y_v = (x_d^b y_v)x_d^{a-b}$.

- $x_d^a y_v,\ d \in \{1,\ldots,j+1\}$.

$$\frac{\operatorname{lcm}(x_d^a y_v,(x_1\cdots x_{j+1})^{a-(j+1)b}w^{j+1})}{(x_1\cdots x_{j+1})^{a-(j+1)b}w^{j+1}} = x_d^{(j+1)b}y_v$$

  Note that $x_d^{(j+1)b}y_v = (x_d^b y_v)x_d^{jb}$.

Degree s $(2 \le s \le j)$:

- $(x_{q_1}\cdots x_{q_s})^{a-sb}w^s$, with $\{q_1 < \cdots < q_s\} \subset \{1,\ldots,j+1\}$.

$$\frac{\operatorname{lcm}((q_1\cdots x_{q_s})^{a-sb}w^s,(x_1\cdots x_{j+1})^{a-(j+1)b}w^{j+1})}{(x_1\cdots x_{j+1})^{a-(j+1)b}w^{j+1}} = (x_{q_1}\cdots x_{q_s})^{(j+1-s)b}$$

- $(x_{q_1}\cdots x_{q_r}x_{d_1}\cdots x_{d_{s-r}})^{a-sb}w^s,\ \{q_1 < \cdots < q_r\} \subset \{1,\ldots,j+1\},\ d_1 < \ldots < d_{s-r}$ with $j+1 < d_1$.

$$\begin{aligned}&\frac{\operatorname{lcm}((x_{q_1}\cdots x_{q_r}x_{d_1}\cdots x_{d_{s-r}})^{a-sb}w^s,(x_1\cdots x_{j+1})^{a-(j+1)b}w^{j+1})}{(x_1\cdots x_{j+1})^{a-(j+1)b}w^{j+1}}\\ &= \ (x_{q_1}\cdots x_{q_r})^{(j+1-s)b}(x_{d_1}\cdots x_{d_{s-r}})^{a-sb}\end{aligned}$$

- $(x_{d_1}\cdots x_{d_s})^{a-sb}w^s,\ d_1 < \ldots < d_{s-r}$ with $j+1 < d_1$.

$$\frac{\operatorname{lcm}((x_{d_1}\cdots x_{d_s})^{a-sb}w^s,(x_1\cdots x_{j+1})^{a-(j+1)b}w^{j+1})}{(x_1\cdots x_{j+1})^{a-(j+1)b}w^{j+1}} = (x_{d_1}\cdots x_{d_s})^{a-sb}$$

To conclude the present case of degree jump, we stress the limit situation where the degree jumps to the highest possible degree of a Sylvester form. It is convenient to separate the two basic settings:

SETTING $a > nb$.

The expected outcome is $\operatorname{in}(\mathcal{H}(2,\ldots,n) : \operatorname{in}(H_n^{1,\ldots,n}) = (x_1,\ldots,x_n)^{(n-1)b}S$ and the calculation of the required least common multiples is included in the general calculation above, setting $j = n$.

SETTING $a \le nb,\ (p-1)b < a \le pb$.

Here $H_p^{1,\dots,p} = w^p - (x_{p+1}\cdots x_n)^{pb}(x_1\cdots x_p)^{pb-a}y_1\cdots y_p$. The typical expected generator has one of the following forms

$$(x_{r_1}\cdots x_{r_s})^{a-sb} \text{ and } (x_{q_1}\cdots x_{q_r}x_{d_1}\cdots x_{d_{s-r}})^{a-sb},$$

where $s \in \{2,\dots,p-1\}$, $r \in \{0,\dots,s-1\}$, $\{r_1 < \cdots < r_s\} \subset \{1,\dots,p\}$, $\{q_1 < \cdots < q_r\} \subset \{1,\dots,p\}$, $\{d_1 < \cdots < d_{s-r}\} \subset \{p+1,\dots,n\}$.

Here is the calculation for this setting, according to the external degrees of the generating monomials:

Degree 1:

- $x_d^{a-b}w$, $d \notin \{1,\dots,p\}$.
$$\frac{\operatorname{lcm}(x_d^{a-b}w, w^p)}{w^p} = x_d^{a-b}$$
- $x_d^{a-b}w$, $d \in \{1,\dots,p\}$.
$$\frac{\operatorname{lcm}(x_d^{a-b}w, w^p)}{w^p} = x_d^{a-b}$$
- $x_d^a y_v$, $d \notin \{1,\dots,p\}$.
$$\frac{\operatorname{lcm}(x_d^a y_v, w^p)}{w^p} = x_d^a y_v$$
  Note that $x_d^a y_v = (x_d^b y_v)x_d^{a-b}$.
- $x_d^a y_v$, $d \in \{1,\dots,p\}$.
$$\frac{\operatorname{lcm}(x_d^a y_v, w^p)}{w^p} = x_d^a y_v$$
  Note that $x_d^a y_v = (x_d^b y_v)x_d^{a-b}$.

Degree s $(2 \le \mathbf{s} \le p-1)$:

- $(x_{q_1}\cdots x_{q_s})^{a-sb}w^s$, $\{q_1 < \cdots < q_s\} \subset \{1,\dots,p\}$
$$\frac{\operatorname{lcm}((q_1\cdots x_{q_s})^{a-sb}w^s, w^p)}{w^p} = (x_{q_1}\cdots x_{q_s})^{a-sb}$$
- $(x_{q_1}\cdots x_{q_r}x_{d_1}\cdots x_{d_{s-r}})^{a-sb}w^s$, $\{q_1 < \cdots < q_s\} \subset \{1,\dots,p\}$, $d_1 < \ldots < d_{s-r}$ with $p < d_1$
$$\frac{\operatorname{lcm}((x_{q_1}\cdots x_{q_r}x_{d_1}\cdots x_{d_{s-r}})^{a-sb}w^s, w^p)}{w^p} = (x_{q_1}\cdots x_{q_r}x_{d_1}\cdots x_{d_{s-r}})^{a-sb}$$
- $(x_{d_1}\cdots x_{d_s})^{a-sb}w^s$, $d_1 < \ldots < d_s$ with $p < d_1$
$$\frac{\operatorname{lcm}((x_{d_1}\cdots x_{d_s})^{a-sb}w^s, w^p)}{w^p} = (x_{d_1}\cdots x_{d_s})^{a-sb}$$

This concludes the proof of the proposition.

2.3.3 Proof of Proposition 2.2.4

We just have to prove the inclusion $\supset$ since the inclusion $\subset$ follows from it by applying Lemma 2.2.1 (ii).

Again, we deal with two cases, according to the established sets of generators for the right hand side of the stated equality in either (a) or (b) of Proposition 2.2.2.

Same degree

- $x_s^{a-b},\ s = k_j + 1, \ldots, n.$

$$x_s^{a-b} H_j^{k_1,\ldots,k_j} = (x_{k_1} \cdots x_{k_j})^{a-jb} w^{j-1} L_s + x_{k_1}^{a-(j-1)b} P(k_1, \ldots, k_j, s)^b y_s H_{j-1}^{k_2,\ldots,k_j}$$
$$+ P(k_1, \ldots, k_j, s)^{jb} x_s^{(j-1)b} y_{k_2} \cdots y_{k_j} K_{k_1,s}.$$

- $x_{k_s}^{(j-1)b},\ s = 1, \ldots, j.$

$$x_{k_s}^{(j-1)b} H_j^{k_1,\ldots,k_j} = (x_{k_1} \cdots \widehat{x_{k_s}} \cdots x_{k_j})^{a-jb} w^{j-1} L_{k_s} + P(k_1, \ldots, k_s, \ldots, k_j)^b y_{k_s} H_{j-1}^{k_1,\ldots,\widehat{k_s},\ldots,k_j}.$$

- $(x_{r_1} \cdots x_{r_s})^{(j-s)b},\ s \in \{1, \ldots, j-1\},\ \{r_1 < \cdots < r_s\} \subset \{k_1, \ldots, k_j\}.$

$$(x_{r_1} \cdots x_{r_s})^{(j-s)b} H_j^{k_1,\ldots,k_j} = (x_{k_1} \cdots \widehat{x_{r_1}} \cdots \widehat{x_{r_s}} \cdots x_{k_j})^{a-jb} w^{j-s} H_s^{r_1,\ldots,r_s} +$$
$$P(k_1, \ldots, r_1, \ldots, r_s, \ldots, k_j)^{sb} y_{r_1} \cdots y_{r_s} H_{j-s}^{k_1,\ldots,\widehat{r_1},\ldots,\widehat{r_s},\ldots,k_j}$$

- $x_r^{a-jb},\ r < k_1.$

$$x_r^{a-jb} H_j^{k_1,\ldots,k_j} = x_{k_1}^{a-jb} H_j^{r,k_2,\ldots,k_j} - P(r, k_1, \ldots, k_j)^{jb} y_{k_2} \cdots y_{k_j} K_{r,k_1}.$$

- $x_r^{a-jb}, k_i < r < k_{i+1},\ i = 1, \ldots, j-1$.

$$\begin{aligned} x_r^{a-jb} H_j^{k_1,\ldots,k_j} &= x_{k_{i+1}}^{a-jb} H_j^{k_1,k_2,\ldots,k_i,r,\widehat{k_{i+1}},\ldots,k_j} \\ &- P(k_1, \ldots, k_i, r, k_{i+1}, \ldots, k_j)^{jb} y_{k_1} \cdots y_{k_i} \widehat{y_{k_{i+1}}} y_{k_{i+2}} \cdots y_{k_j} K_{r,k_{i+1}}. \end{aligned}$$

- $(x_{q_1} \cdots x_{q_r})^{(j-s)b} (x_{d_1} \cdots x_{d_{s-r}})^{a-sb}.$

Since this case is a lot more involved than the previous ones, we chose to formulate it as a lemma.

Lemma 2.3.1. *Fix an integer $2 \leq j \leq p-1$ and an ordered subset $\{k_1, \ldots, k_j\} \subset \{1, \ldots, n\}$. Let there be given integers $s \in \{1, \ldots, j-1\}$, $r \in \{0, \ldots, s-1\}$ and ordered subsets $\{q_1, \ldots, q_r\} \subset$*

$\{k_1,\ldots,k_j\}$ and $\{d_1,\ldots,d_{s-r}\}\subset\{1,\ldots,n\}\setminus\{k_1,\ldots,k_j\}$, with $k_j<d_1$. Consider a 2-partition of $\{k_1,\ldots,k_j\}\setminus\{q_1,\ldots,q_r\}$ by ordered subsets $\{k_{m_1},\ldots,k_{m_{j-s}}\}$ and $\{n_1,\ldots,n_{s-r}\}$. Set

$$Q:=P(k_1,\ldots,k_j,\,d_1,\ldots,d_{s-r})^{jb}\,(x_{q_1}\cdots x_{q_r})^{(j-s)b}y_{q_1}\cdots y_{q_r}y_{k_{m_1}}\cdots y_{k_{m_{j-s}}}.$$

Then

$$\begin{aligned}(x_{q_1}\cdots x_{q_r})^{(j-s)b}(x_{d_1}\cdots x_{d_{s-r}})^{a-sb}H_j^{k_1,\ldots,k_j} &= (x_{k_{m_1}}\cdots x_{k_{m_{j-s}}}\,x_{n_1}\cdots x_{n_{s-r}})^{a-jb}\\ &\cdot w^{j-s}H_s^{q_1,\ldots,q_r,d_1,\ldots,d_{s-r}}\\ &+P(k_1,\ldots,k_j,d_1,\ldots,d_{s-r})^{sb}\cdot(x_{n_1}\cdots x_{n_{s-r}})^{a-(j-s)b}\\ &\cdot y_{q_1}\cdots y_{q_r}y_{d_1}\cdots y_{d_{s-r}}H_{j-s}^{k_{m_1},\ldots,k_{m_{j-s}}}\\ &+\sum_{c=1}^{s-r}(x_{n_{c+1}}\cdots x_{n_{s-r}})^{a}(x_{d_1}\cdots x_{d_{c-1}})^{a+(j-s)b}(x_{d_c}\cdots x_{d_{s-r}})^{(j-s)b}\,Q\\ &\cdot y_{d_{c+1}}\cdots y_{d_{s-r}}y_{n_1}\cdots y_{n_{c-1}}K_{n_c,d_c},\end{aligned}$$

with the convention that $x_{d_0}=y_{n_0}=x_{n_{s-r+1}}=x_{d_{s-r+1}}=1$.

Proof. Although the above expression is verifiable by expanding the right hand side, the idea to get at it is by no means obvious. Since similar expressions will appear in the sequel, we will now explain its main core. Thus, first write

$$\begin{aligned}(x_{q_1}\cdots x_{q_r})^{(j-s)b}(x_{d_1}\cdots x_{d_{s-r}})^{a-sb}H_j^{k_1,\ldots,k_j} &= (x_{q_1}\cdots x_{q_r})^{(j-s)b}(x_{d_1}\cdots x_{d_{s-r}})^{a-sb}\\ &\cdot\big((x_{k_1}\cdots x_{k_j})^{a-jb}w^j-P(k_1,\ldots,k_j)^{jb}\,y_{k_1}\cdots y_{k_j}\big)\\ &=(x_{k_1}\cdots\widehat{x_{q_1}}\cdots\widehat{x_{q_r}}\cdots x_{k_j})^{a-jb}\,(x_{q_1}\cdots x_{q_r})^{a-sb}\,(x_{d_1}\cdots x_{d_{s-r}})^{a-sb}w^j \qquad (2.9)\\ &-(x_{q_1}\cdots x_{q_r})^{(j-s)b}\,(x_{d_1}\cdots x_{d_{s-r}})^{a+(j-s)b}\,P(k_1,\ldots,k_j,d_1,\ldots,d_{s-r})^{jb}y_{k_1}\cdots y_{k_j} \qquad (2.10)\end{aligned}$$

Next, we rewrite each of the numbered expressions above. Using the partition explained above, one can write

$$\begin{aligned}&(x_{k_{m_1}}\cdots x_{k_{m_{j-s}}}\,x_{n_1}\cdots x_{n_{s-r}})^{a-jb}\,w^{j-s}H_s^{q_1,\ldots,q_r,\,d_1,\ldots,d_{s-r}}\\ &=(x_{k_{m_1}}\cdots x_{k_{m_{j-s}}}\,x_{n_1}\cdots x_{n_{s-r}})^{a-jb}\,w^{j-s}\,\big((x_{q_1}\cdots x_{q_r}\,x_{d_1}\cdots x_{d_{s-r}})^{a-sb}w^s\\ &-P(q_1,\ldots,q_r,d_1,\ldots,d_{s-r})^{sb}y_{q_1}\cdots y_{q_r}\,y_{d_1}\cdots y_{d_{s-r}}\big)\\ &=(x_{k_1}\cdots\widehat{x_{q_1}}\cdots\widehat{x_{q_r}}\cdots x_{k_j})^{a-jb}\,(x_{q_1}\cdots x_{q_r})^{a-sb}\,(x_{d_1}\cdots x_{d_{s-r}})^{a-sb}w^j \qquad (2.11)\\ &-P(k_1,\ldots,k_j,d_1,\ldots,d_{s-r})^{sb}\,(x_{k_{m_1}}\cdots x_{k_{m_{j-s}}}\,x_{n_1}\cdots x_{n_{s-r}})^{a-(j-s)b}\,w^{j-s} \qquad (2.12)\\ &\cdot\,y_{q_1}\cdots y_{q_r}\,y_{d_1}\cdots y_{d_{s-r}}.\end{aligned}$$

The first numbered expression above is exactly the first numbered expression in the previous display. However, the second numbered expression above does not coincide with the second numbered expression in the previous display, so there is a little more to pursue in order to cancel this expression by bringing up an expression involving another Sylvester form:

$$
\begin{aligned}
&P(k_1,\dots,k_j,d_1,\dots,d_{s-r})^{sb}\,(x_{n_1}\cdots x_{n_{s-r}})^{a-(j-s)b} y_{q_1}\cdots y_{q_r}\,y_{d_1}\cdots y_{d_{s-r}} H_{j-s}^{k_{m_1}\dots,k_{m_{j-s}}}\\
&\qquad = P(k_1,\dots,k_j,d_1,\dots,d_{s-r})^{sb}\,(x_{n_1}\cdots x_{n_{s-r}})^{a-(j-s)b} y_{q_1}\cdots y_{q_r}\,y_{d_1}\cdots y_{d_{s-r}}\\
&\qquad \cdot \Big((x_{k_{m_1}}\cdots x_{k_{m_{j-s}}})^{a-(j-s)b}\,w^{j-s} - P(k_{m_1},\dots,k_{m_{j-s}})^{(j-s)b}\,y_{k_{m_1}}\cdots y_{k_{m_{j-s}}}\Big)\\
&\qquad = P(k_1,\dots,k_j,d_1,\dots,d_{s-r})^{sb}\,(x_{k_{m_1}}\cdots x_{k_{m_{j-s}}}\,x_{n_1}\cdots x_{n_{s-r}})^{a-(j-s)b}\,w^{j-s}\\
&\qquad \cdot y_{q_1}\cdots y_{q_r}\,y_{d_1}\cdots y_{d_{s-r}}\\
&\qquad - P(k_1,\dots,k_j,d_1,\dots,d_{s-r})^{jb}\,(x_{q_1}\cdots x_{q_r})^{(j-s)b}\,(x_{n_1}\cdots x_{n_{s-r}})^{a}\,(x_{d_1}\cdots x_{d_{s-r}})^{(j-s)b}\\
&\qquad \cdot y_{q_1}\cdots y_{q_r}\,y_{k_{m_1}}\cdots y_{k_{m_{j-s}}}\,y_{d_1}\cdots y_{d_{s-r}}\\
&\qquad = P(k_1,\dots,k_j,d_1,\dots,d_{s-r})^{sb}\,(x_{k_{m_1}}\cdots x_{k_{m_{j-s}}}\,x_{n_1}\cdots x_{n_{s-r}})^{a-(j-s)b}\,w^{j-s} \qquad (2.13)\\
&\qquad \cdot y_{q_1}\cdots y_{q_r}\,y_{d_1}\cdots y_{d_{s-r}}\\
&\qquad - (x_{n_1}\cdots x_{n_{s-r}})^{a}\,(x_{d_1}\cdots x_{d_{s-r}})^{(j-s)b}\,Q\,y_{d_1}\cdots y_{d_{s-r}}. \qquad (2.14)
\end{aligned}
$$

Now, expression numbered (14) is same as expression numbered (13), but expression (15) still has way to go. In the subsequent steps we resort to Koszul generators as tags, namely, firstly,

$$
\begin{aligned}
&(x_{n_2}\cdots x_{n_{s-r}})^{a}\,\underbrace{x_{d_0}^{a+(j-s)b}}_{=1}(x_{d_1}\cdots x_{d_{s-r}})^{(j-s)b}\,Q\,y_{d_2}\cdots y_{d_{s-r}}\,\underbrace{y_{n_0}}_{=1}\,K_{n_1,d_1}\\
&\qquad = (x_{n_1}\cdots x_{n_{s-r}})^{a}\,(x_{d_1}\cdots x_{d_{s-r}})^{(j-s)b}\,Q\,y_{d_1}\cdots y_{d_{s-r}} \qquad (2.15)\\
&\qquad - (x_{n_2}\cdots x_{n_{s-r}})^{a}\,x_{d_1}^{a+(j-s)b}(x_{d_2}\cdots x_{d_{s-r}})^{(j-s)b}\,Q\,y_{d_2}\cdots y_{d_{s-r}}\,y_{n_1}
\end{aligned}
$$

The procedure establishes an inductive argument by which one monomial term is canceled against a next term in an expression involving a further down Koszul form. To obtain the final combination in terms of earlier Sylvester forms and Koszul forms, one resorts to a summation of expressions of the same type where the first summand is the expression in the last line of the last display and the last summand recovers (11). This explains the final form of the required expression as stated. □

Degree jump

Now $j' = j+1$.

- x_s^{a-b}, $s = j+2,\dots,n$.

$$
\begin{aligned}
x_s^{a-b} H_{j+1}^{1,\dots,j,j+1} &= (x_1\cdots x_j x_{j+1})^{a-(j+1)b} w^j L_s\\
&+ x_1^{a-jb}(x_{j+2}\cdots x_{s-1}\,\widehat{x}_s\,x_{s+1}\cdots x_n)^b y_s H_j^{2,\dots,j+1}\\
&+ (x_{j+2}\cdots x_{s-1}\,\widehat{x}_s\,x_{s+1}\cdots x_n)^{(j+1)b} x_s^{jb} y_2\cdots y_{j+1} K_{1,s}.
\end{aligned}
$$

- $x_s^{jb},\ s=1,\ldots,j+1.$

$$x_s^{jb}H_{j+1}^{1,\ldots,j,j+1} = (x_1\cdots x_{s-1}\,\widehat{x_s}\,x_{s+1}\cdots x_{j+1})^{a-(j+1)b}w^jL_s - (x_{j+2}\cdots x_n)^b y_s H_j^{1,\ldots,\widehat{s},\ldots,j+1}.$$

- $(x_{r_1}\cdots x_{r_s})^{(j+1-s)b}$, where $s\in\{1,\ldots,j\}$ and $\{r_1,\ldots,r_s\}$ is an ordered subset of $\{1,\ldots,j,j+1\}$.

$$\begin{aligned}(x_{r_1}\cdots x_{r_s})^{(j+1-s)b}H_{j+1}^{1,\ldots,j,j+1} &= P_{j+1}(r_1,\ldots,r_s)^{a-(j+1)b}w^{j+1-s}H_s^{r_1,\ldots,r_s}\\ &+ P(1,\ldots,j+1)^{sb}y_{r_1}\cdots y_{r_s}H_{j+1-s}^{1,\ldots,\widehat{r_1},\ldots,\widehat{r_s},\ldots,j,j+1},\end{aligned}$$

where the lower index $j+1$ in the first P indicates that the product of the variables is over the complement of $\{r_1,\ldots,r_s\}$ in $\{1,\ldots,j,j+1\}$ (and not in the entire $\{1,\ldots,n\}$.)

- $(x_{q_1}\cdots x_{q_r})^{(j+1-s)b}(x_{d_1}\cdots x_{d_{s-r}})^{a-sb}$.

One applies the hypotheses and the conclusion of Lemma 2.3.1 with the following changes in the numerology:
$\{k_1,\ldots.k_j\}\rightsquigarrow\{1,\ldots,j+1\}$
$\{k_{m_1},\ldots,k_{m_{j-s}}\}\rightsquigarrow\{m_1,\ldots,m_{j+1-s}\}$
$j\rightsquigarrow j+1$, in all appearances of j in a subscript or exponent.

Finally, we stress the calculation when the degree jumps to the highest degree in the sequence of Sylvester forms. Once more we only display the case where $a\le nb$, $(p-1)b<a\le pb$, since when $a>nb$ the result is embedded in the general discussion of this case.

- $x_r^{a-b},\ r=1,\ldots,p.$

$$x_r^{a-b}H_p^{1,\ldots,p} = w^{p-1}L_r + (x_{p+1}\cdots x_n)^b(x_1\cdots\widehat{x_r}\cdots x_p)^{pb-a}y_rH_{p-1}^{1,\ldots,\widehat{r},\ldots,p}.$$

- $x_s^{a-b},\ s=p+1,\ldots,n.$

$$\begin{aligned}x_s^{a-b}H_p^{1,\ldots,p} &= w^{p-1}L_s + (x_1\cdots x_{p-1})^{pb-a}(x_px_{p+1}\cdots\widehat{x_s}\cdots x_n)^b y_sH_{p-1}^{1,\ldots,p-1}\\ &\quad + x_s^{(p-1)b}(x_1\cdots x_p)^{pb-a}(x_{p+1}\cdots\widehat{x_s}\cdots x_n)^{pb}y_1\cdots y_{p-1}K_{p,s}.\end{aligned}$$

- $(x_1\cdots\widehat{x_r}\cdots x_p)^{a-(p-1)b},\ r=1,\ldots,p.$

$$(x_1\cdots\widehat{x_r}\cdots x_p)^{a-(p-1)b}H_p^{1,\ldots,p} = wH_{p-1}^{1,\ldots,\widehat{r},\ldots,p} + x_r^{pb-a}(x_{p+1}\cdots x_n)^{(p-1)b}y_1\cdots\widehat{y_r}\cdots y_pL_r$$

- $(x_{q_1}\cdots x_{q_s})^{a-sb}$, $\{q_1<\cdots<q_s\}\subset\{1,\ldots,p\}$.

$$\begin{aligned}(x_{q_1}\cdots x_{q_s})^{a-sb}H_p^{1,\ldots,p} &= w^{p-s}H_s^{q_1,\ldots,q_s}\\ &+\ (x_1\cdots\widehat{x_{q_1}}\cdots\widehat{x_{q_s}}\cdots x_p)^{pb-a}(x_{p+1}\cdots x_n)^{sb}y_{q_1}\cdots y_{q_s}H_{p-s}^{1,\ldots,\widehat{q_1},\ldots,\widehat{q_s},\ldots,p}\end{aligned}$$

- $(x_{q_1}\cdots x_{q_r}x_{d_1}\cdots x_{d_{s-r}})^{a-sb}$, where $s\in\{2,\ldots,p-1\}, r\in\{0,\ldots,s-1\}, \{q_1<\ldots<q_r\}\subset\{1,\ldots,p\}, p<d_1<\cdots<d_{s-r}$.

  This case follows the pattern of Lemma 2.3.1, with the following changes:

  $\{k_1,\ldots,k_j\}\rightsquigarrow\{1,\ldots,p\}$

  $\{m_1,\ldots,m_{j-s}\}\rightsquigarrow\{m_1,\ldots,m_{p-s}\}$

  $j\rightsquigarrow p$, in all appearances of j in a subscript or exponent.

  However, there are some changes in the coefficients of the final expression. We write this expression for the sake of completeness:

$$
\begin{aligned}
(x_{q_1}\cdots x_{q_r}x_{d_1}\cdots x_{d_{s-r}})^{a-sb}H_p^{1,\ldots,p} &= w^{p-s}H_s^{q_1,\ldots,q_r,d_1,\ldots,d_{s-r}}\\
&+P(q_1,\ldots,q_r,m_1,\ldots,m_{p-s},d_1,\ldots,d_{s-r})^{sb}(x_{m_1}\cdots x_{m_{p-s}})^{pb-a}\\
&\cdot y_{q_1}\cdots y_{q_r}y_{d_1}\cdots y_{d_{s-r}}H_{p-s}^{m_1,\ldots,m_{p-s}}+\sum_{c=1}^{s-r}(x_{p+1}\cdots\widehat{x_{d_1}}\cdots\widehat{x_{d_{s-r}}}\cdots x_n)^{pb}\\
&\cdot(x_{d_{c-1}}\cdots x_{d_{s-r}})^{a}(x_{d_1}\cdots x_{d_{s-r}})^{(p-s)b}(x_{m_1}\cdots x_{m_{p-s}})^{pb-a}\\
&\cdot(x_{n_{c+1}}\cdots x_{n_{s-r}})^{pb}(x_{n_1}\cdots x_{n_c})^{pb-a}(x_{q_1}\cdots x_{q_r})^{(p-s)b}\\
&\cdot y_{q_1}\cdots y_{q_r}y_{m_1}\cdots y_{m_{p-s}}y_{n_1}\cdots y_{n_{c-1}}y_{d_{c+1}}\cdots y_{d_{s-r}}K_{n_c,d_c},
\end{aligned}
$$

  with the convention that $x_{d_0}=y_{n_0}=x_{n_{s-r+1}}=x_{d_{s-r+1}}=1$.

- $(x_1\cdots x_j\,x_{r_{j+1}}\cdots x_{r_{p-1}})^{a-(p-1)b}$, with $j\in\{0,\ldots,p-2\},\ k\in\{j+1,\ldots,p-1\}$ and $r_k\in\{p+1,\ldots,n\}$.

This case still follows the general shape afforded by the result of Lemma 2.3.1, except that, besides changes in the **x**-coefficients, the Sylvester tag of one of the terms degenerates into a syzygy generator L_{j+1}.

$$
\begin{aligned}
&(x_1\cdots x_jx_{r_{j+1}}\cdots x_{r_{p-1}})^{a-(p-1)b}H_p^{1,\ldots,p} = wH_{p-1}^{1,\ldots,j,r_{j+1},\ldots,r_{p-1}}\\
&+x_{j+1}^{pb-a}(x_{j+2}\cdots\widehat{x_{r_{j+1}}}\cdots\widehat{x_{r_{p-1}}}\cdots x_n)^{(p-1)b}\,y_1\cdots y_jy_{r_{j+1}}\cdots y_{r_{p-1}}L_{j+1}\\
&+\sum_{c=2}^{p-1-j}(x_1\cdots x_j)^b(x_{j+1}\cdots x_{j+c})^{pb-a}(x_{j+c+1}\cdots x_p)^{pb}(x_{r_{j+1}}\cdots x_{r_{j+c-2}})^{a+b}(x_{r_{j+c-1}}\cdots x_{r_{p-1}})^b\\
&\cdot(x_{p+1}\cdots\widehat{x_{r_{j+1}}}\cdots\widehat{x_{r_{p-1}}}\cdots x_n)^{pb}y_1\cdots y_{j+c-1}y_{r_{j+c}}\cdots y_{r_{p-1}}K_{j+c,r_{j+c-1}}\\
&+(x_1\cdots x_j)^b(x_{j+1}\cdots x_p)^{pb-a}(x_{r_{j+1}}\cdots x_{r_{p-2}})^{a+b}x_{r_{p-1}}^b(x_{p+1}\cdots\widehat{x_{r_{j+1}}}\cdots\widehat{x_{r_{p-1}}}\cdots x_n)^{pb}\\
&\cdot y_1\cdots y_{p-1}K_{p,r_{p-1}}.
\end{aligned}
$$

This concludes the proof of the proposition. □

Chapter 3

b-Uniform case

In the general case of [22, Conjecture 4.15], the ideal has the form

$$I = (x_1^{a_1}, \dots, x_n^{a_n}, x_1^{b_1} \cdots x_n^{b_n}) \subset R = k[x_1, \dots, x_n],$$

where a_i, b_i are integers such that $0 \leq b_i < a_i$, for $1 \leq i \leq n$.

We will informally refer to the integers a_i, b_i as the *defining exponents* or, simply, *exponents*, of I.

3.1 Reshaping

As a preliminary, we wish to show how to reduce/rescale the given shape to easier ones at the cost of changing the ideal exponents or the ambient polynomial ring.

3.1.1 Reduction to non-absent b_i's

It may happen that some b_i vanishes. This preliminary lack of symmetry is a nuisance in the calculations, so we would like to deal out with it. Next is a procedure that aims at the impact on the various pertinent algebraic structures.

Let us assume, without loss of generality, that $b_n = 0$.

Setting $\tilde{I} := (x_1^{a_1}, \dots, x_{n-1}^{a_{n-1}}, x_1^{b_1} \cdots x_{n-1}^{b_{n-1}}) \subset \tilde{R} := k[x_1, \dots, x_{n-1}]$, one has $I = (\tilde{I}, x_n^{a_n})R$. Writing

$$\mathcal{R}_{\tilde{R}}(\tilde{I}) = \tilde{R}\,[x_1^{a_1}T, \dots, x_n^{a_{n-1}}T, x_1^{b_1} \cdots x_{n-1}^{b_{n-1}}T] \subset \tilde{R}[T],$$

one has

$$\mathcal{R}_R(I) = (\mathcal{R}_{\tilde{R}}(\tilde{I}))[x_n, x_n^{a_n}T].$$

In terms of external presentation, letting $t_i \mapsto x_i^{a_i}T\,(1 \leq i \leq n)$ and $w \mapsto x_1^{b_1} \cdots x_{n-1}^{b_{n-1}}T$, one obtains

$$\mathcal{R}_{\tilde{R}}(\tilde{I}) \simeq \tilde{R}\,[t_1, \dots, t_{n-1}, w]/\widetilde{\mathcal{J}}, \quad \mathcal{A} := \left(\tilde{R}\,[t_1, \dots, t_{n-1}, w]/\widetilde{\mathcal{J}}\right)\,[x_n, t_n] \twoheadrightarrow \mathcal{R}_R(I).$$

Since the Koszul relations $\mathcal{K} := \{x_i^{a_i}t_n - x_n^{a_n}t_i (1 \leq i \leq n-1), x_1^{b_1} \cdots x_{n-1}^{b_{n-1}}t_n - x_n^{a_n}w\}$ are relations of generators of I, one has a surjection

$$\mathcal{A}/(\mathcal{K}) \twoheadrightarrow \mathcal{R}_R(I)$$

of finitely generated k-algebras, the target being a domain.

Conjectured claim: This surjection is an isomorphism.

To see this one shows that the Koszul ideal $(\mathcal{K})$ is prime on $\mathcal{A}$. For this purpose we recall some further structure. Namely, set

$$\mathcal{B} := (\tilde{R}\,[t_1, \ldots, t_{n-1}, w]/\widetilde{\mathcal{J}})\,[x_n^{a_n}, t_n] \subset \mathcal{A}.$$

Then $\mathcal{B}$ is still a polynomial ring in the new variables $x_n^{a_n}, t_n$ and $(\mathcal{K})$ is an extended ideal from $\mathcal{B}$. As an ideal in $\mathcal{B}$ it is the presentation ideal on $\mathcal{B}$ of the symmetric algebra of the cokernel $\mathcal{C}$ of the $(\tilde{R}\,[t_1, \ldots, t_{n-1}, w]/\widetilde{\mathcal{J}})$-module map defined by the matrix

$$\mathcal{M} := \begin{pmatrix} x_1^{a_1} & \ldots & x_{n-1}^{a_{n-1}} & x_1^{b_1} \cdots x_{n-1}^{b_{n-1}} \\ -t_1 & \ldots & -t_{n-1} & -w \end{pmatrix}$$

This matrix has rank one on $\tilde{R}\,[t_1, \ldots, t_{n-1}, w]/\widetilde{\mathcal{J}}$ because its 2-minors are elements of the presentation ideal $\widetilde{\mathcal{J}}$. Therefore, $\mathcal{K} : I_1(\mathcal{M})^\infty$ is the presentation of the Rees algebra of $\mathcal{C}$ on $\mathcal{B}$. However, $\mathcal{K}$ has codimension one, while $I_1(\mathcal{M})$ has codimension $2(n-1)+1-(n-1) = n > 1$. If we prove that $\mathcal{K}$ has no associated prime of codimension $\geq n$, then $\mathcal{K} : I_1(\mathcal{M})^\infty = \mathcal{K}$. This gives that $\mathcal{C}$ is of linear type, and hence $\mathcal{K}$ is prime since $\mathcal{B}$ is a domain.

Thus, it remains to prove:

(a) Any associated prime of $\mathcal{K}$ on $\mathcal{B}$ has codimension at most $n-1$

(b) The extension of $\mathcal{K}$ to $\mathcal{A}$ remains prime.

The proof of (b) should hopefully follow from the fact that the extension $\mathcal{B} \subset \mathcal{A}$ is integral. As for (a) one needs a subtler look.

Conjecture 3.1.1. $\operatorname{depth}(\mathcal{R}_R(I)) = \operatorname{depth}(\mathcal{R}_{\tilde{R}}(\tilde{I})) + 1$.

Since the corresponding dimensions differ by 1 as well, a consequence of the conjecture is that the question of almost Cohen–Macaulayness is inductive, and hence one may conjecturally assume heretofore that $b_i \neq 0$ for every $1 \leq i \leq n$.

3.2 The top associated a-uniform ideal

This part does not describe a true reduction. One introduces an a-uniform ideal closely associated to the original ideal I as above. (As discussed previously, one is lead to assume that no b_i vanishes, although this will play no essential role in this part.)

Namely, given $I = (x_1^{a_1}, \dots, x_n^{a_n}, x_1^{b_1} \cdots x_n^{b_n})$, we assume that $a_1 \leq \cdots \leq a_n := a$. Then introduce the subideal $I^{\rm top} := (x_1^a, \dots, x_n^a, x_1^{b_1} \cdots x_n^{b_n}) \subset I$, to be called the *top associated a-uniform* ideal – note that, if I were b-uniform then $I^{\rm top}$ would be fully uniform.

We proceed the relation between the respective Rees algebras.

Let $y_1, \dots, y_n, w$ (respectively, $Y_1, \dots, Y_n, W$) denote external variables for I (respectively, $I^{\rm top}$). Then the presentation surjection $R[Y_1, \dots, Y_n, W] \twoheadrightarrow R[x_1^a T, \dots, x_n^a T, x_1^{b_1} \cdots x_n^{b_n} T]$, with $Y_j \mapsto x_j^a T$, $W \mapsto x_1^{b_1} \cdots x_n^{b_n} T$, factors as

$$\begin{aligned} R[Y_1, \dots, Y_n, W] &\stackrel{\rho}{\twoheadrightarrow} R[x_1^{a-a_1} y_1, \dots, x_{n-1}^{a-a_{n-1}} y_{n-1}, y_n, w] \\ &\twoheadrightarrow R[x_1^a T, \dots, x_n^a T, x_1^{b_1} \cdots x_n^{b_n} T], \end{aligned} \tag{3.1}$$

where ρ is the R-homomorphism defined by $Y_j \mapsto x_j^{a-a_j} y_j$, $W \mapsto w$, while the second one is the restriction to $R[x_1^{a-a_1} y_1, \dots, x_{n-1}^{a-a_{n-1}} y_{n-1}, y_n, w]$ of the presentation surjection $R[y_1, \dots, y_n, w] \twoheadrightarrow R[x_1^{a_1} T, \dots, x_n^{a_n} T, x_1^{b_1} \cdots x_n^{b_n} T]$, noting that the image of this restriction is indeed the Rees algebra $R[x_1^a T, \dots, x_n^a T, x_1^{b_1} \cdots x_n^{b_n} T]$ of $I^{\rm top}$.

Now, ρ in (3.1) is in fact injective since the image is generated by $n+1$ algebraically independent elements over R. Since the second surjection is restriction, then up to the identification through the first isomorphism the presentation ideal of the Rees algebra of $I^{\rm top}$ is the pullback of the presentation ideal $\mathcal{J}$ of the Rees algebra of I. In other words, the presentation ideal of the Rees algebra of $I^{\rm top}$ on $R[Y_1, \dots, Y_n, W]$ is the lifting through ρ of the ideal

$$\mathcal{J} \cap R[x_1^{a-a_1} y_1, \dots, x_{n-1}^{a-a_{n-1}} y_{n-1}, y_n, w].$$

It remains to clarify the latter ideal and derive some comparison between the respective homological invariants of the two ideals.

3.3 Reduction to the b-uniform shape

In this part, given generators of the ideal $I \subset R = k[x_1, \dots, x_n]$ with arbitrary exponents, say, $I = (x_1^{a_1}, \dots, x_n^{a_n}, x_1^{b_1} \cdots x_n^{b_n})$, we reduce to the case where $b_1 = \cdots = b_n = b$ – we call the obtained format a *b-shape*.

Inspired from the previous sections, we will assume that $b_i \neq 0$ for every $1 \leq i \leq n$.

The background idea is simple: one introduces the following endomorphism of k-algebras $\mathfrak{u} : R \to R$ defined by the assignment $\mathfrak{u}(x_i) = x_i^{\mathfrak{o}(b_i)}$, where $\mathfrak{o}(b_i) := \prod_{j \neq i} b_j$. Obviously, $\mathfrak{u}$ is injective since the image is a k-subalgebra generated by n algebraically independent elements.

Clearly, $\mathfrak{u}(x_i^{b_i}) = x_i^b$, for every $1 \leq i \leq n$, where $b = \prod_{1 \leq i \leq n} b_i$. It follows immediately that the ideal generated by the image of I is the ideal $I' := (x_1^{a_1'}, \dots, x_n^{a_n'}, (x_1 \cdots x_n)^b)$, where $a_i' := \mathfrak{o}(b_i)\, a_i$.

Lemma 3.3.1. *With the above notation, extend the endomorphism $\mathfrak{u}$ to a k-endomorphism $\mathfrak{U}$ of $R[\mathbf{y}] = R[y_1, \dots, y_n, w]$ by setting $\mathfrak{U}(y_j) = y_j$, $\mathfrak{U}(w) = w$. If $\mathcal{J}$ denotes the presentation ideal of $\mathcal{R}_R(I)$ then:*

(a) *The ideal of* $R[\mathbf{y}]$ *generated by* $\mathfrak{U}(\mathcal{J})$ *is the presentation ideal of* $\mathcal{R}_R(I')$.

(b) *The two presentation ideals have the same number of minimal binomial generators with the same* $\mathbf{y}$*-degrees throughout.*

(c) *The ideals* I *and* I' *have the same reduction number.*

Proof. The argument follows closely that of [38, Lemma 2.1]. Note, however, that here one proceeds in the reverse sense: while there the exponents were blown-down, here one blows them up.

(a) Set $S := R[\mathbf{T}]$. Let $\mathcal{K}(I') := I_1(\mathbf{T}\cdot\mathcal{K}_2(I'))$ (respectively, $\mathcal{K}(I) := I_1(\mathbf{T}\cdot\mathcal{K}_2(I))$) stand for the ideal generated by the set of binomials coming from the Koszul relations of I' (respectively, I), where $\mathcal{K}_2(I')$ (respectively, $\mathcal{K}_2(I)$) denotes the matrix of the Koszul syzygies of the generators of I' (respectively, I).

Recall once more the assumption that $b_i \neq 0$, $\forall\ i = 1,\ldots,n$. Since $\mathcal{J}' = \mathcal{K}(I') : I'^{\infty}$ (respectively, $\mathcal{J} = \mathcal{K}(I) : I^{\infty}$) then it is easy to see that $\mathcal{J}' = \mathcal{K}(I') : (x_1^{a_1'})^{\infty}$ (respectively, $\mathcal{J} = \mathcal{K}(I) : (x_1^{a_1})^{\infty}$). On the other hand, it is clear, by definition, that $\mathfrak{U}(\mathcal{K}(I)) = \mathcal{K}(I')$. Therefore, $\mathcal{J}' = \mathfrak{U}(\mathcal{K}(I)) : (x_1^{a_1'})^{\infty}$.

Clearly, then $\mathfrak{U}(\mathcal{J}) = \mathfrak{U}(\mathcal{K}(I) : (x_1^{a_1})^{\infty}) \subset \mathcal{J}'$. Conversely, we claim that every binomial $H' \in \mathcal{J}'$ is of the form $H' = \mathfrak{U}(H)$, for some binomial $H \in \mathcal{J}$. Thus, let $H' \in \mathcal{J}'$. Without loss of generality on can assume

$$H' = x_1^{\alpha_1'}\cdots x_p^{\alpha_p'}T_{n+1}^{\gamma}T_{p+1}^{\beta_{p+1}}\cdots T_n^{\beta_n} - x_{p+1}^{\alpha_{p+1}'}\cdots x_n^{\alpha_n'}T_1^{\beta_1}\cdots T_p^{\beta_p},$$

where $\alpha_i', \beta_j, \gamma$ are non-negative integers (some possibly zero) satisfying the exponent equations:

$$\begin{array}{rcll} \alpha_s' + \gamma b &=& \beta_s a_s', & \text{for } 1\leq s\leq p\\ \alpha_t' &=& \beta_t a_t' + \gamma b, & \text{for } p+1\leq t\leq n. \end{array}$$

Since $a_i' := \mathfrak{o}(b_i)\, a_i$, $\forall\ i = 1,\ldots,n$ and $b := \prod_{1\leq i\leq n} b_i$, then $\alpha_i' = \mathfrak{o}(b_i)\alpha_i$, simplifying by $\mathfrak{o}(b_i)$ all of the above exponent equations, one obtains

$$\begin{array}{rcll} \alpha_s + \gamma b_s &=& \beta_s a_s, & \text{for } 1\leq s\leq p\\ \alpha_t &=& \beta_t a_t + \gamma b_t, & \text{for } p+1\leq t\leq n. \end{array}$$

meaning that

$$H' = \mathfrak{U}(H),$$

where

$$H = x_1^{\alpha_1}\cdots x_p^{\alpha_p}T_{n+1}^{\gamma}T_{p+1}^{\beta_{p+1}}\cdots T_n^{\beta_n} - x_{p+1}^{\alpha_{p+1}}\cdots x_n^{\alpha_n}T_1^{\beta_1}\cdots T_p^{\beta_p} \in \mathcal{J}.$$

(b) Let $G \in \mathcal{J}$ be a binomial minimal generator. Write

$$\mathfrak{U}(G) = g_1\mathfrak{U}(G_1) + \cdots + g_m\mathfrak{U}(G_m),$$

where G_j are binomials in $\mathcal{J}$ and $g_j \in S$.

Keeping in mind that $\mathfrak{U}(G)$ is a binomial, let $\mathrm{in}(\mathfrak{U}(G)) = cx_1^{\alpha'_1}\cdots x_p^{\alpha'_p}T_{n+1}^{\gamma}T_{p+1}^{\beta_{p+1}}\cdots T_n^{\beta_n}$ stand for the leading monomial of $\mathfrak{U}(G)$ under some monomial order on S, where $c \in k\setminus\{0\}$. Then this monomial must be multiple of some monomial M' in S and another monomial N' in S effectively appearing in one of the binomials G_j. Since N' divides $\mathrm{in}(\mathfrak{U}(G))$, then it must be of the form

$$N' = x_1^{\overline{\alpha'}_1}\cdots x_p^{\overline{\alpha'}_p}T_{n+1}^{\overline{\gamma}}T_{p+1}^{\overline{\beta}_{p+1}}\cdots T_n^{\overline{\beta}_n},$$

where $\alpha'_s \geq \overline{\alpha'}_s \geq 0,\ \gamma \geq \overline{\gamma} \geq 0$, and $\beta_t \geq \overline{\beta}_t \geq 0$. Then

$$M' = cx_1^{\alpha'_1-\overline{\alpha'}_1}\cdots x_p^{\alpha'_p-\overline{\alpha'}_p}T_{n+1}^{\gamma-\overline{\gamma}}T_{p+1}^{\beta_{p+1}-\overline{\beta}_{p+1}}\cdots T_n^{\beta_n-\overline{\beta}_n}.$$

From the exponent equations for α' and $\overline{\alpha'}$, for $1 \leq s \leq p$ one has $\alpha'_s + \gamma b = \beta_s a'_s$ and $\overline{\alpha'_s} + \overline{\gamma}b = \overline{\beta}_s a'_s$, giving

$$\alpha'_s - \overline{\alpha'_s} = (\beta_s - \overline{\beta}_s)a'_s - (\gamma - \overline{\gamma})b,\ 1 \leq s \leq p.$$

Again, since $a'_i := \mathfrak{o}(b_i)\, a_i,\ \forall\ i = 1,\ldots,n$ and $b := \prod_{1\leq i\leq n} b_i$, then $\alpha'_s - \overline{\alpha'}_s = \mathfrak{o}(b_i)\rho_s$. So we obtain

$$M' = \mathfrak{U}(M),$$

where

$$M = cx_1^{\rho_1}\cdots x_p^{\rho_p}T_{n+1}^{\gamma-\overline{\gamma}}T_{p+1}^{\beta_{p+1}-\overline{\beta}_{p+1}}\cdots T_n^{\beta_n-\overline{\beta}_n}.$$

Since $\mathfrak{U}$ is a ring homomorphism we get

$$\mathfrak{U}(G - MG_1) = (g_1 - M')\mathfrak{U}(G_1) + g_2\mathfrak{U}(G_2) + \cdots + g_m\mathfrak{U}(G_m),$$

with $\mathrm{in}(\mathfrak{U}(G - MG_1)) < \mathrm{in}(\mathfrak{U}(G))$. Inducting, one gets $G \in (G_1,\ldots,G_m)$, contradicting the minimality of G. So $\mathfrak{U}(G)$ is also a part of a minimal generating set for $\mathcal{J}'$.

(c) Let us assume that $J := (x_1^{a_1},\ldots,x_n^{a_n})$ is a reduction of I with reduction number r. Set $f = x_1^{b_1}\cdots x_n^{b_n}$. Since $I = (J,f)$, then $I^{r+1} = (JI^r, f^{r+1})$, hence $f^{r+1} \in JI^r$. Similarly, $I'^{r+1} = (J'I'^r, f'^{r+1})$, where $J' = (x_1^{a'_1},\ldots,x_n^{a'_n})$ and $f' = (x_1\cdots x_n)^b$. Applying $\mathfrak{U}$ to $f^{r+1} \in JI^r$ yields $f'^{r+1} \in J'I'^r$. Therefore, J' is a reduction of I' with reduction number at most r. But since $\mathfrak{U}$ is injective, we can reverse any inclusion $f'^{s+1} \in J'I'^s$ back to $f^{s+1} \in JI^s$. Therefore, the respective reduction numbers coincide, as stated. □

3.4 b-Uniform with exponents $> nb$

In this section we take up the b-uniform case as obtained from the general case assuming the non-vanishing of the b_i's. Thus, one has $I = (x_1^{a_1},\ldots,x_n^{a_n},(x_1\cdots x_n)^b)$, where a_i, b are integers such that $0 \leq b < a_i$, for $1 \leq i \leq n$. Moreover, without loss of generality, suppose $b < a_1 \leq \cdots \leq a_n$.

The goal is to prove that the Rees algebra of I is again almost Cohen–Macaulay under the additional assumption that $a_i > nb,\ \forall\ i = 1, \ldots, n$. The gist is that with this extra hypothesis the proofs follow pretty closely the ones in the completely uniform case dealt with in Chapter 2. We actually follow the script of the proofs used in the previous chapter pointing the very few modifications.

Next are the main results needed to prove that the Rees algebra is almost Cohen–Macaulay. We will comment on the slight differences between the proofs of the present results as compared to the ones argued in Chapter 2.

Proposition 3.4.1. *Let* $I = (x_1^{a_1}, \ldots, x_n^{a_n}, (x_1 \cdots x_n)^b)$, *with* $0 < b < a_1 \leq \cdots \leq a_n$ *and* $a_i > nb,\ \forall\ i = 1, \ldots, n$, *then* $Q = (x_1^{a_1} - x_n^{a_n}, \ldots, x_{n-1}^{a_{n-1}} - x_n^{a_n}, (x_1 \cdots x_n)^b)$ *is a minimal reduction of* I *and* $\mathrm{red}_Q(I) = n - 1$.

Caveat: both the statement and its proof follow the pattern of Proposition 2.1.2, (b). Note the slight difference in the form of the generators of the reduction Q.

Next is the basic theorem on the shape of the binomial generators of the Rees algebra. This theorem generalizes Theorem 2.1.4, (b) – since an explicit proof of the latter result has not been given, we now furnish a proof in the present context.

We will use the following lemma, which is an analogue of Lemma 2.1.3.

Lemma 3.4.2. *Any binomial in* $\mathcal{I}$ *belonging to a set of minimal generators thereof is of the form*

$$\mathbf{m}(\mathbf{x})w^\delta - \mathbf{n}(\mathbf{x})y_{i_1}^{\alpha_{i_1}} \cdots y_{i_s}^{\alpha_{i_s}},$$

where $\mathbf{m}(\mathbf{x}), \mathbf{n}(\mathbf{x})$ *are relatively prime monomials in* $\mathbf{x} = x_1, \ldots, x_n$ *and* $1 \leq i_1 < \cdots < i_s \leq n$, $\alpha_{i_j} > 0$.

Caveat: The proof is the same as the one of Lemma 2.1.3, with a modified grading attributing the following weights to the variables: $\deg(x_i) = 1$, $\deg(w) = 1, \deg(y_j) = a_j - nb + 1$ since $a_j \geq nb,\ \forall\ j = 1, \ldots, n$. Note that with these weights the basic binomials

$$K_{i,j} = x_i^{a_i} y_j - x_j^{a_j} y_i,\ i, j \in \{1, \ldots, n\},\ i < j,$$

$$L_i = x_i^{a_i - b} w - P(i)^b y_i,\ i \in \{1, \ldots, n\}$$

are homogeneous in $S := R[y_1, \ldots, y_n, w]$.

Theorem 3.4.3. Let $I \subset R = k[x_1, \ldots, x_n]$ be a ideal b-uniform. Then the polynomial ring $S := R[y_1, \ldots, y_n, w]$ admits a grading such that the presentation ideal $\mathcal{I}$ of the Rees algebra of I over it is generated by homogeneous binomials in this grading. Moreover, any minimal binomial generator of external degree δ can be written in the form

$$x_{i_1}^{a_{i_1} - \delta b} \cdots x_{i_\delta}^{a_{i_\delta} - \delta b}\, w^\delta - P(i_1, \ldots, i_\delta)^{\delta b}\, y_{i_1} \cdots y_{i_\delta}, \tag{3.2}$$

where $\delta \leq n$. (no convention needed in this case since for $\delta = n$, there is no $\mathbf{x}$-term on the right hand side).

Proof. Start with generators of the presentation ideal of the symmetric algebra of I. It is easy to see that the syzygies of I are generated by the Koszul relations of the pure powers $x_1^{a_1}, \ldots, x_n^{a_n}$ and by the reduced relations of $(x_1 \cdots x_n)^b$ with each one of the pure powers. In other words, $\mathcal{L} \subset S = R[y_1, \ldots, y_n, w]$ is generated by the binomials

$$K_{i,j} = x_i^{a_i} y_j - x_j^{a_j} y_i, \ i, j \in \{1, \ldots, n\}, \ i < j,$$

$$L_i = x_i^{a_i - b} w - P(i)^b y_i, \ i \in \{1, \ldots, n\}.$$

Now, these binomials are homogeneous in S by attributing the following weights to the variables: $\deg(x_i) = 1$, $\deg(w) = 1, \deg(y_j) = a_j - nb + 1$ since $a_j \geq nb$, $\forall \ j = 1, \ldots, n$. Therefore, $\mathcal{L}$ is homogeneous for these weights. Since $\mathcal{I} = \mathcal{L} : I^\infty$ and I is monomial, it follows that $\mathcal{I}$ is generated by binomials which are homogeneous as well under the same weights. Indeed, one has the string of inclusions

$$\mathcal{I} = \mathcal{L} : I^\infty \subset \mathcal{L} : (x_1)^\infty \subset \mathcal{I} : (x_1)^\infty = \mathcal{I},$$

the last equality because $\mathcal{I}$ is a prime ideal. Then by [11, Corollary 1.9], $\mathcal{I}$ is generated by binomials and hence by homogeneous binomials as x_1 is homogeneous of degree 1.

By Lemma 3.4.2, a binomial in $\mathcal{I}$ belonging to a set of minimal generators thereof is of the form

$$\mathbf{m}(\mathbf{x}) w^\delta - \mathbf{n}(\mathbf{x}) y_{i_1}^{\alpha_{i_1}} \cdots y_{i_s}^{\alpha_{i_s}},$$

with $1 \leq i_1 < \cdots < i_s \leq n$, $\alpha_{i_j} > 0$, and $\mathbf{m}(\mathbf{x}), \mathbf{n}(\mathbf{x})$ suitable monomials in R such that $\gcd\{\mathbf{m}(\mathbf{x}), \mathbf{n}(\mathbf{x})\} = 1$.

We now observe in addition the following three principles:

• w corresponds to a monomial that involves all variables of R; this implies that the monomial $\mathbf{n}(\mathbf{x})$ must involve the variables indexed by the complementary subset $\{j_{s+1}, \ldots, j_n\} := \{1, \ldots, n\} \setminus \{i_1, \ldots, i_s\}$ and, since $\gcd\{\mathbf{m}(\mathbf{x}), \mathbf{n}(\mathbf{x})\} = 1$, the variables effectively involved in $\mathbf{m}(\mathbf{x})$ must be indexed by a subset of $\{i_1, \ldots, i_s\}$. Therefore, the monomial has the form

$$x_{i_1}^{d_{i_1}} \cdots x_{i_s}^{d_{i_s}} w^\delta - x_{j_{s+1}}^{c_{i_{s+1}}} \cdots x_{i_n}^{c_{jn}} y_{i_1}^{\alpha_{i_1}} \cdots y_{i_s}^{\alpha_{i_s}}$$

for suitable exponents $d_{i_l} \geq 0$, for $l = 1, \ldots, s$ (some of which may vanish) and c_{j_k}, for $k = s+1, \ldots, n$ (which are positive).

• Weighted homogeneity implies the equalities

$$\delta + \sum_{l=1}^{s} d_{i_l} = \sum_{l=1}^{s} (a_{i_l} - nb + 1)\alpha_{i_l} + \sum_{k=s+1}^{n} c_{i_k} \tag{3.3}$$

since $a_i \geq nb$, $\forall \ i = 1, \ldots, n$.

Moreover, since upon evaluation the powers $x_{j_k}^{c_{j_k}}$ on the right hand side can only cancel against the ones coming from w^δ on the left hand side, we see that $c_{j_k} = \delta b$ for every k. By the same token, $d_{i_l} = a_{i_l}\alpha_{i_l} - \delta b$ for every l.

• Lastly, since the Rees algebra $\mathcal{R}_R(I)$ is also standard graded over $R = \mathcal{R}_R(I)_0$, we may assume that the binomial is homogeneous with respect to the external variables (however, we warn that $\mathcal{R}_R(I)$ is *standard* bigraded over k if and only if $a_i = nb, \ \forall\ i = 1, \ldots, n$). This means that $\delta = \sum_{l=1}^{s} \alpha_{i_l}$, a formula already found in the above lemma.

So we can assume our binomial to look like

$$x_{r_1}^{a_{r_1}\alpha_{r_1}-\delta b} \cdots x_{r_s}^{a_{r_s}\alpha_{r_s}-\delta b} w^{\delta} - (x_{r_{s+1}} \cdots x_{r_n})^{\delta b} y_{r_1}^{\alpha_{r_1}} \cdots y_{r_s}^{\alpha_{r_s}},\ r_c \in \{1, \ldots, n\} \text{ and } \alpha_i \geq 1.$$

Denote the form above by H, in this case H can be assumed as one the types:

$$\text{Type } i(j_1, \ldots, j_i), 0 \leq i \leq n : x_{j_1}^{\beta_{j_1}} \cdots x_{j_i}^{\beta_{j_i}} w^{\delta} - x_{j_{i+1}}^{\beta_{j_{i+1}}} \cdots x_{j_n}^{\beta_{j_n}} y_1^{\alpha_1} \cdots y_n^{\alpha_n}, \quad \beta_{j_1} > 0, \ldots, \beta_{j_i} > 0,$$

with $\beta_j < a_j,\ j = 1, \ldots, n$, otherwise replace β_j of $\beta_j - a_j$.

Suppose that $\delta > n$.

Note that H is not of the form $H = w^{\delta} - x_1^{\beta_1} \cdots x_n^{\beta_n} y_1^{\alpha_1} \cdots y_n^{\alpha_n}$. Because,

$$\delta b = \beta_i + a_i \alpha_i,\ \forall\ i = 1, \ldots, n,$$

then,

$$n\delta b = \sum_{i=1}^{n} \beta_i + \sum_{i=1}^{n} a_i \alpha_i \geq \sum_{i=1}^{n} \beta_i + a_1 (\sum_{i=1}^{n} \alpha_i) \geq \sum_{i=1}^{n} \beta_i + a_1 \delta,$$

then,

$$\sum_{i=1}^{n} \beta_i \leq \delta(nb - a_1) < 0,$$

absurd.

If H is the Type $i(j_1, \ldots, j_i), 1 \leq i \leq n$, we have

$$\begin{array}{ccccccc}
\beta_{j_1} & + & \delta b & = & \alpha_{j_1} a_{j_1} & & \\
\vdots & & \vdots & & & & \\
\beta_{j_i} & + & \delta b & = & \alpha_{j_i} a_{j_i} & & \\
 & & \delta b & = & \beta_{j_{i+1}} & + & \alpha_{j_{i+1}} a_{j_{i+1}} \\
 & & \vdots & & \vdots & & \\
 & & \delta b & = & \beta_{j_n} & + & \alpha_{j_n} a_{j_n}
\end{array}$$

Note that $\alpha_{j_r} \geq 1,\ r = 1, \ldots, i$, otherwise, $\beta_{j_r} = 0$, absurd.

If $\alpha_{j_k} \geq 1,\ k = i+1, \ldots, n$ then replace $y_1^{\alpha_1} \cdots y_n^{\alpha_n}$ in H with

$$y_1^{\alpha_1 - 1} \cdots y_n^{\alpha_n - 1}(y_1 \cdots y_n - x_1^{a_1 - nb} \cdots x_n^{a_n - nb} w^n + x_1^{a_1 - nb} \cdots x_n^{a_n - nb} w^n),$$

i.e.,

$$H = x_{j_1}^{\beta_{j_1}} \cdots x_{j_i}^{\beta_{j_i}} w^{\delta} - x_{j_{i+1}}^{\beta_{j_{i+1}}} \cdots x_{j_n}^{\beta_{j_n}} y_1^{\alpha_1} \cdots y_n^{\alpha_n}$$

$$= x_{j_1}^{\beta_{j_1}} \cdots x_{j_i}^{\beta_{j_i}} w^{\delta} - x_{j_{i+1}}^{\beta_{j_{i+1}}} \cdots x_{j_n}^{\beta_{j_n}} y_1^{\alpha_1 - 1} \cdots y_n^{\alpha_n - 1}(y_1 \cdots y_n - x_1^{a_1 - nb} \cdots x_n^{a_n - nb} w^n + x_1^{a_1 - nb} \cdots x_n^{a_n - nb} w^n)$$

$$= -x_{j_{i+1}}^{\beta_{j_{i+1}}} \cdots x_{j_n}^{\beta_{j_n}} y_1^{\alpha_1 - 1} \cdots y_n^{\alpha_n - 1}(y_1 \cdots y_n - x_1^{a_1 - nb} \cdots x_n^{a_n - nb} w^n)$$
$$+ w^n (x_{j_1}^{\beta_{j_1}} \cdots x_{j_i}^{\beta_{j_i}} w^{\delta - n} - x_{j_1}^{a_{j_1} - nb} \cdots x_{j_i}^{a_{j_i} - nb} x_{j_{i+1}}^{a_{j_{i+1}} + \beta_{j_{i+1}} - nb} \cdots x_{j_n}^{a_{j_n} + \beta_{j_n} - nb} y_1^{\alpha_1 - 1} \cdots y_n^{\alpha_n - 1}).$$

Since that $\mathcal{I}$ is prime,

$$(x_{j_1}^{\beta_{j_1}} \cdots x_{j_i}^{\beta_{j_i}} w^{\delta - n} - x_{j_1}^{a_{j_1} - nb} \cdots x_{j_i}^{a_{j_i} - nb} x_{j_{i+1}}^{a_{j_{i+1}} + \beta_{j_{i+1}} - nb} \cdots x_{j_n}^{a_{j_n} + \beta_{j_n} - nb} y_1^{\alpha_1 - 1} \cdots y_n^{\alpha_n - 1}) \in \mathcal{I},$$

the result follows of induction on δ.

If $\alpha_{j_{m_k}} = 0, \ k = 1, \ldots, q \leq (n - i - 1)$ with $m_k \in \{i+1, \ldots, n\}$ and $\alpha_{j_{m'_e}} \geq 1, \ e = 1, \ldots, (n - i - 1 - q) \leq n$ with $m'_e \in \{i+1, \ldots, n\} \setminus \{m_1, \ldots, m_q\}$, then

$$\beta_{j_{m_k}} = \delta b > nb > (n - q)b.$$

Therefore, replace $x_{j_{m_1}}^{\beta_{j_{m_1}}} \cdots x_{j_{m_q}}^{\beta_{j_{m_q}}} y_1^{\alpha_1} \cdots y_n^{\alpha_n}$ in H with

$$x_{j_{m_1}}^{\beta_{j_{m_1}} - (n-q)b} \cdots x_{j_{m_q}}^{\beta_{j_{m_q}} - (n-q)b} y_1^{\alpha_1 - 1} \cdots y_{j_{m_1}}^{\alpha_{j_{m_1}}} \cdots y_{j_{m_q}}^{\alpha_{j_{m_q}}} \cdots y_{j_{m'_1}}^{\alpha_{j_{m'_1}} - 1} \cdots y_{j_{m'_{n-i-1-q}}}^{\alpha_{j_{m'_{n-i-1-q}}} - 1} \cdots y_n^{\alpha_n - 1} [(x_{j_{m_1}} \cdots x_{j_{m_q}})^{(n-q)b}$$

$$(y_1 \cdots \widehat{y_{j_{m_1}}} \cdots \widehat{y_{j_{m_q}}} \cdots y_n) - (x_1 \cdots \widehat{x_{j_{m_1}}} \cdots \widehat{x_{j_{m_q}}} \cdots x_n)^{a - (n-q)b} w^{n-q} + (x_1 \cdots \widehat{x_{j_{m_1}}} \cdots \widehat{x_{j_{m_q}}} \cdots x_n)^{a - (n-q)b} w^{n-q}]$$

and proceed as in the previous case.

Thus, we assume that $\delta \leq n$:
Type 0: $H = w^{\delta} - x_1^{\beta_1} \cdots x_n^{\beta_n} y_1^{\alpha_1} \cdots y_n^{\alpha_n}$. We have seen that this type doesn't occur.
Type $i(j_1, \ldots, j_i), 1 \leq i \leq n - 1$:

$$H = x_{j_1}^{\beta_{j_1}} \cdots x_{j_i}^{\beta_{j_i}} w^{\delta} - x_{j_{i+1}}^{\beta_{j_{i+1}}} \cdots x_{j_n}^{\beta_{j_n}} y_1^{\alpha_1} \cdots y_n^{\alpha_n}, \quad \beta_{j_1} > 0, \ldots, \beta_{j_i} > 0.$$

Then we have

$$\begin{array}{rcl} \beta_{j_1} + \delta b &=& \alpha_{j_1} a_{j_1} \\ \vdots \quad \ \vdots & & \\ \beta_{j_i} + \delta b &=& \alpha_{j_i} a_{j_i} \\ \delta b &=& \beta_{j_{i+1}} + \alpha_{j_{i+1}} a_{j_{i+1}} \\ \vdots & & \vdots \\ \delta b &=& \beta_{j_n} + \alpha_{j_n} a_{j_n} \end{array}$$

Again, note that $\alpha_{j_k} \geq 1, \ k = 1, \ldots, i$, otherwise, $\beta_{j_k} = 0$, absurd.

If $\delta < i$: this case doesn't occur. Indeed, $\delta = \sum_{m=1}^{n} \alpha_{j_m} \geq i$.
If $\delta = i$: in this case we have $\alpha_{j_1} = \cdots = \alpha_{j_i} = 1$ and $\alpha_{j_{i+1}} = \cdots = \alpha_{j_n} = 0$. Consequently, $\beta_{j_1} = a_{j_1} - ib, \ldots, \beta_{j_i} = a_{j_i} - ib$ and $\beta_{j_{i+1}} = \cdots = \beta_{j_n} = ib$. Therefore,

$$H = x_{j_1}^{a_{j_1}-ib} \cdots x_{j_i}^{a_{j_i}-ib} w^i - (x_{j_{i+1}} \cdots x_{j_n})^{ib} y_{j_1} \cdots y_{j_i}.$$

If $\delta > i$: this case doesn't occur. Indeed, since $\delta \leq n$, we have $\alpha_{j_m} = 0,\ \forall\ m = i+1, \ldots, n$, otherwise, $\beta_{j_m} = \delta b - \alpha_{j_m} a_{j_m} < 0$, absurd. But, being $\delta > i$ and $\alpha_{j_k} \geq 1,\ \forall\ k = 1, \ldots, i$, we have that there $r \in \{1, \ldots, i\}$ such that $\alpha_{j_r} > 1$. Consequently, $\beta_{j_r} = \alpha_{j_r} a_{j_r} - \delta b = a_{j_r} + (\alpha_{j_r} - 1)a_{j_r} - \delta b > a_{j_r}$, (since $a_{j_r} > nb$) absurd.
• Let H be of Type n.
Then we have

$$\begin{aligned} \beta_1 + \delta b &= \alpha_1 a_1 \\ &\vdots \\ \beta_n + \delta b &= \alpha_n a_n \end{aligned}$$

Again, note that $\alpha_j \geq 1,\ j = 1, \ldots, n$, otherwise, $\beta_j = 0$, absurd. Consequently, $\delta = n$ and $\alpha_j = 1,\ \forall\ j = 1, \ldots, n$, since $\delta = \sum_{j=1}^{n} \alpha_j$. Therefore, $\beta_j = a_j - nb,\ \forall\ j = 1, \ldots, n$ and

$$H = x_1^{a_1-nb} \cdots x_n^{a_n-nb} w^n - y_1 \cdots y_n.$$

This concludes the proof of the theorem. □

Theorem 3.4.4. *Let $I \subset R$ be a b-uniform monomial ideal as above and let r denote its reduction number as established in* Proposition 3.4.1*. Then:*

(a) *$\mathcal{I}$ is generated by*

$$\binom{n}{2} + \sum_{\delta=1}^{r} \binom{n}{\delta} + 1,$$

quasi-homogeneous binomials, where r is the reduction number of I; of these, $\binom{n}{2}$ are the Koszul syzygies of the generators of I and the remaining ones are each a binomial of the form

$$H_{\delta}^{i_1,\ldots,i_\delta} := x_{i_1}^{a_{i_1}-\delta b} \cdots x_{i_\delta}^{a_{i_\delta}-\delta b} w^{\delta} - P(i_1, \ldots, i_\delta)^{\delta b} y_{i_1} \cdots y_{i_\delta},$$

where $1 \leq \delta \leq r+1$.

(b) *Moreover, each binomial in the previous item is a Sylvester form obtained in an iterative form out of the syzygy forms.*

(c) *The relation type of I is $r+1$.*

Thus, as in the uniform case, the above results show that the Rees ideal is generated by Sylvester forms. Using the same notation and same ordering of the Sylvester forms as in Section 2.2, the remaining statements hold, namely:

Proposition 3.4.5. *Let $2 \leq j \leq n-1$, where $n-1$ is the reduction number of the ideal $I \subset S = k[x_1, \ldots, x_n]$, and let $i_1 < \cdots < i_j$ be an ordered subset of $\{1, \ldots, n\}$. The set*

$$\Sigma(i_1, \ldots, i_j) := \{K_{i,k}\,(1 \leq i < k \leq n), L_i\,(1 \leq i \leq n), H_2^{1,2}, \ldots, H_j^{i_1,\ldots,i_j}\}$$

is a Gröbner basis of the ideal $\mathcal{H}(i_1, \ldots, i_j) := (\mathcal{L}, H_2^{1,2}, \ldots, H_j^{i_1,\ldots,i_j})$ in the lexicographic order on $w > x_n > \cdots > x_1 >> \cdots$. In particular, the initial ideal of $\mathcal{H}(i_1, \ldots, i_j)$ is generated by

$$\{x_i^{a_i} y_k\,(1 \leq i < k \leq n),\, x_i^{a_i - b} w\,(1 \leq i \leq n),\, x_1^{a_1-2b} x_2^{a_2-2b} w^2, \ldots, x_{i_1}^{a_{i_1}-jb} \cdots x_{i_j}^{a_{i_j}-jb} w^j\}, \tag{3.4}$$

where j and $\{x_{i_1}, \ldots, x_{i_j}\}$ flow as in the statement.

Proposition 3.4.6. *With the above setting, let $H_{j'}^{k_1,\cdots,k_{j'}} \in S$ denote the first Sylvester form succeeding the Sylvester from $H_j^{i_1,\cdots,i_j} \in S$ in the prescribed ordering of these forms.*

(a) *If $j = j'$, one has*

$$\mathrm{in}(\mathcal{H}(i_1, \ldots, i_j)) : \mathrm{in}(H_j^{k_1,\ldots,k_j}) = \Big(x_{k_1}^{(j-1)b}, \ldots, x_{k_j}^{(j-1)b}, x_u^{a_u - jb}, x_{k_j+1}^{a_{k_j+1}-b}, \ldots, x_n^{a_n - b}, (x_{r_1} \cdots x_{r_s})^{(j-s)b},$$
$$(x_{q_1} \cdots x_{q_r})^{(j-s)b} x_{d_1}^{a_{d_1}-sb} \cdots x_{d_{s-r}}^{a_{d_{s-r}}-sb}\Big) S,$$

for all $k_i < u < k_{i+1}, i = 1, \ldots, j-1$ and all choices of indices $s \in \{1, \ldots, j-1\}$, $r \in \{0, \ldots, s-1\}$, of ordered subsets $\{r_1 < \cdots < r_s\} \subset \{k_1, \ldots, k_j\}$, $\{q_1 < \cdots < q_r\} \subset \{k_1, \ldots, k_j\}$ and of an ordered set $d_1 < \cdots < d_{s-r}$ with $k_j < d_1$.

(b) *If $j' = j+1$ (and hence $\{k_1, \cdots, k_{j'}\} = \{1, \ldots, j+1\}$), one has*

$$\mathrm{in}(\mathcal{H}(i_1, \ldots, i_j)) : \mathrm{in}(H_{j+1}^{1,\ldots,j+1}) = \Big(x_1^{jb}, \ldots, x_{j+1}^{jb}, x_{j+2}^{a_{j+2}-b}, \ldots, x_n^{a_n-b}, (x_{r_1} \cdots x_{r_s})^{(j+1-s)b},$$
$$(x_{q_1} \cdots x_{q_r})^{(j+1-s)b} x_{d_1}^{a_{d_1}-sb} \cdots x_{d_{s-r}}^{a_{d_{s-r}}-sb}\Big) S,$$

for all choices of indices $s \in \{1, \ldots, j\}$, $r \in \{0, \ldots, s-1\}$, of ordered subsets $\{r_1 < \cdots < r_s\} \subset \{1, \ldots, j+1\}$, $\{q_1 < \cdots < q_r\} \subset \{1, \ldots, j+1\}$ and of an ordered set $d_1 < \cdots < d_{s-r}$ with $j+1 < d_1$.

(In both cases, we adopt the convention that $x_{q_0} = 1$.)

Proposition 3.4.7. *With the previously established notation, one has*

$$\mathcal{H}(i_1, \ldots, i_j) : H_{j'}^{k_1,\ldots,k_{j'}} = \mathrm{in}(\mathcal{H}(i_1, \ldots, i_j)) : \mathrm{in}(H_{j'}^{k_1,\ldots,k_{j'}}).$$

In particular, the colon ideal on the left hand side is a monomial ideal.

Theorem 3.4.8. *Let $I \subset R = k[x_1, \ldots, x_n]$ denote a b-uniform monomial ideal with exponents satisfying $a_i > nb\ \forall i$. Then $S/\mathcal{H}(i_1, \ldots, i_j)$ has depth at least n for every tuple $i_1 < \cdots < i_j$. In particular, the Rees algebra $\mathcal{R}_R(I)$ of I is an almost Cohen–Macaulay ring.*

The relevant proofs are in Appendix A.

Chapter 4

Binary general forms

Throughout this chapter $R := k[x, y]$ denotes a standard graded polynomial ring in two variables over an infinite field k. Let $I \subset R$ stand for an (x, y)-primary ideal generated by 3 forms of the same degree $d \geq 2$.

We recall from previous chapters that the depth of $\mathcal{R}(I)$ is computed with respect to its maximal graded ideal $(x, y, \bigoplus_{\ell\geq 1} I^{\ell})$.

The following conjecture is due to A. Simis. Some of the preliminaries introduced here are due to Simis and S. Tohăneanu.

Conjecture 4.0.9. *Let $I \subset R := k[x, y]$ denote a codimension 2 ideal generated by three general forms f_1, f_2, f_3 of degree $d \geq 5$. Then the Rees algebra of I is not almost Cohen–Macaulay.*

In other words, in the stated hypotheses, one is claiming that $\text{depth}\mathcal{R}_R(I) = 1$.

The objective of this chapter is to expand on positive cases of the conjecture (very low degrees) and to give sufficient evidence for its validity as a consequence of other conjectures and methods. The two important tools used in this part are the Huckaba-Marley criterion and the Ratliff-Rush filtration. It would seem to us that the conjecture may depend on a refined analysis of the Hilbert function of the second power of I.

4.1 Preliminaries on three binary general forms

Set $R := k[x, y]$, a standard graded polynomial ring in two variables over an infinite field k. Let $I \subset R$ denote a codimension 2 ideal generated by 3 forms of the same degree $d \geq 2$. We consider a minimal free resolution of I

$$0 \to R(-(d + r)) + R(-(d + s)) \xrightarrow{\varphi} R(-d)^3 \longrightarrow I \to 0, \tag{4.1}$$

where $1 \leq s \leq r$ denote the standard degrees of the columns of a matrix of φ. We observe that I is the ideal generated by the 2×2 minors of φ, and hence one often says that φ is the *Hilbert–Burch matrix associated* to I and s, r are its standard degrees. Note that the r, s are numerical invariants of I adding up to d.

One usually says that a set of forms in a polynomial ring over an infinite field is *general* in the sense that the total collection of the coefficients of the forms is general in the parameter space of the coefficients. Typically, in a concrete situation, this is understood in the sense that one avoids a contextualized closed set in the parameter space, Such a property is often hard to work with due to its instability under ordinary algebraic operations. Even the common perception that over three or more variables a general form is irreducible over k becomes elusive in the present case of two variables since in this case any form factors into linear forms over an algebraic closure of k.

The surest way to get going is typified by taking random coefficients throughout. Alas, we need to perform some operations with the given forms which, by and large, destroy randomness. Therefore, we have to work with the geometric notion of general as usual, making sure that at the nearest event the forms still have coefficients sufficiently general as the ones we started with.

Note the usual action of $GL(2, k)$ on forms in $R = k[x, y]$ by means of k-linear change of variables; this action changes individual forms, hence also ideals of R, but preserves most algebraic-geometric properties of varieties and ideals.

Fixing an integer $m \geq 1$, there is also the action of $GL(m, k)$ on sets of m forms of the same degree. The latter acts by means of k-linear transformations on the vector space of m-tuples of forms and give the ordinary k-elementary operations of sets of generators of an ideal. These do not change the ideal, only its sets of generators.

Since both actions are in terms of matrices over k it makes sense again to talk about general instances of these actions.

Lemma 4.1.1. *Let $I \subset R$ denote a codimension 2 ideal generated by 3 general forms of the same degree $d \geq 2$. Then the syzygy module of I is generated in degrees $d + \lfloor d/2 \rfloor$ and $d + \lceil d/2 \rceil$.*

Proof. We will induct on d. Since d may be even or odd and our inductive process will pass from $d - 2$ to d, we need to start the induction in the cases where $d = 2$ and $d = 3$, respectively. Obviously, there is nothing to show in these two cases as the statement holds by default for any set of forms in these degrees.

We now proceed to the inductive step, assuming $d \geq 4$ for the even case and $d \geq 5$ for the odd case. Up to general k-linear combinations we can assume that the given forms have the shape

$$\begin{aligned} f_1 &= x^d + y^3 g_1 \\ f_2 &= x^{d-1} y + y^3 g_2 \\ f_3 &= x^{d-2} y^2 + y^3 g_3. \end{aligned} \tag{4.2}$$

Note that the set $\{g_1, g_2, g_3\}$ of forms preserve almost all coefficients of the original f's, hence is general. Their common degree is $d - 3$.

Suppose $Pf_1 + Qf_2 + Rf_3 = 0$ is a syzygy. Since f_2, f_3 are divisible by y, one has $\boxed{P = yP'}$, for some $P' \in k[x,y]$. Plugging this in the syzygy equation, and canceling y yields

$$P'(x^d + y^3 g_1) + Q(x^{d-1} + y^2 g_2) + R(x^{d-2} y + y^2 g_3) = 0.$$

Then y divides $x^d P' + Q x^{d-1} = x^{d-1}(Q + xP')$, giving $\boxed{Q = yQ' - xP'}$ for some $Q' \in k[x,y]$. Plugging this back into the above equation, after canceling $P'x^d$, and after simplifying by y, one has

$$P' y^2 g_1 + Q' x^{d-1} + y^2 Q' g_2 - xyP' g_2 + R x^{d-2} + yRg_3 = 0.$$

Thus, y divides $Q'x^{d-1} + Rx^{d-2} = x^{d-2}(R + xQ')$, giving $\boxed{R = yR' - xQ'}$, for some $R' \in k[x,y]$.

Again, plugging this back, canceling $Q'x^{d-1}$, and simplifying by y yields

$$P'(yg_1 - xg_2) + Q'(yg_2 - xg_3) + R'(x^{d-2} + yg_3) = 0.$$

Note that the coefficients of the three forms $\{yg_1 - xg_2, yg_2 - xg_3, x^{d-2} + yg_3\}$ of degree $d-2$ are almost all of the original general coefficients or sums of these, hence this set is again a general set of forms of degrees $d-2$. By the inductive hypothesis on d they generate an ideal whose syzygy module is generated in standard degrees $\lfloor (d-2)/2 \rfloor$ and $\lceil (d-2)/2 \rceil$.

Using the formulas in the boxes, we get two independent syzygies on (f_1, f_2, f_3) of standard degrees $\lfloor d/2 \rfloor$ and $\lceil d/2 \rceil$. $\square$

Let us read the Hilbert function $H_I(t)$ of I off the graded free resolution of I implied by Lemma 4.1.1:

Lemma 4.1.2. *Let $I \subset R$ denote a codimension 2 ideal generated by 3 general forms of the same degree $d \geq 2$. Then*

$$H_I(t) = 3H_R(t-d) - H_R(t - d - \lfloor d/2 \rfloor) - H_R(t - d - \lceil d/2 \rceil).$$

Consequently, setting $s := \lfloor d/2 \rfloor$, one has:

(Odd) *If d is odd then* $\dim(R/I)_{3s} = 1$ *and* $\dim(R/I)_{3s+1} = 0$.

(Even) *If d is even then* $\dim(R/I)_{3s-2} = 2$ *and* $\dim(R/I)_{3s-1} = 0$.

Moreover, up to k-linear transformations and change of variables, we can assume the following:

- *If $d = 2s+1$, then y^{3s} spans $(R/I)_{3s}$;*
- *If $d = 2s$, then $\{y^{3s-2}, xy^{3s-3}\}$ span $(R/I)_{3s-2}$.*

Proof. The Hilbert function easily reads off the minimal graded resolution

$$0 \to R(-d - \lfloor d/2 \rfloor) \oplus R(-d - \lceil d/2 \rceil) \longrightarrow R(-d)^3 \longrightarrow I \to 0$$

afforded by Lemma 4.1.1.

Therefore, the full Hilbert function of R/I is

$$\begin{array}{cccccccc} \text{dimension} & \ldots 2s+1 & 2s-1 & 2s-3 & \ldots & 3 & 1 & 0 \\ \text{degree} & \ldots 2s & 2s+1 & 2s+2 & \ldots & 3s-1 & 3s & 3s+1 \end{array},$$

if $d = 2s+1$, and

$$\begin{array}{cccccccc} \text{dimension} & \ldots 2s & 2s-2 & 2s-4 & \ldots & 4 & 2 & 0 \\ \text{degree} & \ldots 2s-1 & 2s & 2s+1 & \ldots & 3s-3 & 3s-2 & 3s-1 \end{array},$$

if $d = 2s$.

In order to prove the supplementary assertion, we focus on the odd case $d = 2s+1$, the even case being similar. Consider the k-vector space $I_{3s} = (x,y)_{s-1}I_{2s+1}$ and recall that it has diimension $3s$. By a similar procedure as (4.2) in the proof of Lemma 4.1.1, we can apply a suitable action of $GL(3s,k)$ to obtaining a k-basis of I_{3s} in a "triangular" form

$$\begin{array}{rcl} F_1 & = & \alpha_{3s,0}x^{3s} + \alpha_{3s-1,1}x^{3s-1}y + \cdots \\ F_2 & = & \beta_{3s-1,1}x^{3s-1}y + \beta_{3s-2,2}x^{3s-2}y^2 + \cdots \\ F_3 & = & \gamma_{3s-2,2}x^{3s-3}y^2 + \beta_{3s-3,3}x^{3s-3}y^3 + \cdots \\ \vdots & & \vdots \\ F_{3s-1} & = & \mu_{2,3s-2}x^2y^{3s-2} + \mu_{1,3s-1}xy^{3s-1} + \mu_{3s}y^{3s} \\ F_{3s} & = & \nu_{1,3s-1}xy^{3s-1} + \nu_{3s}y^{3s} \end{array}$$

Applying the change of variables

$$x \mapsto \frac{1}{\nu_{1,3s-1}}x - \frac{\nu_{3s}}{\nu_{1,3s-1}}, \quad y \mapsto y$$

transforms F_{3s} in the above basis into xy^{3s-1}. Let $\tilde{I} \subset R$ denote the ideal obtained by applying this change of variables to the forms generating I. Then, since $I_{3s} = (x,y)_{s-1}I_{2s+1}$ and (x,y) is invariant under this change, it follows that $\tilde{I}_{3s} = (x,y)_{s-1}\tilde{I}_{2s+1}$ contains the monomial xy^{3s-1}. On the other hand, the change of variables did not affect the term $\mu_{3s}y^{3s}$ of the basis element F_{3s-1}. If $y^{3s} \in \tilde{I}$ then we can clean bottom up all monomials, thus implying that I_{3s} is spanned by all monomials of this degree - an absurd. □

4.2 Degree ≤ 12

We observe that the Rees algebra of an (x,y)-primary ideal of $k[x,y]$ generated by three arbitrary forms of degree $d \leq 4$ is almost Cohen–Macaulay: when I admits a linear syzygy, the result is part of [22, Theorem 4.4] (see also [26, Theorem 4.4] and [38, Proposition 2.3]). The balanced case $d = 4$ is proved in [22, Proposition 4.3].

For very low degrees larger than 4 one can handle the criterion of Huckaba–Marley in order to show that the summation of the lengths $\lambda(I^\ell/JI^{\ell-1})$, for $\ell \geq 1$, exceeds the Chern number $e_1(I)$. As a matter of precision, the criterion of Huckaba–Marley is stated for local rings. Here, as commonly done, whenever using the local tool, we harmlessly pass to the local ring $k[x,y]_{\mathfrak{m}}$, where $\mathfrak{m} = (x,y)$, and consider the extended ideal $I_{\mathfrak{m}}$. We will make no distinction in the notation.

Theorem 4.2.1. *Let $I \subset R$ denote a codimension 2 ideal generated by 3 general forms of degree $5 \leq d \leq 12$. Then $\mathcal{R}_R(I)$ is not almost Cohen–Macaulay.*

Proof. One has to show that $\mathcal{R}_R(I)$ has depth 1, which will follow from proving that the associated graded ring of I has depth 0, according to the second observation after Theorem 1.5.1.

According to the Huckaba–Marley criterion (Theorem 1.5.1) one has to prove the strict inequality

$$\sum_{\ell \geq 1} \lambda(I^\ell/JI^{\ell-1}) > e_1(I), \tag{4.3}$$

where $\lambda(\_)$ denotes length over R, $J \subset I$ is a minimal reduction and $e_1(I)$ is the second coefficient of the Hilbert–Samuel polynomial of R/I in its combinatorial expression. Since the generators of I are general forms, any two of them will generate a minimal reduction $J \subset I$ – this is a first weak use of general forms.

We recall the easy general fact that, since $I = (J, f)$, for some $f \in I$, then $I^\ell = (JI^{\ell-1}, f^\ell)$, for any $\ell \geq 1$. It follows that

$$\begin{aligned} I^\ell/JI^{\ell-1} &\simeq (JI^{\ell-1}, f^\ell)/JI^{\ell-1} \simeq (f^\ell)/JI^{\ell-1} \cap (f^\ell) \\ &\simeq (f^\ell)/(JI^{\ell-1} : f^\ell)(f^\ell) \simeq R/JI^{\ell-1} : f^\ell, \end{aligned}$$

hence $\lambda(I^\ell/JI^{\ell-1}) = \lambda(R/JI^{\ell-1} : f^\ell)$.

Now, $JI^{\ell-1} : f^\ell$ coincides with the ideal $\mathfrak{a}_\ell \subset R$ generated by the last coordinates of the syzygies of I^ℓ, which is an (x,y)-primary ideal. Thus, $\lambda(I^\ell/JI^{\ell-1}) = \lambda(R/\mathfrak{a}_\ell)$. It then follows immediately from Lemma 4.1.1 that

$$\lambda(I/J) = \begin{cases} \frac{d^2}{4} & \text{if } d \text{ is even} \\ \frac{d^2-1}{4} & \text{if } d \text{ is odd.} \end{cases}$$

Next, by computing the Hilbert function of $R/I_1(\varphi)$ using Lemma 4.1.1 again, one has:

$$\lambda(R/I_1(\varphi)) = \binom{\lfloor \frac{d}{2} \rfloor + 1}{2} + \begin{cases} \max\{\frac{d}{2} - 5, 0\} & \text{if } d \text{ is even} \\ \lfloor \frac{d}{2} \rfloor - 2 & \text{if } d \text{ is odd.} \end{cases}$$

(A second weak use of general forms)

On the other hand, by Proposition D.16 one has $\lambda(I^2/JI) = \lambda(I/J) - \lambda(R/I_1(\varphi))$. Therefore, it obtains

$$\lambda(I^2/JI) = \binom{\lceil \frac{d}{2} \rceil}{2} - \begin{cases} \max\{\frac{d}{2}-5,0\} & \text{if } d \text{ is even} \\ (\lfloor \frac{d}{2} \rfloor - 2) & \text{if } d \text{ is odd.} \end{cases}$$

We next claim that the rational map $\mathbb{P}^1 \dashrightarrow \mathbb{P}^2$ defined by the generators of I is birational. Indeed, one can reduce the problem to the affine situation by setting $t := x/y$ in the usual way after dividing all three generators of I by y^d and then taking the fractions with same denominator (one of the three). This way we get a rational map $\mathbb{A}^1 \dashrightarrow \mathbb{A}^2$ defined by rational functions $F_1(t)/F_3(t), F_2(t)/F_3(t)$, where $\deg F_i(t) = d$. Since the involved terms are general polynomials in t (k being infinite), for a general point $(a_1, a_2) \in \mathbb{A}^2$ the system of equations $F_1(t)/F_3(t) = a_1, F_2(t)/F_3(t) = a_2$ admits exactly one solution, including multiplicity (algebraically: t is a rational fraction in $F_1(t)/F_3(t), F_2(t)/F_3(t)$).

(Yet another use of general forms)

As a consequence, one has $e_1(I) = \frac{1}{2}(d^2 - d) = \binom{d}{2}$ and $\deg(R/I) = d$ (see Proposition D.17). It also follows that the reduction number of I is $d-1$.

By the Huckaba–Marley criterion we have to prove that for $d \geq 5$

$$\sum_{\ell=1}^{d-1} \lambda_\ell > \binom{d}{2},$$

where $\lambda_\ell := \lambda_\ell(I^\ell/JI^{\ell-1})$. Subtracting the sum $\lambda_1 + \lambda_2$ as computed above, one has to show that

$$\sum_{\ell=3}^{d-1} \lambda_\ell > \begin{cases} \frac{d(d-2)}{8} + \max\left\{\frac{d}{2} - 5, 0\right\} & \text{if } d \text{ is even} \\ \frac{(d+1)(d-1)}{8} - 2 & \text{if } d \text{ is odd} \end{cases}$$

As it turns out, for $5 \leq d \leq 12$ it will suffice to compute λ_3 and bound below each of the remaining λ_i's ($i \geq 4$) by 1. We remark that for higher values of d (e.g., $d = 13$) more precise bounds for λ_4, etc., may be required for the conclusion.

As remarked earlier, finding λ_3 depends on determining a minimal set of generators of the ideal $\mathfrak{a}_3$ generated by the last coordinates of the syzygies of I^3. Computing with *Macaulay* [2] by employing random forms and using the previously established values of λ_1, λ_2, our findings are as follows:

$d = 5$: $\mathfrak{a}_3 = (x, y) \Rightarrow \lambda_3 + \lambda_4 \geq 2$; since $\lambda_1 = d^2 - 1/4 = 6$ and $\lambda_2 = 3$ the total sum is at least $11 > 10 = \binom{5}{2}$

$d = 6$: $\mathfrak{a}_3 = (x, y)^2 \Rightarrow \lambda_3 + \lambda_4 + \lambda_5 \geq 5$; since $\lambda_1 = d^2/4 = 9$ and $\lambda_2 \geq \lambda_3$ the total sum is at least $9 + 3 + 5 = 17 > 15 = \binom{6}{2}$

$d = 7$: $\mathfrak{a}_3 = (x, y)^2 \Rightarrow \lambda_3 + \cdots + \lambda_6 \geq 3 + 3 = 6$; since $\lambda_1 = d^2 - 1/4 = 12$ and $\lambda_2 = 5$ the total sum is at least $23 > 21 = \binom{7}{2}$

$d = 8$: $\mathfrak{a}_3 = (x, y)^2 \Rightarrow \lambda_3 = 3 \Rightarrow \lambda_3 + \cdots + \lambda_7 \geq 7$; since $\lambda_1 = 64/4 = 16$ and $\lambda_2 = 6$ for $d = 8$, we find the total sum is at least $29 > 28 = \binom{8}{2}$

$d = 9$: $\mathfrak{a}_3 = (x, y)^3 \Rightarrow \lambda_3 = 6 \Rightarrow \lambda_3 + \cdots + \lambda_8 \geq 6 + 5 = 11$; since $\lambda_1 = d^2 - 1/4 = 20$ and

$\lambda_2 = 8$ the total sum is at least $39 > 36 = \binom{9}{2}$

$d = 10$: $\mathfrak{a}_3 = (x,y)^3 \Rightarrow \lambda_3 + \cdots \lambda_9 \geq 6 + 6 = 12$; since $\lambda_1 = 100/4 = 25$ and $\lambda_2 = 10$, the total sum is at least $47 > 45 = \binom{10}{2}$

$d = 11$: $\mathfrak{a}_3$ is generated by 3 cubics $\Rightarrow \lambda_3 = 7 \Rightarrow \lambda_3 + \cdots + \lambda_{10} \geq 14$; since $\lambda_1 = 120/4 = 30$ and $\lambda_2 = 12$, the total sum is at least $56 > 55 = \binom{11}{2}$.

$d = 12$: $\mathfrak{a}_3 = (x,y)^4$, hence $\lambda_3 = 10$. Since $\lambda_1 = 36$ and $\lambda_2 = 14$, the first three lengths sum up to 60. On the other hand, the reduction number being 11 the remaining lengths sum up to at least $11 - 4 + 1 = 8$. Thus, the total sum $68 > 66 = e_1(I)$.

□

Remark 4.2.2. We observe that the expected result to the effect that, for $\ell \geq 3$, $\mathfrak{a}_\ell = (x,y)^{s_\ell}$ for some $s_\ell \geq 1$, fails for $\ell = 3, d = 13$. In addition, the conclusive analysis in the above cases fails as well. Indeed, one has to resort to $\lambda_4 = 6$ to conclude that $\sum_{\ell=1}^{d-1} \lambda_\ell > 78 = \binom{13}{2}$, without computing the remaining lengths. Clearly, one can go on computing like this for higher d. This calculation possibly indicates that there exists a cutting edge index ℓ_0, which is much less than $d - 1 = \mathrm{red}_J(I)$, such that adding the exact values of the lengths past it becomes needless. Having a previous control or estimate on this index could lead us to a full proof of the Huckaba–Marley inequality.

Discussion A purely theoretical argument, without the computer, even for $d = 6$, is quite subtle. Here for the failure of the almost Cohen–Macaulay property we need to show the bound $\lambda_3 + \lambda_4 + \lambda_5 > 3$, so it suffices to show that $\lambda_3 \geq 2$ since the reduction number is 5. Equivalently, we are to show that I^3 has at most one linear syzygy because then, since $\mathfrak{a}_i$ is always an ideal of finite colength, $\lambda_3 \geq 2$. Contradicting this takes us to a long discussion about the degrees of the syzygies of I^3 which eventually abuts at the following: since I^3 is a perfect ideal with 10 generators and generated in degree 18, its minimal presentation matrix has 7 columns whose degrees $r_3 \leq \cdots \leq r_9$ add up to 16. By elementary column operations with pivot the last coordinates of the two assumed linear syzygies, we may assume that the last entry of any of the other 7 columns are zero. Then they are syzygies of JI^2. Therefore, $r_9 \geq \cdots \geq r_3 \geq 2$. On the other hand, no minimal syzygy has degree ≥ 4 since the presentation matrix of I is in degree 3. To add up to 16, the only way is $r_3 = \cdots = r_7 = 2$, $r_8 = r_9 = 3$. This structure of the Hilbert–Burch matrix of I^3 is realizable if the entries of the Hilbert–Burch matrix of I are not general - see Example 4.2.3 (b) below.

The theoretical side of the discussion for $d \geq 7$ eludes the eye.

The following examples show that, already for $d = 6$, the statement of Theorem 4.2.1 is no longer true if the generators of the ideal are not general enough, even when the degrees of the syzygies are as in Lemma 4.1.1.

Example 4.2.3. (a) The first example keeps some of the properties above: 1) The associated rational map is birational (onto the image), hence $e_1(I) = \binom{6}{2} = 15$ and the reduction number

$\text{red}_J(I) = 5$; (2) $\lambda(I/J) = 9$, but $\lambda(I^2/JI) = 2$ instead, a degenerate value; (3) The "expected" value $\lambda_\ell = \binom{s_\ell+1}{2}$ for suitable s_ℓ fails for $\ell = 3$ as here $\lambda_3 = 2$.

The Hilbert–Burch matrix of this example is

$$\begin{pmatrix} -xy^2 & -y^3 \\ x^3 & 0 \\ y^3 & x^3 \end{pmatrix}$$

A minimal reduction is $(x^6, x^4y^2 - y^6)$. Note that a linear syzygy appears all too soon among the forms $x^6(x^4y^2 - y^6)$, $x^6.x^3y^3$, $(x^3y^3)^2$. One gets here

$$\sum_{\ell=1}^{5} \lambda_\ell = 9 + 2 + 2 + 1 + 1 = 15.$$

(b) The second example has Hilbert–Burch matrix

$$\begin{pmatrix} x^3 & xy^2 \\ x^2y & x^3 - y^3 \\ y^3 & -x^2y + xy^2 \end{pmatrix}$$

The subideal $J := (x^6 - 2x^3y^3, x^4y^2 - y^6)$ is a minimal reduction. The example is really on the edge since we can check that: (1) The associated rational map is again birational, hence $e_1(I) = 15$ and $\text{red}_J(I) = 5$; (2) The lengths $\lambda(I/J), \lambda(I^2/JI)$ are as stated in the above preliminaries; (3) The expected value $\lambda_\ell = \binom{s_\ell}{2}$ for $\ell \geq 3$ holds here with $s_3 = 1$ (i.e., $JI^2 : I^3 = (x, y)$).

Since $\lambda_3 = 1$ then $\lambda_5 = \lambda_4 = 1$, where $\text{red}_J(I) = 5$, thus yielding

$$\sum_{\ell=1}^{5} \lambda_\ell = 9 + 3 + 1 + 1 + 1 = 15.$$

Therefore, in both examples the Rees algebra of I has depth ≥ 2, i.e., it is almost Cohen–Macaulay.

4.3 Stronger conjectures

In this part we state a couple of conjectures, in increasing strength, that imply the main conjecture in this chapter.

4.3.1 The annihilator conjecture

The first of these is inspired from the Ratliff–Rush filtration theory.

Conjecture 4.3.1. *Let $I = \subset R := k[x, y]$ denote a codimension 2 ideal generated by a set of general forms f_1, f_2, f_3 of degree $d \geq 13$. Then $I^2 : I \not\subset I$.*

Note that an affirmative answer to this conjecture, coupled with the previous result on degrees ≤ 12, implies that Conjecture 4.0.9 is true. Indeed, since the Ratliff–Rush closure of I is strictly larger than I, by Theorem 1.6.1 the associated graded ring of I has depth zero and hence the Rees algebra of I has depth 1.

Next is a proof of the latter conjecture, modulo some assumption on the behavior of the general coefficients involved in the calculations.

Since a change of variables does not affect either the hypothesis or the statement, we can assume by Lemma 4.1.2 that

- If $d = 2s + 1$, then y^{3s} spans $(R/I)_{3s}$;
- If $d = 2s$, then $\{y^{3s-2}, xy^{3s-3}\}$ span $(R/I)_{3s-2}$.

Now, in the proof of Lemma 4.1.1 we had

$$\begin{aligned} f_1 &= x^d + y^3 g_1 \\ f_2 &= x^{d-1}y + y^3 g_2 \\ f_3 &= x^{d-2}y^2 + y^3 g_3, \end{aligned}$$

where g_1, g_2, g_3 are general forms of degree $d-3$.

In the same proof we obtained three general forms of degree $d-2$

$$\begin{aligned} h_1 &:= yg_1 - xg_2, \\ h_2 &:= yg_2 - xg_3, \\ h_3 &:= x^{d-2} + yg_3. \end{aligned}$$

Let $I' = (h_1, h_2, h_3)$.

As a result, the following relations are satisfied:

$$\begin{aligned} f_1 &= y^2 h_1 + xyh_2 + x^2 h_3 \\ f_2 &= y^2 h_2 + xyh_3 \\ f_3 &= y^2 h_3, \end{aligned}$$

and

$$\begin{aligned} y^3 h_1 &= yf_1 - xf_2 \\ y^3 h_2 &= yf_2 - xf_3 \\ y^2 h_3 &= f_3. \end{aligned} \tag{4.4}$$

We induct on d, or rather on $s = \lfloor d/2 \rfloor$.

We will focus on the odd case $d = 2s + 1$. Keeping in mind that we are assuming $d \geq 13$ for the odd case, we induct on $s \geq 6$ to show that $M = y^{3s} \in I^2 : I$. The induction starts at $s = 6$ – this case is directly checked in the computer by taking f_1, f_2, f_3 to be random forms of degree 13.

Thus, assume that $s \geq 7$.

We are to show that $Mf_u \in I^2$, for $u = 1, 2, 3$. We will produce here the argument for $u = 2, 3$, postponing the calculation for $u = 1$ and the even case to Appendix B as they are a bit lengthy, but very similar.

$\underline{u = 3}$. By the inductive hypothesis, we have $y^{3(s-1)}h_3 = \sum \alpha_{i,j}h_ih_j \in I'^2$, for suitable $\alpha_{i,j} \in R_{s-2}$. Also note from (4.4) that $y^3h_j \in I$, for $j = 1, 2$ and $y^2h_3 \in I$. Therefore,

$$\begin{aligned} y^{3s}f_3 &= y^{3s}y^2h_3 = y^5(y^{3(s-1)}h_3) = y^5(\sum \alpha_{i,j}h_ih_j) = \sum \alpha_{i,j}y^2h_iy^3h_j \\ &\equiv \alpha_{1,1}y^2h_1y^3h_1 + \alpha_{1,2}y^2h_1y^3h_2 + \alpha_{2,2}y^2h_2y^3h_2 \pmod{I^2} \\ &\equiv \alpha_{1,1}y^2h_1(-xf_2) + \alpha_{1,2}y^2(-xf_2)h_2 + \alpha_{2,2}y^2h_2(-xf_3) \pmod{I^2} \\ &\equiv (-\alpha_{1,1}xy^2h_1)f_2 + (-\alpha_{1,2}xy^2h_2)f_2 + (-\alpha_{2,2}xy^2h_2)f_3 \pmod{I^2} \\ &\equiv (-\alpha_{1,1}xy^2h_1 - \alpha_{1,2}xy^2h_2)f_2 + (-\alpha_{2,2}xy^2h_2)f_3 \pmod{I^2} \end{aligned} \tag{4.5}$$

where we have used the explicit relations in (4.4).

Note that $-\alpha_{1,1}xy^2h_1 - \alpha_{1,2}xy^2h_2$ and $-\alpha_{2,2}xy^2h_2$ are both forms of degree $3s$. Since y^{3s} spans $(R/I)_{3s}$, these forms are scalar multiples of y^{3s} modulo I_{3s}; say, $-\alpha_{1,1}xy^2h_1 - \alpha_{1,2}xy^2h_2 = \eta y^{3s} + F$ and $-\alpha_{2,2}xy^2h_2 = \zeta y^{3s} + G$, for certain $\eta, \zeta \in k$ and $F, G \in I_{3s} \subset I$. Substituting for these in the last congruence of (B.1) yields $y^{3s}f_3 \equiv \eta y^{3s}f_2 + \zeta y^{3s}f_3 \pmod{I^2}$. It follows that

$$-\eta y^{3s}f_2 + (1 - \zeta)y^{3s}f_3 \in I^2 \tag{4.6}$$

$\underline{u = 2}$. In a pretty similar way, by the inductive hypothesis and the relations (4.4), one obtains

$$\begin{aligned} y^{3s}f_2 &= y^{3s}(y^2h_2 + xyh_3) = y^5(y^{3(s-1)}h_2) + xy^4(y^{3(s-1)}h_3) \\ &= y^5(\sum \alpha_{i,j}h_ih_j) + xy^4(\sum \beta_{i,j}h_ih_j) = \sum \alpha_{i,j}y^2h_iy^3h_j + \sum \beta_{i,j}xyh_iy^3h_j \\ &\equiv \alpha_{1,1}y^2h_1y^3h_1 + \alpha_{1,2}y^2h_1y^3h_2 + \alpha_{2,2}y^2h_2y^3h_2 + \beta_{1,1}xyh_1y^3h_1 \\ &+ \beta_{1,2}xyh_1y^3h_2 + \beta_{1,3}xy^2h_1y^2h_3 + \beta_{2,2}xyh_2y^3h_2 + \beta_{2,3}xy^2h_2y^2h_3 \pmod{I^2} \\ &\equiv \alpha_{1,1}y^2h_1(-xf_2) + \alpha_{1,2}y^2(-xf_2)h_2 + \alpha_{2,2}y^2h_2(-xf_3) \\ &+ \beta_{1,1}xyh_1(-xf_2) + \beta_{1,2}xy(-xf_2)h_2 + \beta_{1,3}xy^2h_1f_3 + \beta_{2,2}xyh_2(-xf_3) + \beta_{2,3}xy^2h_2f_3 \pmod{I^2} \\ &\equiv (-\alpha_{1,1}xy^2h_1)f_2 + (-\alpha_{1,2}xy^2h_2)f_2 + (-\alpha_{2,2}xy^2h_2)f_3 + (-\beta_{1,1}x^2yh_1)f_2 + (-\beta_{1,2}x^2yh_2)f_2 \\ &+ (\beta_{1,3}xy^2h_1)f_3 + (-\beta_{2,2}x^2yh_2)f_3 + (\beta_{2,3}xy^2h_2)f_3 \pmod{I^2} \\ &\equiv (-\alpha_{1,1}xy^2h_1 - \alpha_{1,2}xy^2h_2 - \beta_{1,1}x^2yh_1 - \beta_{1,2}x^2yh_2)f_2 \\ &+ (-\alpha_{2,2}xy^2h_2 + \beta_{1,3}xy^2h_1 - \beta_{2,2}x^2yh_2 + \beta_{2,3}xy^2h_2)f_3 \pmod{I^2} \end{aligned}$$

By the same token as in the case of $u = 3$, since $-\alpha_{1,1}xy^2h_1 - \alpha_{1,2}xy^2h_2 - \beta_{1,1}x^2yh_1 - \beta_{1,2}x^2yh_2$ and $-\alpha_{2,2}xy^2h_2 + \beta_{1,3}xy^2h_1 - \beta_{2,2}x^2yh_2 + \beta_{2,3}xy^2h_2$ are forms of degree $3s$, we obtain

$$(1 - \zeta')y^{3s}f_2 - \eta' y^{3s} f_3 \in I^2, \tag{4.7}$$

for certain $\eta', \zeta' \in k$.

Now consider (4.6) and (4.7) together as a system of linear equations over R/I^2 in the unknowns $y^{3s}f_2$ and $y^{3s}f_3$. By Cramer, one has $\Delta y^{3s} f_2 \in I^2$ and $\Delta y^{3s} f_3 \in I^2$, where $\Delta = \eta\eta' - (1 - \zeta)(1 - \zeta')$ is the determinant of the system. Provided $\Delta \neq 0$ we can conclude that $y^{3s}f_2 \in I^2$ and $y^{3s}f_3 \in I^2$. But, since the h_i's are general forms, hence have general coefficients, then $\eta, \eta', \zeta, \zeta'$ are general as well from their definition. Therefore, cannot constitute the coordinates of a point lying on the quadric of equation $X_1X_2 - (1 - X_3)(1 - X_4)$ in affine space $\mathbb{A}^4$.

For additional complete calculative evidence of this conjecture see Appendix B.

4.3.2 The tilde conjecture

Will focus on the odd case: $d = 2s + 1$.

We establish the technical preliminaries. Given a form

$$f = \alpha_{d,0}x^d + \alpha_{d-1,1}x^{d-1}y + \cdots + \alpha_{2,d-1}x^2y^{d-2} + \alpha_{1,d-1}xy^{d-1} + \alpha_{0,d}y^d$$

of degree d we associate to it the unique form $\delta(f)$ of degree $d - 2$ in the following way: divide the first $d - 1$ terms by x^2 (keep same coefficients) and omit the last two terms. It is clear that this association preserves many properties of the original form, including that of being general provided $d \geq 4$. Now, given three general forms f_1, f_2, f_3 of degree d, in the sense that the $(d+1) \times 3$ total matrix of the coefficients has general entries, we let $g_1 := \delta(f_1), g_2 : \delta(f_2), g_3 := \delta(f_3)$ denote the respective forms of degree $d - 2$ as explained. Let I and J stand for the corresponding ideals they generate in $R = k[x, y]$.

By construction, $I \subset \widetilde{I} := (x^2J, xy^{d-1}, y^d)$; more precisely, this is an inclusion induced by an inclusion of the corresponding linear systems spanned in the common degree d.

Throughout we denote $\mathfrak{m} := (R_+) = (x, y)$.

Conjecture 4.3.2. *Let $d = 2s + 1$ be odd. Then $\mathfrak{m}^{s-1}(\widetilde{I})^2 \subset I^2$.*

We now state a more precise version of the annihilator conjecture as a consequence of the above conjecture.

Proposition 4.3.3. *Let $d = 2s + 1$ be odd. Then the annihilator of the R/I-module I/I^2 contains a monomial spanning $(R/I)_{3s}$, and hence $I^2 : I \neq I$.*

Proof. By the inductive hypothesis, we may assume that $y^{3(s-1)} \in J^2 : J$. More precisely, by a degree count one has $y^{3(s-1)}J \subset \mathfrak{m}^{s-2}J^2$.

We know that in the Hilbert function above <u>any</u> monomial of degree $3s$ spans $(R/I)_{3s}$ – the proof is the same as in the first section. We have seen in the Hilbert function of R/I that x^2y^{3s-2} spans $(R/I)_{3s}$. For this monomial one has:

$$x^2y^{3s-2}I \subset x^2y^{3s-2}\widetilde{I} = x^2y^{3s-2}(x^2J, xy^{2s}, y^{2s+1}).$$

Treat each piece separately:

(i) $x^2y^{3s-2}x^2J = x^4y(y^{3s-3}J) \subset x^4y(\mathfrak{m}^{s-2}J^2) = ym^{s-2}(x^2J)^2 \subset ym^{s-2}(\widetilde{I})^2 \subset \mathfrak{m}^{s-1}(\widetilde{I})^2 \subset I^2$, the last inclusion by the conjecture.

(ii) $x^2y^{3s-2}\, xy^{2s} = xy^{s-2}(x^2y^{4s}) \subset \mathfrak{m}^{s-1}(\widetilde{I})^2 \subset I^2$.

(iii) $x^2y^{3s-2}\, y^{2s+1} = x^2y^{s-3}(y^{2(2s+1)}) \subset \mathfrak{m}^{s-1}(\widetilde{I})^2 \subset I^2$. $\square$

Appendix

Appendix A

Case: b-uniform

A.1 Proof of Proposition 3.4.5

The proof will compute all $S-$pairs of elements in the set $\Sigma = \Sigma(i_1, \ldots, i_j)$. As usual, pairs F, G such $\gcd(\text{in}(F), \text{in}(G)) = 1$ will be overlooked.

Case 1. $S(K_{i,k}, K_{i',k'})$.
In this case, $\text{in}(K_{i,k}) = x_k^{a_k} y_i$ and $\text{in}(K_{i',k'}) = x_{k'}^{a_{k'}} y_{i'}$.

Case 1.1. Let $i < k < i' < k'$. No action here since $\text{in}(K_{i,k})$ and $\text{in}(K_{i',k'})$ are relatively prime.

Case 1.2. Let $i < i' < k < k'$. No action here since $\text{in}(K_{i,k})$ and $\text{in}(K_{i',k'})$ are relatively prime.

Case 1.3. Let $i' < i < k < k'$. No action here since $\text{in}(K_{i,k})$ and $\text{in}(K_{i',k'})$ are relatively prime.

Case 1.4. Let $k = k'$. Then

$$S(K_{i,k}, K_{i',k}) = \frac{x_k^{a_k} y_i y_{i'}}{-x_k^{a_k} y_i} K_{i,k} - \frac{x_k^{a_k} y_i y_{i'}}{-x_k^{a_k} y_{i'}} K_{i',k} = -y_k(x_i^{a_k} y_{i'} - x_{i'}^{a_k} y_i) \equiv 0 \mod \Sigma.$$

Case 1.5. Let $i < k = i' < k'$. No action here since $\text{in}(K_{i,k})$ and $\text{in}(K_{k,k'})$ are relatively prime.

Case 1.6. Let $i' < k' = i < k$. No action here since $\text{in}(K_{i,k})$ and $\text{in}(K_{k,i})$ are relatively prime.

Case 1.7. Let $i = i'$. Then

$$S(K_{i,k}, K_{i,k'}) = \frac{x_k^{a_k} x_{k'}^{a_{k'}} y_i}{-x_k^{a_k} y_i} K_{i,k} - \frac{x_k^{a_k} x_{k'}^{a_{k'}} y_i}{-x_{k'}^{a_{k'}} y_i} K_{i,k'} = -x_i^{a_i}(x_{k'}^{a_{k'}} y_k - x_k^{a_k} y_{k'}) \equiv 0 \mod \Sigma.$$

Case 2. $S(L_j, L_{j'})$, with $j < j'$.
In this case, $\text{in}(L_j) = x_j^{a_j - b} w$ and $\text{in}(L_{j'}) = x_{j'}^{a_{j'} - b} w$.
Then

$$S(L_j, L_{j'}) = \frac{x_j^{a_j - b} x_{j'}^{a_{j'} - b} w}{x_j^{a_j - b} w} L_j - \frac{x_j^{a_j - b} x_{j'}^{a_{j'} - b} w}{x_{j'}^{a_{j'} - b} w} L_{j'} = P(j, j')^b K_{j,j'} \equiv 0 \mod \Sigma.$$

Case 3. $S(L_j, K_{i,k})$.
In this case, $\text{in}(L_j) = x_j^{a-b} w$ and $\text{in}(K_{i,k}) = x_k^a y_i$.

Case 3.1. Let $j < i < k$. No action here since $\mathrm{in}(L_j)$ and $\mathrm{in}(K_{i,k})$ are relatively prime.

Case 3.2. Let $i < j < k$. No action here since $\mathrm{in}(L_j)$ and $\mathrm{in}(K_{i,k})$ are relatively prime.

Case 3.3. Let $i < k < j$. No action here since $\mathrm{in}(L_j)$ and $\mathrm{in}(K_{i,k})$ are relatively prime.

Case 3.4. Let $j = i < k$. No action here since $\mathrm{in}(L_i)$ and $\mathrm{in}(K_{i,k})$ are relatively prime.

Case 3.5. Let $i < j = k$. Then

$$S(L_k, K_{i,k}) = \frac{x_k^{a_k} w y_i}{x_k^{a_k-b} w} L_k - \frac{x_k^{a_k} w y_i}{-x_k^{a_k} y_i} K_{i,k} = x_i^b y_k L_i \equiv 0 \mod \Sigma.$$

Case 4. $S(K_{u,k}, H_j^{i_1,\ldots,i_j})$.

In this case, $\mathrm{in}(K_{u,k}) = x_k^{a_k} y_u$ and $\mathrm{in}(H_j^{i_1,\ldots,i_j}) = x_{i_1}^{a_{i_1}-jb} \cdots x_{i_j}^{a_{i_j}-jb} w^j$.

Case 4.1. Let $u < k$, $u, k \notin \{i_1, \ldots, i_j\}$. No action here since $\mathrm{in}(K_{u,k})$ and $\mathrm{in}(H_j^{i_1,\ldots,i_j})$ are relatively prime.

Case 4.2. Let $u < k$, $u \in \{i_1, \ldots, i_j\}$ and $k \notin \{i_1, \ldots, i_j\}$. No action here since $\mathrm{in}(K_{u,k})$ and $\mathrm{in}(H_j^{i_1,\ldots,i_j})$ are relatively prime.

Case 4.3. Let $u < k$, $u \notin \{i_1, \ldots, i_j\}$, and $k \in \{i_1, \ldots, i_j\}$. Then

$$S(K_{u,k}, H_j^{i_1,\ldots,i_j}) = \frac{x_k^{a_k} y_u x_{i_1}^{a_{i_1}-jb} \cdots \widehat{x_k^{a_k-jb}} \cdots x_{i_j}^{a_{i_j}-jb} w^j}{-x_k^{a_k} y_u} K_{u,k} - \frac{x_k^{a_k} y_u x_{i_1}^{a_{i_1}-jb} \cdots \widehat{x_k^{a_k-jb}} \cdots x_{i_j}^{a_{i_j}-jb} w^j}{x_{i_1}^{a_{i_1}-jb} \cdots x_k^{a_k-jb} \cdots x_{i_j}^{a_{i_j}-jb} w^j} H_j^{i_1,\ldots,i_j}$$

$$= (-x_u^{jb} y_k) H_j^{I'} \equiv 0 \mod \Sigma, \text{ where } I' = (\{i_1, \ldots, i_j\} \setminus \{k\}) \cup \{u\}.$$

Case 4.4. Let $u < k$, $u, k \in \{i_1, \ldots, i_j\}$. Then

$$S(K_{u,k}, H_j^{i_1,\ldots,i_j}) = \frac{x_k^{a_k} y_u x_{i_1}^{a_{i_1}-jb} \cdots \widehat{x_k^{a_k-jb}} \cdots x_{i_j}^{a_{i_j}-jb} w^j}{-x_k^{a_k} y_u} K_{u,k} - \frac{x_k^{a_k} y_u x_{i_1}^{a_{i_1}-jb} \cdots \widehat{x_k^{a_k-jb}} \cdots x_{i_j}^{a_{i_j}-jb} w^j}{x_{i_1}^{a_{i_1}-jb} \cdots x_k^{a_k-jb} \cdots x_{i_j}^{a_{i_j}-jb} w^j} H_j^{i_1,\ldots,i_j}$$

$$= -y_k [x_u^{a_u} x_{i_1}^{a_{i_1}-jb} \cdots \widehat{x_k^{a_k-jb}} \cdots x_{i_j}^{a_{i_j}-jb} w^j - x_k^{jb} y_u P(i_1, \ldots, i_j)^{jb} y_{i_1} \cdots y_u \cdots \widehat{y_k} \cdots y_{i_j}].$$

Since $x_u^{a_u-b} w = L_u + P(u)^b y_u$, it obtains

$$S(K_{u,k}, H_j^{i_1,\ldots,i_j}) \equiv -P(i_1, \ldots, \widehat{k}, \ldots, i_j)^b y_u y_k H_{j-1}^{I''} \equiv 0 \mod \Sigma,$$

where $I'' = \{i_1, \ldots, i_j\} \setminus \{k\}$.

Case 5. $S(L_u, H_j^{i_1,\ldots,i_j})$.

In this case, $\mathrm{in}(L_u) = x_u^{a_u-b} w$ and $\mathrm{in}(H_j^{i_1,\ldots,i_j}) = x_{i_1}^{a_{i_1}-jb} \cdots x_{i_j}^{a_{i_j}-jb} w^j$.

Case 5.1. Let $u \notin \{i_1, \ldots, i_j\}$. Then

$$\begin{aligned} S(L_u, H_j^{i_1,\ldots,i_j}) &= -P(i_1, \ldots, i_j, u)^b [x_{i_1}^{a_{i_1}-(j-1)b} \cdots x_{i_j}^{a_{i_j}-(j-1)b} w^{j-1} y_u \\ &- x_u^{a_u+(j-1)b} P(i_1, \ldots, i_j, u)^{(j-1)b} y_{i_1}, \ldots, y_{i_j}]. \end{aligned}$$

Pick any subset $I'' \subset I' = \{i_1, \dots, i_j\}$, with $|I''| = j-1$ and reduce modulo $H_{j-1}^{I''}$ the monomial with w occurring within the square brackets. The result is a binomial not involving w. By the same argument as before, we conclude that this pair reduces to 0 modulo Σ.

Case 5.2. Let $u \in \{i_1, \dots, i_j\}$. Then

$$S(L_u, H_j^{i_1,\dots,i_j}) = -P(i_1, \dots, i_j)^b y_u H_{j-1}^{i_1,\dots,\widehat{u},\dots,i_j} \equiv 0 \mod \Sigma.$$

Case 6. Consider $H_m^{i_1,\dots,i_m}$ and $H_{m'}^{j_1,\dots,j_{m'}}$, with the respective external degrees $m \leq m'$. Denote $L := \{i_1, \dots, i_m\}$, $J := \{j_1, \dots, j_{m'}\}$ and $\{k_1, \dots, k_s\} = L \cap J$, for some $s \in \{0, \dots, m\}$.

Under the given order, the two leading terms of the two binomials are $\text{in}(H_m^{i_1,\dots,i_m}) = x_{i_1}^{a_{i_1}-mb} \cdots x_{i_m}^{a_{i_m}-mb} w^m$ and $\text{in}(H_{m'}^{j_1,\dots,j_{m'}}) = x_{j_1}^{a_{j_1}-m'b} \cdots x_{j_{m'}}^{a_{j_{m'}}-m'b} w^{m'}$, so their least common multiple is $w^{m'}(x_{j_1}^{a_{j_1}-m'b} \cdots \widehat{x_{i_1}^{a_{i_1}-m'b}} \cdots \widehat{x_{i_m}^{a_{i_m}-m'b}} \cdots x_{j_{m'}}^{a_{j_{m'}}-m'b})(x_{i_1}^{a_{i_1}-mb} \cdots x_{i_m}^{a_{i_m}-mb})$. Therefore

$$\begin{aligned} S(H_m^{i_1,\dots,i_m}, H_{m'}^{j_1,\dots,j_{m'}}) &= -P(i_1, \dots, j_1, \dots, i_m, \dots, j_{m'})^{mb} y_{k_1} \cdots y_{k_s} \\ &\cdot \Big[w^{m'-m} x_{j_1}^{a_{j_1}-m'b+mb} \cdots x_{i_1}^{a_{i_1}\widehat{-m'b}+mb} \cdots x_{i_m}^{a_{i_m}\widehat{-m'b}+mb} \cdots x_{j_{m'}}^{a_{j_{m'}}-m'b+mb} y_{i_1} \cdots \widehat{y_{j_1}} \cdots \widehat{y_{j_{m'}}} \cdots y_{i_m} \\ &- x_{i_1}^{a_{i_1}-mb+m'b} \cdots x_{j_1}^{a_{j_1}\widehat{-mb}+m'b} \cdots x_{j_{m'}}^{a_{j_{m'}}\widehat{-mb}+m'b} \cdots x_{i_m}^{a_{i_m}-mb+m'b} (x_{k_1} \cdots x_{k_s})^{(m'-m)b} \\ &\cdot P(i_1, \dots, j_1, \dots, i_m, \dots, j_{m'})^{(m'-m)b} y_{j_1} \cdots \widehat{y_{i_1}} \cdots \widehat{y_{i_m}} \cdots y_{j_{m'}} \Big]. \end{aligned}$$

If $m' = m$, then the binomial inside the square brackets does not involve w and therefore reduces to 0 modulo Σ by previous cases.

If $m' > m$, then $|J| = m' > m = |L|$, and therefore $|J \setminus L| \geq m' - m \geq 1$. Let $\tilde{I} \subset J \setminus L$ with $|\tilde{I}| = m' - m$. Reducing the binomial inside the square brackets modulo $H_{m'-m}^{\tilde{I}} \in \Sigma$, will result in the cancellation of the monomial involving w, hence we are back to the previous situation. □

A.2 Proof of Proposition 3.4.6

For both items, we will apply Lemma 2.2.1 (i), by which one is to compute a minimal set of generators of the intersection of the two initial ideals on the left hand side, then divide each generator by the initial term of $H_{j'}^{k_1,\dots,k_{j'}}$. To get a minimal set of generators of the intersection we use a well-known principle, by which this set is the set of the least common multiples of $\text{in}(H_{j'}^{k_1,\dots,k_{j'}})$ and each minimal generator of $\text{in}(\mathcal{H}(i_1, \dots, i_j))$.

We separate the two cases, according as to whether $j' = j$ or $j' = j + 1$.

(a) Same degree: $j = j'$

One has $\text{in}(H_j^{k_1,\dots,k_j}) = x_{k_1}^{a_{k_1}-jb} \cdots x_{k_j}^{a_{k_j}-jb} w^j$. Drawing upon (3.4), according to the external degree of a monomial, we have

Degree 1:

- $x_d^{a_d-b}w,\ d \notin \{k_1,\dots,k_j\}$ and $d < k_j$ (coming from $L_d \in \mathcal{L}$)

$$\frac{\operatorname{lcm}(x_d^{a_d-b}w, x_{k_1}^{a_{k_1}-jb}\cdots x_{k_j}^{a_{k_j}-jb}w^j)}{x_{k_1}^{a_{k_1}-jb}\cdots x_{k_j}^{a_{k_j}-jb}w^j} = x_d^{a_d-b}.$$

As $j \le n-1$, and $a > nb$, then $a - jb > 0$. But then $x_d^{a_d-b} = x_d^{(j-1)b}x_d^{a_d-jb}$, and $x_d^{a_d-jb}$ is among the generators listed in the right hand side monomial ideal.

- $x_d^{a_d-b}w,\ d \notin \{k_1,\dots,k_j\}$ and $d > k_j$ (coming from $L_d \in \mathcal{L}$)

$$\frac{\operatorname{lcm}(x_d^{a_d-b}w, x_{k_1}^{a_{k_1}-jb}\cdots x_{k_j}^{a_{k_j}-jb}w^j)}{x_{k_1}^{a_{k_1}-jb}\cdots x_{k_j}^{a_{k_j}-jb}w^j} = x_d^{a_d-b}.$$

As above, $x_d^{a_d-jb}$ is among the generators listed in the right hand side monomial ideal.

- $x_d^{a_d-b}w,\ d \in \{k_1,\dots,k_j\}$ (coming from $L_d \in \mathcal{L}$)

$$\frac{\operatorname{lcm}(x_d^{a_d-b}w, x_{k_1}^{a_{k_1}-jb}\cdots x_{k_j}^{a_{k_j}-jb}w^j)}{x_{k_1}^{a_{k_1}-jb}\cdots x_{k_j}^{a_{k_j}-jb}w^j} = x_d^{(j-1)b},$$

which is among the generators listed in the right hand side monomial ideal.

- $x_d^{a_d}y_v,\ d \notin \{k_1,\dots,k_j\}$ and $d < k_j$ (coming from $K_{d,v} \in \mathcal{L}$)

$$\frac{\operatorname{lcm}(x_d^{a_d}y_v, x_{k_1}^{a_{k_1}-jb}\cdots x_{k_j}^{a_{k_j}-jb}w^j)}{x_{k_1}^{a_{k_1}-jb}\cdots x_{k_j}^{a_{k_j}-jb}w^j} = x_d^{a_d}y_v$$

Note that $x_d^{a_d}y_v = (x_d^{jb}y_v)x_d^{a_d-jb}$, while $x_d^{a_d-jb}$ is among the generators listed in the right hand side monomial ideal.

- $x_d^{a_d}y_v,\ d \notin \{k_1,\dots,k_j\}$ and $d > k_j$ (coming from $K_{d,v} \in \mathcal{L}$)

$$\frac{\operatorname{lcm}(x_d^{a_d}y_v, x_{k_1}^{a_{k_1}-jb}\cdots x_{k_j}^{a_{k_j}-jb}w^j)}{x_{k_1}^{a_{k_1}-jb}\cdots x_{k_j}^{a_{k_j}-jb}w^j} = x_d^{a_d}y_v$$

One has $x_d^{a_d}y_v = (x_d^{b}y_v)x_d^{a_d-b}$, while $x_d^{a_d-b}$ is among the generators listed in the right hand side monomial ideal.

- $x_d^{a_d}y_v,\ d \in \{k_1,\dots,k_j\}$ (coming from $K_{d,v} \in \mathcal{L}$)

$$\frac{\operatorname{lcm}(x_d^{a_d}y_v, x_{k_1}^{a_{k_1}-jb}\cdots x_{k_j}^{a_{k_j}-jb}w^j)}{x_{k_1}^{a_{k_1}-jb}\cdots x_{k_j}^{a_{k_j}-jb}w^j} = x_d^{jb}y_v$$

Note that $x_d^{jb} y_v = (x_d^{(j-1)b} y_v) x_d^b$, so once more we get a generator listed in the right hand side monomial ideal.

Degree s $(2 \leq s \leq j-1)$:

- $x_{d_1}^{a_{d_1}-sb} \cdots x_{d_r}^{a_{d_r}-sb} x_{q_1}^{a_{q_1}-sb} \cdots x_{q_{s-r}}^{a_{q_{s-r}}-sb} w^s$, $\{d_1 < \ldots < d_r\} \cap \{k_1, \ldots, k_j\} = \emptyset$, $d_1 < k_j$ and $\{q_1 < \cdots < q_{s-r}\} \subset \{k_1, \ldots, k_j\}$.

$$\begin{aligned} & \frac{\operatorname{lcm}(x_{d_1}^{a_{d_1}-sb} \cdots x_{d_r}^{a_{d_r}-sb} x_{q_1}^{a_{q_1}-sb} \cdots x_{q_{s-r}}^{a_{q_{s-r}}-sb} w^s, x_{k_1}^{a_{k_1}-jb} \cdots x_{k_j}^{a_{k_j}-jb} w^j)}{x_{k_1}^{a_{k_1}-jb} \cdots x_{k_j}^{a_{k_j}-jb} w^j} \\ = \; & x_{d_1}^{a_{d_1}-sb} \cdots x_{d_r}^{a_{d_r}-sb} (x_{q_1} \cdots x_{q_{s-r}})^{(j-s)b} \end{aligned}$$

Note that $x_{d_1}^{a_{d_1}-sb}$ is a factor thereof factoring further as $x_{d_1}^{a_{d_1}-sb} = x_{d_1}^{(j-s)b} x_{d_1}^{a_{d_1}-jb}$, while $x_{d_1}^{a_{d_1}-jb}$ is among the generators listed in the right hand side monomial ideal since $d_1 \notin \{k_1, \ldots, k_j\}$ and $d_1 < k_j$.

- $x_{q_1}^{a_{q_1}-sb} \cdots x_{q_s}^{a_{q_s}-sb} w^s$, $\{q_1 < \cdots < q_s\} \subset \{k_1, \ldots, k_j\}$.

$$\frac{\operatorname{lcm}(x_{q_1}^{a_{q_1}-sb} \cdots x_{q_s}^{a_{q_s}-sb} w^s, x_{k_1}^{a_{k_1}-jb} \cdots x_{k_j}^{a_{k_j}-jb} w^j)}{x_{k_1}^{a_{k_1}-jb} \cdots x_{k_j}^{a_{k_j}-jb} w^j} = (x_{q_1} \cdots x_{q_s})^{(j-s)b}$$

- $x_{q_1}^{a_{q_1}-sb} \cdots x_{q_r}^{a_{q_r}-sb} x_{d_1}^{a_{d_1}-sb} \cdots x_{d_{s-r}}^{a_{d_{s-r}}-sb} w^s$, $\{q_1 < \cdots < q_r\} \subset \{k_1, \ldots, k_j\}$, $d_1 < \ldots < d_{s-r}$ with $k_j < d_1$.

$$\begin{aligned} & \frac{\operatorname{lcm}(x_{q_1}^{a_{q_1}-sb} \cdots x_{q_r}^{a_{q_r}-sb} x_{d_1}^{a_{d_1}-sb} \cdots x_{d_{s-r}}^{a_{d_{s-r}}-sb} w^s, x_{k_1}^{a_{k_1}-jb} \cdots x_{k_j}^{a_{k_j}-jb} w^j)}{x_{k_1}^{a_{k_1}-jb} \cdots x_{k_j}^{a_{k_j}-jb} w^j} \\ = \; & (x_{q_1} \cdots x_{q_r})^{(j-s)b} x_{d_1}^{a_{d_1}-sb} \cdots x_{d_{s-r}}^{a_{d_{s-r}}-sb} \end{aligned}$$

- $x_{d_1}^{a_{d_1}-sb} \cdots x_{d_s}^{a_{d_s}-sb} w^s$, $d_1 < \ldots < d_s$ with $k_j < d_1$.

$$\frac{\operatorname{lcm}(x_{d_1}^{a_{d_1}-sb} \cdots x_{d_s}^{a_{d_s}-sb} w^s, x_{k_1}^{a_{k_1}-jb} \cdots x_{k_j}^{a_{k_j}-jb} w^j)}{x_{k_1}^{a_{k_1}-jb} \cdots x_{k_j}^{a_{k_j}-jb} w^j} = x_{d_1}^{a_{d_1}-sb} \cdots x_{d_s}^{a_{d_s}-sb}.$$

In all three cases above the resulting monomial is among the generators listed in the right hand side monomial ideal.

Degree j :

- $x_{k_1}^{a_{k_1}-jb} \cdots \widehat{x_{k_c}^{a_{k_c}-jb}} \ldots x_{k_j}^{a_{k_j}-jb} x_d{}^{a_d-jb} w^j, d \notin \{k_1, \ldots, k_j\}$, $d < k_j$, $c \in \{1, \ldots, j\}$.

$$\frac{\operatorname{lcm}(x_{k_1}^{a_{k_1}-jb} \cdots \widehat{x_{k_c}^{a_{k_c}-jb}} \ldots x_{k_j}^{a_{k_j}-jb} x_d{}^{a_d-jb} w^j, x_{k_1}^{a_{k_1}-jb} \cdots x_{k_j}^{a_{k_j}-jb} w^j)}{x_{k_1}^{a_{k_1}-jb} \cdots x_{k_j}^{a_{k_j}-jb} w^j} = x_d^{a_d-jb}.$$

Again we conclude as before.

(b) DEGREE JUMP: $j' = j+1$

We now consider the case where the degree goes up, that is, one is dealing with

$$\mathrm{in}(H_{j+1}^{1,\ldots,j+1}) = x_1^{a_1-(j+1)b} \cdots x_{j+1}^{a_{j+1}-(j+1)b} w^{j+1}.$$

We go through similar calculations as before. In each case below the resulting monomial is among the generators listed in the right hand side monomial ideal.

Degree 1:

- $x_d^{a_d-b}w,\ d \notin \{1,\ldots,j+1\}$.

$$\frac{\mathrm{lcm}(x_d^{a_d-b}w, x_1^{a_1-(j+1)b} \cdots x_{j+1}^{a_{j+1}-(j+1)b} w^{j+1})}{x_1^{a_1-(j+1)b} \cdots x_{j+1}^{a_{j+1}-(j+1)b} w^{j+1}} = x_d^{a_d-b}$$

- $x_d^{a_d-b}w,\ d \in \{1,\ldots,j+1\}$.

$$\frac{\mathrm{lcm}(x_d^{a_d-b}w, x_1^{a_1-(j+1)b} \cdots x_{j+1}^{a_{j+1}-(j+1)b} w^{j+1})}{x_1^{a_1-(j+1)b} \cdots x_{j+1}^{a_{j+1}-(j+1)b} w^{j+1}} = x_d^{jb}$$

- $x_d^{a_d}y_v,\ d \notin \{1,\ldots,j+1\}$.

$$\frac{\mathrm{lcm}(x_d^{a_d}y_v, x_1^{a_1-(j+1)b} \cdots x_{j+1}^{a_{j+1}-(j+1)b} w^{j+1})}{x_1^{a_1-(j+1)b} \cdots x_{j+1}^{a_{j+1}-(j+1)b} w^{j+1}} = x_d^{a_d}y_v$$

  Note that $x_d^{a_d}y_v = (x_d^b y_v)x_d^{a_d-b}$.

- $x_d^{a_d}y_v,\ d \in \{1,\ldots,j+1\}$.

$$\frac{\mathrm{lcm}(x_d^{a_d}y_v, x_1^{a_1-(j+1)b} \cdots x_{j+1}^{a_{j+1}-(j+1)b} w^{j+1})}{x_1^{a_1-(j+1)b} \cdots x_{j+1}^{a_{j+1}-(j+1)b} w^{j+1}} = x_d^{(j+1)b}y_v$$

  Note that $x_d^{(j+1)b}y_v = (x_d^b y_v)x_d^{jb}$.

Degree s $(2 \le s \le j)$:

- $x_{q_1}^{a_{q_1}-sb} \cdots x_{q_s}^{a_{q_s}-sb} w^s$, with $\{q_1 < \cdots < q_s\} \subset \{1,\ldots,j+1\}$.

$$\frac{\mathrm{lcm}(x_{q_1}^{a_{q_1}-sb} \cdots x_{q_s}^{a_{q_s}-sb} w^s, x_1^{a_1-(j+1)b} \cdots x_{j+1}^{a_{j+1}-(j+1)b} w^{j+1})}{x_1^{a_1-(j+1)b} \cdots x_{j+1}^{a_{j+1}-(j+1)b} w^{j+1}} = (x_{q_1} \cdots x_{q_s})^{(j+1-s)b}$$

- $x_{q_1}^{a_{q_1}-sb} \cdots x_{q_r}^{a_{q_r}-sb} x_{d_1}^{a_{d_1}-sb} \cdots x_{d_{s-r}}^{a_{d_{s-r}}-sb} w^s,\ \{q_1 < \cdots < q_r\} \subset \{1, \ldots, j+1\},\ d_1 < \ldots < d_{s-r}$ with $j+1 < d_1$.

$$\begin{aligned} &\frac{\operatorname{lcm}(x_{q_1}^{a_{q_1}-sb} \cdots x_{q_r}^{a_{q_r}-sb} x_{d_1}^{a_{d_1}-sb} \cdots x_{d_{s-r}}^{a_{d_{s-r}}-sb} w^s, x_1^{a_1-(j+1)b} \cdots x_{j+1}^{a_{j+1}-(j+1)b} w^{j+1})}{x_1^{a_1-(j+1)b} \cdots x_{j+1}^{a_{j+1}-(j+1)b} w^{j+1}} \\ = \ & (x_{q_1} \cdots x_{q_r})^{(j+1-s)b} x_{d_1}^{a_{d_1}-sb} \cdots x_{d_{s-r}}^{a_{d_{s-r}}-sb} \end{aligned}$$

- $x_{d_1}^{a_{d_1}-sb} \cdots x_{d_s}^{a_{d_s}-sb} w^s,\ d_1 < \ldots < d_{s-r}$ with $j+1 < d_1$.

$$\frac{\operatorname{lcm}(x_{d_1}^{a_{d_1}-sb} \cdots x_{d_s}^{a_{d_s}-sb} w^s, x_1^{a_1-(j+1)b} \cdots x_{j+1}^{a_{j+1}-(j+1)b} w^{j+1})}{x_1^{a_1-(j+1)b} \cdots x_{j+1}^{a_{j+1}-(j+1)b} w^{j+1}} = x_{d_1}^{a_{d_1}-sb} \cdots x_{d_s}^{a_{d_s}-sb}$$

To conclude the present case of degree jump, we stress the limit situation where the degree jumps to the highest possible degree of a Sylvester form.

Setting $a_i > nb,\ \forall\ i = 1, \ldots, n.$

The expected outcome is $\operatorname{in}(\mathcal{H}(2, \ldots, n) : \operatorname{in}(H_n^{1,\ldots,n}) = (x_1, \ldots, x_n)^{(n-1)b} S$ and the calculation of the required least common multiples is included in the general calculation above, setting $j = n$.

This concludes the proof of the proposition. □

A.3 Proof of Proposition 3.4.7

We just have to prove the inclusion $\supset$ since the inclusion $\subset$ follows from it by applying Lemma 2.2.1 (ii).

Again, we deal with two cases, according to the established sets of generators for the right hand side of the stated equality in either (a) or (b) of Proposition 3.4.5.

Same degree

- $x_s^{a_s-b},\ s = k_j+1, \ldots, n.$

$$\begin{aligned} x_s^{a_s-b} H_j^{k_1,\ldots,k_j} = x_{k_1}^{a_{k_1}-jb} \cdots x_{k_j}^{a_{k_j}-jb} w^{j-1} L_s + x_{k_1}^{a_{k_1}-(j-1)b} P(k_1, \ldots, k_j, s)^b y_s H_{j-1}^{k_2,\ldots,k_j} \\ + P(k_1, \ldots, k_j, s)^{jb} x_s^{(j-1)b} y_{k_2} \cdots y_{k_j} K_{k_1,s}. \end{aligned}$$

- $x_{k_s}^{(j-1)b},\ s = 1, \ldots, j.$

$$x_{k_s}^{(j-1)b} H_j^{k_1,\ldots,k_j} = x_{k_1}^{a_{k_1}-jb} \cdots \widehat{x_{k_s}^{a_{k_s}-jb}} \cdots x_{k_j}^{a_{k_j}-jb} w^{j-1} L_{k_s} + P(k_1, \ldots, k_s, \ldots, k_j)^b y_{k_s} H_{j-1}^{k_1,\ldots,\widehat{k_s},\ldots,k_j}.$$

- $(x_{r_1} \cdots x_{r_s})^{(j-s)b},\ s \in \{1, \ldots, j-1\},\ \{r_1 < \cdots < r_s\} \subset \{k_1, \ldots, k_j\}.$

$$(x_{r_1} \cdots x_{r_s})^{(j-s)b} H_j^{k_1,\ldots,k_j} = x_{k_1}^{a_{k_1}-jb} \cdots \widehat{x_{r_1}^{a_{r_1}-jb}} \cdots \widehat{x_{r_s}^{a_{r_s}-jb}} \cdots x_{k_j}^{a_{k_j}-jb} w^{j-s} H_s^{r_1,\ldots,r_s} +$$

$$P(k_1,\ldots,r_1,\ldots,r_s,\ldots,k_j)^{sb}y_{r_1}\cdots y_{r_s}H_{j-s}^{k_1,\ldots,\widehat{r_1},\ldots,\widehat{r_s},\ldots,k_j}$$

- $x_r^{a_r-jb},\ r<k_1$.

$$x_r^{a_r-jb}H_j^{k_1,\ldots,k_j}=x_{k_1}^{a_{k_1}-jb}H_j^{r,k_2,\ldots,k_j}-P(r,k_1,\ldots,k_j)^{jb}y_{k_2}\cdots y_{k_j}K_{r,k_1}.$$

- $x_r^{a_r-jb}, k_i<r<k_{i+1},\ i=1,\ldots,j-1$.

$$\begin{aligned} x_r^{a_r-jb}H_j^{k_1,\ldots,k_j} &= x_{k_{i+1}}^{a_{k_{i+1}}-jb}H_j^{k_1,k_2,\ldots,k_i,r,\widehat{k_{i+1}},\ldots,k_j} \\ &- P(k_1,\ldots,k_i,r,k_{i+1},\ldots,k_j)^{jb}y_{k_1}\cdots y_{k_i}\widehat{y_{k_{i+1}}}y_{k_{i+2}}\cdots y_{k_j}K_{r,k_{i+1}}. \end{aligned}$$

- $(x_{q_1}\cdots x_{q_r})^{(j-s)b}x_{d_1}^{a_{d_1}-sb}\cdots x_{d_{s-r}}^{a_{d_{s-r}}-sb}$.

Since this case is a lot more involved than the previous ones, we chose to formulate it as a lemma.

Lemma A.1. *Fix an integer $2\le j\le n-1$ and an ordered subset $\{k_1,\ldots,k_j\}\subset\{1,\ldots,n\}$. Let there be given integers $s\in\{1,\ldots,j-1\}$, $r\in\{0,\ldots,s-1\}$ and ordered subsets $\{q_1,\ldots,q_r\}\subset\{k_1,\ldots,k_j\}$ and $\{d_1,\ldots,d_{s-r}\}\subset\{1,\ldots,n\}\setminus\{k_1,\ldots,k_j\}$, with $k_j<d_1$. Consider a 2-partition of $\{k_1,\ldots,k_j\}\setminus\{q_1,\ldots,q_r\}$ by ordered subsets $\{k_{m_1},\ldots,k_{m_{j-s}}\}$ and $\{n_1,\ldots,n_{s-r}\}$. Set*

$$Q:=P(k_1,\ldots,k_j,\,d_1,\ldots,d_{s-r})^{jb}\,(x_{q_1}\cdots x_{q_r})^{(j-s)b}y_{q_1}\cdots y_{q_r}y_{k_{m_1}}\cdots y_{k_{m_{j-s}}}.$$

Then

$$\begin{aligned} (x_{q_1}\cdots x_{q_r})^{(j-s)b}x_{d_1}^{a_{d_1}-sb}\cdots x_{d_{s-r}}^{a_{d_{s-r}}-sb}H_j^{k_1,\ldots,k_j} &= x_{k_{m_1}}^{a_{k_{m_1}}-jb}\cdots x_{k_{m_{j-s}}}^{a_{k_{m_{j-s}}}-jb}x_{n_1}^{a_{n_1}-jb}\cdots x_{n_{s-r}}^{a_{n_{s-r}}-jb} \\ &\cdot w^{j-s}H_s^{q_1,\ldots,q_r,d_1,\ldots,d_{s-r}} \\ &+P(k_1,\ldots,k_j,d_1,\ldots,d_{s-r})^{sb}\cdot x_{n_1}^{a_{n_1}-(j-s)b}\cdots x_{n_{s-r}}^{a_{n_{s-r}}-(j-s)b} \\ &\cdot y_{q_1}\cdots y_{q_r}y_{d_1}\cdots y_{d_{s-r}}H_{j-s}^{k_{m_1},\ldots,k_{m_{j-s}}} \\ &+\sum_{c=1}^{s-r}x_{n_{c+1}}^{a_{n_{c+1}}}\cdots x_{n_{s-r}}^{a_{n_{s-r}}}x_{d_1}^{a_{d_1}+(j-s)b}\cdots x_{d_{c-1}}^{a_{d_{c-1}}+(j-s)b}(x_{d_c}\cdots x_{d_{s-r}})^{(j-s)b}Q \\ &\cdot y_{d_{c+1}}\cdots y_{d_{s-r}}y_{n_1}\cdots y_{n_{c-1}}K_{n_c,d_c}, \end{aligned}$$

with the convention that $x_{d_0}=y_{n_0}=x_{n_{s-r+1}}=x_{d_{s-r+1}}=1$.

Proof. Although the above expression is verifiable by expanding the right hand side, the idea to get at it is by no means obvious. Since similar expressions will appear in the sequel, we will now explain its main core. Thus, first write

$$
\begin{aligned}
(x_{q_1}\cdots x_{q_r})^{(j-s)b} x_{d_1}^{a_{d_1}-sb}\cdots x_{d_{s-r}}^{a_{d_{s-r}}-sb} H_j^{k_1,\ldots,k_j} &= (x_{q_1}\cdots x_{q_r})^{(j-s)b} x_{d_1}^{a_{d_1}-sb}\cdots x_{d_{s-r}}^{a_{d_{s-r}}-sb} \\
&\cdot \left(x_{k_1}^{a_{k_1}-jb}\cdots x_{k_j}^{a_{k_j}-jb} w^j - P(k_1,\ldots,k_j)^{jb}\, y_{k_1}\cdots y_{k_j} \right) \\
&= x_{k_1}^{a_{k_1}-jb}\cdots \widehat{x_{q_1}^{a_{q_1}-jb}}\cdots \widehat{x_{q_r}^{a_{q_r}-jb}}\cdots x_{k_j}^{a_{k_j}-jb}\, x_{q_1}^{a_{q_1}-sb}\cdots x_{q_r}^{a_{q_r}-sb}\, x_{d_1}^{a_{d_1}-sb}\cdots x_{d_{s-r}}^{a_{d_{s-r}}-sb} w^j \quad \text{(A.1)} \\
&-(x_{q_1}\cdots x_{q_r})^{(j-s)b}\, x_{d_1}^{a_{d_1}+(j-s)b}\cdots x_{d_{s-r}}^{a_{d_{s-r}}+(j-s)b}\, P(k_1,\ldots,k_j,d_1,\ldots,d_{s-r})^{jb} y_{k_1}\cdots y_{k_j} \quad \text{(A.2)}
\end{aligned}
$$

Next, we rewrite each of the numbered expressions above. Using the partition explained above, one can write

$$
\begin{aligned}
& x_{k_{m_1}}^{a_{k_{m_1}}-jb}\cdots x_{k_{m_{j-s}}}^{a_{k_{m_{j-s}}}-jb}\, x_{n_1}^{a_{n_1}-jb}\cdots x_{n_{s-r}}^{a_{n_{s-r}}-jb}\, w^{j-s} H_s^{q_1,\ldots,q_r,\,d_1,\ldots,d_{s-r}} \\
&= x_{k_{m_1}}^{a_{k_{m_1}}-jb}\cdots x_{k_{m_{j-s}}}^{a_{k_{m_{j-s}}}-jb}\, x_{n_1}^{a_{n_1}-jb}\cdots x_{n_{s-r}}^{a_{n_{s-r}}-jb}\, w^{j-s} \Big(x_{q_1}^{a_{q_1}-sb}\cdots x_{q_r}^{a_{q_r}-sb}\, x_{d_1}^{a_{d_1}-sb}\cdots x_{d_{s-r}}^{a_{d_{s-r}}-sb} w^s \\
&-P(q_1,\ldots,q_r,d_1,\ldots,d_{s-r})^{sb} y_{q_1}\cdots y_{q_r}\, y_{d_1}\cdots y_{d_{s-r}} \Big) \\
&= x_{k_1}^{a_{k_1}-jb}\cdots \widehat{x_{q_1}^{a_{q_1}-jb}}\cdots \widehat{x_{q_r}^{a_{q_r}-jb}}\cdots x_{k_j}^{a_{k_j}-jb}\, x_{q_1}^{a_{q_1}-sb}\cdots x_{q_r}^{a_{q_r}-sb}\, x_{d_1}^{a_{d_1}-sb}\cdots x_{d_{s-r}}^{a_{d_{s-r}}-sb} w^j \quad \text{(A.3)} \\
&-P(k_1,\ldots,k_j,d_1,\ldots,d_{s-r})^{sb}\, x_{k_{m_1}}^{a_{k_{m_1}}-(j-s)b}\cdots x_{k_{m_{j-s}}}^{a_{k_{m_{j-s}}}-(j-s)b}\, x_{n_1}^{a_{n_1}-(j-s)b}\cdots x_{n_{s-r}}^{a_{n_{s-r}}-(j-s)b}\, w^{j-s} \quad \text{(A.4)} \\
&\cdot y_{q_1}\cdots y_{q_r}\, y_{d_1}\cdots y_{d_{s-r}}.
\end{aligned}
$$

The first numbered expression above is exactly the first numbered expression in the previous display. However, the second numbered expression above does not coincide with the second numbered expression in the previous display, so there is a little more to pursue in order to cancel this expression by bringing up an expression involving another Sylvester form:

$$
\begin{aligned}
& P(k_1,\ldots,k_j,d_1,\ldots,d_{s-r})^{sb}\, x_{n_1}^{a_{n_1}-(j-s)b}\cdots x_{n_{s-r}}^{a_{n_{s-r}}-(j-s)b}\, y_{q_1}\cdots y_{q_r}\, y_{d_1}\cdots y_{d_{s-r}} H_{j-s}^{k_{m_1}\ldots,k_{m_{j-s}}} \\
&= P(k_1,\ldots,k_j,d_1,\ldots,d_{s-r})^{sb}\, x_{n_1}^{a_{n_1}-(j-s)b}\cdots x_{n_{s-r}}^{a_{n_{s-r}}-(j-s)b}\, y_{q_1}\cdots y_{q_r}\, y_{d_1}\cdots y_{d_{s-r}} \\
&\cdot \left(x_{k_{m_1}}^{a_{k_{m_1}}-(j-s)b}\cdots x_{k_{m_{j-s}}}^{a_{k_{m_{j-s}}}-(j-s)b} w^{j-s} - P(k_{m_1},\ldots,k_{m_{j-s}})^{(j-s)b}\, y_{k_{m_1}}\cdots y_{k_{m_{j-s}}} \right) \\
&= P(k_1,\ldots,k_j,d_1,\ldots,d_{s-r})^{sb}\, x_{k_{m_1}}^{a_{k_{m_1}}-(j-s)b}\cdots x_{k_{m_{j-s}}}^{a_{k_{m_{j-s}}}-(j-s)b}\, x_{n_1}^{a_{n_1}-(j-s)b}\cdots x_{n_{s-r}}^{a_{n_{s-r}}-(j-s)b}\, w^{j-s} \\
&\cdot y_{q_1}\cdots y_{q_r}\, y_{d_1}\cdots y_{d_{s-r}} \\
&-P(k_1,\ldots,k_j,d_1,\ldots,d_{s-r})^{jb}\, (x_{q_1}\cdots x_{q_r})^{(j-s)b}\, x_{n_1}^{a_{n_1}}\cdots x_{n_{s-r}}^{a_{n_{s-r}}}\, (x_{d_1}\cdots x_{d_{s-r}})^{(j-s)b} \\
&\cdot y_{q_1}\cdots y_{q_r}\, y_{k_{m_1}}\cdots y_{k_{m_{j-s}}}\, y_{d_1}\cdots y_{d_{s-r}} \\
&= P(k_1,\ldots,k_j,d_1,\ldots,d_{s-r})^{sb}\, x_{k_{m_1}}^{a_{k_{m_1}}-(j-s)b}\cdots x_{k_{m_{j-s}}}^{a_{k_{m_{j-s}}}-(j-s)b}\, x_{n_1}^{a_{n_1}-(j-s)b}\cdots x_{n_{s-r}}^{a_{n_{s-r}}-(j-s)b}\, w^{j-s} \quad \text{(A.5)} \\
&\cdot y_{q_1}\cdots y_{q_r}\, y_{d_1}\cdots y_{d_{s-r}} \\
&-x_{n_1}^{a_{n_1}}\cdots x_{n_{s-r}}^{a_{n_{s-r}}}\, (x_{d_1}\cdots x_{d_{s-r}})^{(j-s)b}\, Q\, y_{d_1}\cdots y_{d_{s-r}}. \quad \text{(A.6)}
\end{aligned}
$$

Now, expression numbered (A.6) is same as expression numbered (A.5), but expression (A.7) still has way to go. In the subsequent steps we resort to Koszul generators as tags, namely, firstly,

$$x_{n_2}^{a_{n_2}}\cdots x_{n_{s-r}}^{a_{n_{s-r}}}\,\underbrace{x_{d_0}^{a_{d_0}+(j-s)b}}_{=1}(x_{d_1}\cdots x_{d_{s-r}})^{(j-s)b}\,Q\,y_{d_2}\cdots y_{d_{s-r}}\,\underbrace{y_{n_0}}_{=1}\,K_{n_1,d_1}$$

$$= x_{n_1}^{a_{n_1}}\cdots x_{n_{s-r}}^{a_{n_{s-r}}}\,(x_{d_1}\cdots x_{d_{s-r}})^{(j-s)b}\,Q\,y_{d_1}\cdots y_{d_{s-r}} \tag{A.7}$$

$$-x_{n_2}^{a_{n_2}}\cdots x_{n_{s-r}}^{a_{n_{s-r}}}\,x_{d_1}^{a_{d_1}+(j-s)b}(x_{d_2}\cdots x_{d_{s-r}})^{(j-s)b}\,Q\,y_{d_2}\cdots y_{d_{s-r}}\,y_{n_1}$$

The procedure establishes an inductive argument by which one monomial term is canceled against a next term in an expression involving a further down Koszul form. To obtain the final combination in terms of earlier Sylvester forms and Koszul forms, one resorts to a summation of expressions of the same type where the first summand is the expression in the last line of the last display and the last summand recovers (A.3). This explains the final form of the required expression as stated. □

Degree jump

Now $j' = j+1$.

- $x_s^{a_s-b}$, $s = j+2, \ldots, n$.

$$\begin{aligned} x_s^{a_s-b}H_{j+1}^{1,\ldots,j,j+1} &= x_1^{a_1-(j+1)b}\cdots x_j^{a_j-(j+1)b}x_{j+1}^{a_{j+1}-(j+1)b}w^jL_s \\ &+ x_1^{a_1-jb}(x_{j+2}\cdots x_{s-1}\,\widehat{x_s}\,x_{s+1}\cdots x_n)^b y_s H_j^{2,\ldots,j+1} \\ &+ (x_{j+2}\cdots x_{s-1}\,\widehat{x_s}\,x_{s+1}\cdots x_n)^{(j+1)b}x_s^{jb}y_2\cdots y_{j+1}K_{1,s}. \end{aligned}$$

- x_s^{jb}, $s = 1, \ldots, j+1$.

$$\begin{aligned} x_s^{jb}H_{j+1}^{1,\ldots,j,j+1} = x_1^{a_1-(j+1)b}\cdots x_{s-1}^{a_{s-1}-(j+1)b}\,\widehat{x_s^{a_s-(j+1)b}}\,x_{s+1}^{a_{s+1}-(j+1)b}\cdots x_{j+1}^{a_{j+1}-(j+1)b}w^jL_s \\ -(x_{j+2}\cdots x_n)^b y_s H_j^{1,\ldots,\widehat{s},\ldots,j+1}. \end{aligned}$$

- $(x_{r_1}\cdots x_{r_s})^{(j+1-s)b}$, where $s\in\{1,\ldots,j\}$ and $\{r_1,\ldots,r_s\}$ is an ordered subset of $\{1,\ldots,j,j+1\}$.

$$\begin{aligned} (x_{r_1}\cdots x_{r_s})^{(j+1-s)b}H_{j+1}^{1,\ldots,j,j+1} = x_1^{a_1-(j+1)b}\cdots \widehat{x_{r_1}^{a_{r_1}-(j+1)b}}\cdots\widehat{x_{rs}^{a_{r_s}-(j+1)b}}\cdots x_j^{a_j-(j+1)b}x_{j+1}^{a_{j+1}-(j+1)b} \\ .\,w^{j+1-s}H_s^{r_1,\ldots,r_s} + P(1,\ldots,j+1)^{sb}y_{r_1}\cdots y_{r_s}H_{j+1-s}^{1,\ldots,\widehat{r_1},\ldots,\widehat{r_s},\ldots,j,j+1}. \end{aligned}$$

- $(x_{q_1}\cdots x_{q_r})^{(j+1-s)b}x_{d_1}^{a_{d_1}-sb}\cdots x_{d_{s-r}}^{a_{d_{s-r}}-sb}$.

One applies the hypotheses and the conclusion of Lemma A.1 with the following changes in the numerology:

$\{k_1, \dots . k_j\} \rightsquigarrow \{1, \dots, j+1\}$
$\{k_{m_1}, \dots, k_{m_{j-s}}\} \rightsquigarrow \{m_1, \dots, m_{j+1-s}\}$
$j \rightsquigarrow j+1$, in all appearances of j in a subscript or exponent.

This concludes the proof of the proposition. $\square$

Appendix B

Binary general forms: degree > 12

B.1 Calculative evidence of the conjecture 4.3.1

Recall the notation introduced in Chapter 4, Section 4.3. For the sake of completeness, we repeat the calculation displayed in that section, after which we introduce the remaining calculations to wrap up the entire argument.

Odd case: $d = 2s + 1$. Keeping in mind that we are assuming $d \geq 13$ for the odd case, we induct on $s \geq 6$ to show that $M = y^{3s} \in I^2 : I$. The induction starts at $s = 6$ – this case is directly checked in the computer by taking f_1, f_2, f_3 to be random forms of degree 13.

Thus, assume that $s \geq 7$. We are to show that $Mf_u \in I^2$, for $u = 1, 2, 3$.

<u>$u = 3$</u>. By the inductive hypothesis, we have $y^{3(s-1)}h_3 = \sum \alpha_{i,j}h_ih_j \in I'^2$, for suitable $\alpha_{i,j} \in R_{s-2}$. Also note from (4.4) that $y^3h_j \in I$, for $j = 1, 2$ and $y^2h_3 \in I$. Therefore,

$$\begin{aligned}
y^{3s}f_3 &= y^{3s}y^2h_3 = y^5(y^{3(s-1)}h_3) = y^5(\sum \alpha_{i,j}h_ih_j) = \sum \alpha_{i,j}y^2h_iy^3h_j \\
&\equiv \alpha_{1,1}y^2h_1y^3h_1 + \alpha_{1,2}y^2h_1y^3h_2 + \alpha_{2,2}y^2h_2y^3h_2 \pmod{I^2} \\
&\equiv \alpha_{1,1}y^2h_1(-xf_2) + \alpha_{1,2}y^2(-xf_2)h_2 + \alpha_{2,2}y^2h_2(-xf_3) \pmod{I^2} \\
&\equiv (-\alpha_{1,1}xy^2h_1)f_2 + (-\alpha_{1,2}xy^2h_2)f_2 + (-\alpha_{2,2}xy^2h_2)f_3 \pmod{I^2} \\
&\equiv (-\alpha_{1,1}xy^2h_1 - \alpha_{1,2}xy^2h_2)f_2 + (-\alpha_{2,2}xy^2h_2)f_3 \pmod{I^2}
\end{aligned} \tag{B.1}$$

where we have used the explicit relations in (4.4).

Note that $-\alpha_{1,1}xy^2h_1 - \alpha_{1,2}xy^2h_2$ and $-\alpha_{2,2}xy^2h_2$ are both forms of degree $3s$. Since y^{3s} spans $(R/I)_{3s}$, these forms are scalar multiples of y^{3s} modulo I_{3s}; say, $-\alpha_{1,1}xy^2h_1 - \alpha_{1,2}xy^2h_2 = \eta y^{3s} + F$ and $-\alpha_{2,2}xy^2h_2 = \zeta y^{3s} + G$, for certain $\eta, \zeta \in k$ and $F, G \in I_{3s} \subset I$. Substituting for these in the last congruence above yields $y^{3s}f_3 \equiv \eta y^{3s}f_2 + \zeta y^{3s}f_3 \pmod{I^2}$. It follows that

$$-\eta y^{3s}f_2 + (1-\zeta)y^{3s}f_3 \in I^2 \tag{B.2}$$

<u>$u = 2$</u>. In a pretty similar way, by the inductive hypothesis and the relations (4.4), one obtains

$$
\begin{aligned}
y^{3s}f_2 &= y^{3s}(y^2h_2+xyh_3) = y^5(y^{3(s-1)}h_2)+xy^4(y^{3(s-1)}h_3)\\
&= y^5(\sum\alpha_{i,j}h_ih_j)+xy^4(\sum\beta_{i,j}h_ih_j) = \sum\alpha_{i,j}y^2h_iy^3h_j+\sum\beta_{i,j}xyh_iy^3h_j\\
&\equiv \alpha_{1,1}y^2h_1y^3h_1+\alpha_{1,2}y^2h_1y^3h_2+\alpha_{2,2}y^2h_2y^3h_2+\beta_{1,1}xyh_1y^3h_1\\
&+ \beta_{1,2}xyh_1y^3h_2+\beta_{1,3}xy^2h_1y^2h_3+\beta_{2,2}xyh_2y^3h_2+\beta_{2,3}xy^2h_2y^2h_3 \pmod{I^2}\\
&\equiv \alpha_{1,1}y^2h_1(-xf_2)+\alpha_{1,2}y^2(-xf_2)h_2+\alpha_{2,2}y^2h_2(-xf_3)\\
&+ \beta_{1,1}xyh_1(-xf_2)+\beta_{1,2}xy(-xf_2)h_2+\beta_{1,3}xy^2h_1f_3+\beta_{2,2}xyh_2(-xf_3)+\beta_{2,3}xy^2h_2f_3 \pmod{I^2}\\
&\equiv (-\alpha_{1,1}xy^2h_1)f_2+(-\alpha_{1,2}xy^2h_2)f_2+(-\alpha_{2,2}xy^2h_2)f_3+(-\beta_{1,1}x^2yh_1)f_2+(-\beta_{1,2}x^2yh_2)f_2\\
&+ (\beta_{1,3}xy^2h_1)f_3+(-\beta_{2,2}x^2yh_2)f_3+(\beta_{2,3}xy^2h_2)f_3 \pmod{I^2}\\
&\equiv (-\alpha_{1,1}xy^2h_1-\alpha_{1,2}xy^2h_2-\beta_{1,1}x^2yh_1-\beta_{1,2}x^2yh_2)f_2\\
&+ (-\alpha_{2,2}xy^2h_2+\beta_{1,3}xy^2h_1-\beta_{2,2}x^2yh_2+\beta_{2,3}xy^2h_2)f_3 \pmod{I^2}
\end{aligned}
$$

By the same token as in the case of $u=3$, since $-\alpha_{1,1}xy^2h_1-\alpha_{1,2}xy^2h_2-\beta_{1,1}x^2yh_1-\beta_{1,2}x^2yh_2$ and $-\alpha_{2,2}xy^2h_2+\beta_{1,3}xy^2h_1-\beta_{2,2}x^2yh_2+\beta_{2,3}xy^2h_2$ are forms of degree $3s$, we obtain

$$
(1-\zeta')y^{3s}f_2-\eta'y^{3s}f_3\in I^2, \tag{B.3}
$$

for certain $\eta',\zeta'\in k$.

Now consider (B.2) and (B.3) together as a system of linear equations over R/I^2 in the unknowns $y^{3s}f_2$ and $y^{3s}f_3$. By Cramer, one has $\Delta y^{3s}f_2\in I^2$ and $\Delta y^{3s}f_3\in I^2$, where $\Delta=\eta\eta'-(1-\zeta)(1-\zeta')$ is the determinant of the system. Provided $\Delta\neq 0$ we can conclude that $y^{3s}f_2\in I^2$ and $y^{3s}f_3\in I^2$. But, since the h_i's are general forms, hence have general coefficients, then η,η',ζ,ζ' are general as well from their definition. Therefore, cannot constitute the coordinates of a point lying on the quadric of equation $X_1X_2-(1-X_3)(1-X_4)$ in affine space $\mathbb{A}^4$.

$\underline{u=1}$. In a pretty similar way, by the inductive hypothesis and the relations (4.4), one obtains

$$\begin{aligned}
y^{3s}f_1 &= y^{3s}(y^2h_1+xyh_2+x^2h_3) = y^5(y^{3(s-1)}h_1)+xy^4(y^{3(s-1)}h_2)+x^2y^3(y^{3(s-1)}h_3)\\
&= y^5(\sum\alpha_{i,j}h_ih_j)+xy^4(\sum\beta_{i,j}h_ih_j)+x^2y^3(\sum\delta_{i,j}h_ih_j)\\
&= \sum\alpha_{i,j}y^2h_iy^3h_j+\sum\beta_{i,j}xyh_iy^3h_j+\sum\delta_{i,j}x^2h_iy^3h_j\\
&\equiv \alpha_{1,1}y^2h_1y^3h_1+\alpha_{1,2}y^2h_1y^3h_2+\alpha_{2,2}y^2h_2y^3h_2+\beta_{1,1}xyh_1y^3h_1\\
&+ \beta_{1,2}xyh_1y^3h_2+\beta_{1,3}xy^2h_1y^2h_3+\beta_{2,2}xyh_2y^3h_2+\beta_{2,3}xy^2h_2y^2h_3\\
&+ \delta_{1,1}x^2h_1y^3h_1+\delta_{1,2}x^2y^3h_1h_2+\delta_{1,3}x^2y^3h_1h_3+\delta_{2,2}x^2h_2y^3h_2\\
&+ \delta_{2,3}x^2yh_2y^2h_3+\delta_{3,3}x^2yh_3y^2h_3 \pmod{I^2}\\
&\equiv \alpha_{1,1}y^2h_1(-xf_2)+\alpha_{1,2}y^2(-xf_2)h_2+\alpha_{2,2}y^2h_2(-xf_3)\\
&+ \beta_{1,1}xyh_1(-xf_2)+\beta_{1,2}xy(-xf_2)h_2+\beta_{1,3}xy^2h_1f_3+\beta_{2,2}xyh_2(-xf_3)+\beta_{2,3}xy^2h_2f_3\\
&+ \delta_{1,1}x^2h_1(-xf_2)+\delta_{1,2}x^2(-xf_2)h_2+\delta_{1,3}x^2(-xf_2)h_3+\delta_{2,2}x^2h_2(-xf_3)\\
&+ \delta_{2,3}x^2yh_2f_3+\delta_{3,3}x^2yh_3f_3 \pmod{I^2}\\
&\equiv (-\alpha_{1,1}xy^2h_1)f_2+(-\alpha_{1,2}xy^2h_2)f_2+(-\alpha_{2,2}xy^2h_2)f_3+(-\beta_{1,1}x^2yh_1)f_2+(-\beta_{1,2}x^2yh_2)f_2\\
&+ (\beta_{1,3}xy^2h_1)f_3+(-\beta_{2,2}x^2yh_2)f_3+(\beta_{2,3}xy^2h_2)f_3\\
&+ (-\delta_{1,1}x^3h_1)f_2+(-\delta_{1,2}x^3h_2)f_2+(-\delta_{1,3}x^3h_3)f_2+(-\delta_{2,2}x^3h_2)f_3\\
&+ (\delta_{2,3}x^2yh_2)f_3+(\delta_{3,3}x^2yh_3)f_3 \pmod{I^2}\\
&\equiv (-\alpha_{1,1}xy^2h_1-\alpha_{1,2}xy^2h_2-\beta_{1,1}x^2yh_1-\beta_{1,2}x^2yh_2-\delta_{1,1}x^3h_1-\delta_{1,2}x^3h_2-\delta_{1,3}x^3h_3)f_2\\
&+ (-\alpha_{2,2}xy^2h_2+\beta_{1,3}xy^2h_1-\beta_{2,2}x^2yh_2+\beta_{2,3}xy^2h_2-\delta_{2,2}x^3h_2+\delta_{2,3}x^2yh_2+\delta_{3,3}x^2yh_3)f_3\\
&\quad \pmod{I^2}
\end{aligned}$$

By the same token as in the case of $u=2$, since $-\alpha_{1,1}xy^2h_1-\alpha_{1,2}xy^2h_2-\beta_{1,1}x^2yh_1-\beta_{1,2}x^2yh_2-\delta_{1,1}x^3h_1-\delta_{1,2}x^3h_2-\delta_{1,3}x^3h_3$ and $-\alpha_{2,2}xy^2h_2+\beta_{1,3}xy^2h_1-\beta_{2,2}x^2yh_2+\beta_{2,3}xy^2h_2-\delta_{2,2}x^3h_2+\delta_{2,3}x^2yh_2+\delta_{3,3}x^2yh_3$ are forms of degree $3s$, we obtain

$$y^{3s}f_1-\zeta''y^{3s}f_2-\eta''y^{3s}f_3\in I^2, \tag{B.4}$$

for certain $\eta'',\zeta''\in k$.

Since $y^{3s}\in I^2:(f_2,f_3)$, follows that $y^{3s}\in I^2:I$.

Even case: $d=2s$. Since d is even, we are assuming $d\geq 12$, hence induct on $s\geq 6$ to show that $N=y^{3s-2}\in I^2:I$. Since we have already disposed of the case $s=6$, we may assume that $s\geq 7$.

We are then to show that for every $u=1,2,3$, $Nf_u\in I^2$.

<u>$u=3$</u>. One has

$$y^{3s-2}f_3=y^{3s-2}y^2h_3=y^5(y^{3(s-1)-2}h_3)=y^5(\sum\alpha_{i,j}h_ih_j)=\sum\alpha_{i,j}y^2h_iy^3h_j,$$

with $\alpha_{i,j}\in R_{s-3}$, where the equality preceding the last one stems from the inductive hypothesis.

Since $y^3h_j \in (f_1, f_2, f_3), \forall\ j = 1, 2$ and $y^2h_3 \in (f_1, f_2, f_3)$ then

$$y^{3s-2}f_3 \equiv \alpha_{1,1}y^2h_1y^3h_1 + \alpha_{1,2}y^2h_1y^3h_2 + \alpha_{2,2}y^2h_2y^3h_2 \pmod{I^2}.$$

Using the relations (4.4) obtains

$$\begin{aligned} y^{3s-2}f_3 &\equiv \alpha_{1,1}y^2h_1(-xf_2) + \alpha_{1,2}y^2(-xf_2)h_2 + \alpha_{2,2}y^2h_2(-xf_3) \pmod{I^2} \\ &\equiv (-\alpha_{1,1}xy^2h_1)f_2 + (-\alpha_{1,2}xy^2h_2)f_2 + (-\alpha_{2,2}xy^2h_2)f_3 \pmod{I^2} \\ &\equiv (-\alpha_{1,1}xy^2h_1 - \alpha_{1,2}xy^2h_2)f_2 + (-\alpha_{2,2}xy^2h_2)f_3 \pmod{I^2}. \end{aligned} \tag{B.5}$$

Note that $v = -\alpha_{1,1}xy^2h_1 - \alpha_{1,2}xy^2h_2$ and $w = -\alpha_{2,2}xy^2h_2$ are forms of degree $3s-2$. Since y^{3s-2}, xy^{3s-3} span $(R/I)_{3s-2}$, v and w are k-linear combinations of these two monomials modulo I_{3s-2}; say, $v = \gamma_1y^{3s-2} + \gamma_2xy^{3s-3} + F$ and $w = \gamma_3y^{3s-2} + \gamma_4xy^{3s-3} + G$ with $\gamma_1, \gamma_2, \gamma_3, \gamma_4 \in k$ and $F, G \in I_{3s-2} \subset I$. Therefore,

$$y^{3s-2}f_3 \equiv (\gamma_1y^{3s-2} + \gamma_2xy^{3s-3})f_2 + (\gamma_3y^{3s-2} + \gamma_4xy^{3s-3})f_3 \pmod{I^2}. \tag{B.6}$$

$\underline{u = 2}$. By a similar token,

$$\begin{aligned} y^{3s-2}f_2 &= y^{3s-2}(y^2h_2 + xyh_3) = y^5(y^{3(s-1)-2}h_2) + xy^4(y^{3(s-1)-2}h_3) \\ &= y^5(\sum \alpha_{i,j}h_ih_j) + xy^4(\sum \beta_{i,j}h_ih_j) = \sum \alpha_{i,j}y^2h_iy^3h_j + \sum \beta_{i,j}xyh_iy^3h_j, \end{aligned}$$

with $\alpha_{i,j}, \beta_{i,j} \in R_{s-3}$, where the equality preceding the last one stems from the inductive hypothesis. Since $y^3h_j \in (f_1, f_2, f_3), j = 1, 2$ and $y^2h_3 \in (f_1, f_2, f_3)$, proceeding in the same scheme as in the previous case yields

$$\begin{aligned} y^{3s-2}f_2 &\equiv \alpha_{1,1}y^2h_1y^3h_1 + \alpha_{1,2}y^2h_1y^3h_2 + \alpha_{2,2}y^2h_2y^3h_2 + \beta_{1,1}xyh_1y^3h_1 \\ &+ \beta_{1,2}xyh_1y^3h_2 + \beta_{1,3}xy^2h_1y^2h_3 + \beta_{2,2}xyh_2y^3h_2 + \beta_{2,3}xy^2h_2y^2h_3 \pmod{I^2} \\ &\equiv \alpha_{1,1}y^2h_1(-xf_2) + \alpha_{1,2}y^2(-xf_2)h_2 + \alpha_{2,2}y^2h_2(-xf_3) + \beta_{1,1}xyh_1(-xf_2) \\ &+ \beta_{1,2}xy(-xf_2)h_2 + \beta_{1,3}xy^2h_1f_3 + \beta_{2,2}xyh_2(-xf_3) + \beta_{2,3}xy^2h_2f_3 \pmod{I^2} \\ &\equiv (-\alpha_{1,1}xy^2h_1)f_2 + (-\alpha_{1,2}xy^2h_2)f_2 + (-\alpha_{2,2}xy^2h_2)f_3 + (-\beta_{1,1}x^2yh_1)f_2 \\ &+ (-\beta_{1,2}x^2yh_2)f_2 + \beta_{1,3}xy^2h_1f_3 + (-\beta_{2,2}x^2yh_2)f_3 + \beta_{2,3}xy^2h_2f_3 \pmod{I^2} \\ &\equiv (-\alpha_{1,1}xy^2h_1 - \alpha_{1,2}xy^2h_2 - \beta_{1,1}x^2yh_1 - \beta_{1,2}x^2yh_2)f_2 \\ &+ (-\alpha_{2,2}xy^2h_2 + \beta_{1,3}xy^2h_1 - \beta_{2,2}x^2yh_2 + \beta_{2,3}xy^2h_2)f_3 \pmod{I^2} \end{aligned}$$

As in the previous case, the forms $v = -\alpha_{1,1}xy^2h_1 - \alpha_{1,2}xy^2h_2 - \beta_{1,1}x^2yh_1 - \beta_{1,2}x^2yh_2$ and $w = -\alpha_{2,2}xy^2h_2 + \beta_{1,3}xy^2h_1 - \beta_{2,2}x^2yh_2 + \beta_{2,3}xy^2h_2$ can be written as $v \equiv \gamma'_1y^{3s-2} + \gamma'_2xy^{3s-3} \pmod{I}$ and $w \equiv \gamma'_3y^{3s-2} + \gamma'_4xy^{3s-3} \pmod{I}$ with $\gamma'_1, \gamma'_2, \gamma'_3, \gamma'_4 \in k$. Thus, one has

$$y^{3s-2}f_2 \equiv (\gamma_1' y^{3s-2} + \gamma_2' xy^{3s-3})f_2 + (\gamma_3' y^{3s-2} + \gamma_4' xy^{3s-3})f_3 \pmod{I^2}$$

Finally, we check that (B.6) and the last relation together imply $y^{3s-2} \in I^2 : (f_2, f_3)$.

The two make up for the linear system

$$\begin{cases} -\gamma_1 y^{3s-2}f_2 - \gamma_2 xy^{3s-3}f_2 + (1-\gamma_3)y^{3s-2}f_3 - \gamma_4 xy^{3s-3}f_3 \equiv 0 \\ (1-\gamma_1')y^{3s-2}f_2 - \gamma_2' xy^{3s-3}f_2 - \gamma_3' y^{3s-2}f_3 - \gamma_4' xy^{3s-3}f_3 \equiv 0 \end{cases} \tag{B.-2}$$

mod I^2 in the uknowns $y^{3s-2}f_2, xy^{3s-3}f_2, y^{3s-2}f_3, y^{3s-2}f_2$.

As in the case where $d = 2s+1$, here too it suffices to know that some 2×2 minor of the scalar matrix

$$\begin{bmatrix} -\gamma_1 & -\gamma_2 & (1-\gamma_3) & -\gamma_4 \\ (1-\gamma_1') & -\gamma_2' & -\gamma_3' & -\gamma_4' \end{bmatrix}$$

does not vanish. But again, since the coefficients of the h_i's are general, then so are the coefficients of the monomials y^{3s-2} and xy^{3s-3} in the expressions of v and w above. Therefore, the entries of the above matrix cannot be the coordinates of a point lying on an intersection of quadrics in $\mathbb{A}^8$.

$\underline{u=1}$. In a pretty similar way, by the inductive hypothesis and the relations (4.4), one obtains

$$
\begin{aligned}
y^{3s-2}f_1 &= y^{3s-2}(y^2h_1+xyh_2+x^2h_3) = y^5(y^{3(s-1)-2}h_1)+xy^4(y^{3(s-1)-2}h_2)+x^2y^3(y^{3(s-1)-2}h_3)\\
&= y^5(\sum \alpha_{i,j}h_ih_j)+xy^4(\sum \beta_{i,j}h_ih_j)+x^2y^3(\sum \delta_{i,j}h_ih_j)\\
&= \sum \alpha_{i,j}y^2h_iy^3h_j+\sum \beta_{i,j}xyh_iy^3h_j+\sum \delta_{i,j}x^2h_iy^3h_j\\
&\equiv \alpha_{1,1}y^2h_1y^3h_1+\alpha_{1,2}y^2h_1y^3h_2+\alpha_{2,2}y^2h_2y^3h_2+\beta_{1,1}xyh_1y^3h_1\\
&+ \beta_{1,2}xyh_1y^3h_2+\beta_{1,3}xy^2h_1y^2h_3+\beta_{2,2}xyh_2y^3h_2+\beta_{2,3}xy^2h_2y^2h_3\\
&+ \delta_{1,1}x^2h_1y^3h_1+\delta_{1,2}x^2y^3h_1h_2+\delta_{1,3}x^2y^3h_1h_3+\delta_{2,2}x^2h_2y^3h_2\\
&+ \delta_{2,3}x^2yh_2y^2h_3+\delta_{3,3}x^2yh_3y^2h_3 \pmod{I^2}\\
&\equiv \alpha_{1,1}y^2h_1(-xf_2)+\alpha_{1,2}y^2(-xf_2)h_2+\alpha_{2,2}y^2h_2(-xf_3)\\
&+ \beta_{1,1}xyh_1(-xf_2)+\beta_{1,2}xy(-xf_2)h_2+\beta_{1,3}xy^2h_1f_3+\beta_{2,2}xyh_2(-xf_3)+\beta_{2,3}xy^2h_2f_3\\
&+ \delta_{1,1}x^2h_1(-xf_2)+\delta_{1,2}x^2(-xf_2)h_2+\delta_{1,3}x^2(-xf_2)h_3+\delta_{2,2}x^2h_2(-xf_3)\\
&+ \delta_{2,3}x^2yh_2f_3+\delta_{3,3}x^2yh_3f_3 \pmod{I^2}\\
&\equiv (-\alpha_{1,1}xy^2h_1)f_2+(-\alpha_{1,2}xy^2h_2)f_2+(-\alpha_{2,2}xy^2h_2)f_3+(-\beta_{1,1}x^2yh_1)f_2+(-\beta_{1,2}x^2yh_2)f_2\\
&+ (\beta_{1,3}xy^2h_1)f_3+(-\beta_{2,2}x^2yh_2)f_3+(\beta_{2,3}xy^2h_2)f_3\\
&+ (-\delta_{1,1}x^3h_1)f_2+(-\delta_{1,2}x^3h_2)f_2+(-\delta_{1,3}x^3h_3)f_2+(-\delta_{2,2}x^3h_2)f_3\\
&+ (\delta_{2,3}x^2yh_2)f_3+(\delta_{3,3}x^2yh_3)f_3 \pmod{I^2}\\
&\equiv (-\alpha_{1,1}xy^2h_1-\alpha_{1,2}xy^2h_2-\beta_{1,1}x^2yh_1-\beta_{1,2}x^2yh_2-\delta_{1,1}x^3h_1-\delta_{1,2}x^3h_2-\delta_{1,3}x^3h_3)f_2\\
&+ (-\alpha_{2,2}xy^2h_2+\beta_{1,3}xy^2h_1-\beta_{2,2}x^2yh_2+\beta_{2,3}xy^2h_2-\delta_{2,2}x^3h_2+\delta_{2,3}x^2yh_2+\delta_{3,3}x^2yh_3)f_3\\
&\quad \pmod{I^2}
\end{aligned}
$$

By the same token as in the case of $u = 2$, since $-\alpha_{1,1}xy^2h_1 - \alpha_{1,2}xy^2h_2 - \beta_{1,1}x^2yh_1 - \beta_{1,2}x^2yh_2 - \delta_{1,1}x^3h_1 - \delta_{1,2}x^3h_2 - \delta_{1,3}x^3h_3$ and $-\alpha_{2,2}xy^2h_2 + \beta_{1,3}xy^2h_1 - \beta_{2,2}x^2yh_2 + \beta_{2,3}xy^2h_2 - \delta_{2,2}x^3h_2 + \delta_{2,3}x^2yh_2 + \delta_{3,3}x^2yh_3$ are forms of degree $3s$, we obtain

$$y^{3s-2}f_1 \equiv (\gamma_1''y^{3s-2}+\gamma_2''xy^{3s-3})f_2+(\gamma_3''y^{3s-2}+\gamma_4''xy^{3s-3})f_3 \pmod{I^2}$$

with $\gamma_1'', \gamma_2'', \gamma_3'', \gamma_4'' \in k$.

Since $y^{3s-2} \in I^2 : (f_2, f_3)$, follows that $y^{3s-2} \in I^2 : I$.

□

Appendix C

Gröbner bases

The proofs and details of the section are contained in ([18]).

The theory of Gröbner bases was used in several branches of mathematics at the end of 1980. This theory is used not only for the purpose of calculations, but also to deduce theoretical results in commutative algebra and combinatorics.

Let $R = k[x_1, \ldots, x_n]$ the polynomial ring in n variables over a field k and $\mathrm{Mon}(R)$ the set of monomials of R. Note that $\mathrm{Mon}(R)$ is a natural k-basis of R, i.e., any polynomial $f \in R$ is a unique k-linear combination of monomials. Write

$$f = \sum_{u \in \mathrm{Mon}(R)} a_u u, \ a_u \in k.$$

Then we call the set

$$\mathrm{supp}(f) = \{u \in \mathrm{Mon}(R) : \ a_u \neq 0\}$$

the support of f.

Let $\mathcal{M}$ be a nonempty subset of $\mathrm{Mon}(R)$. Let $\mathcal{M}^{\min}$ denote the set of minimal elements of $\mathcal{M}$ with respect to divisibility.

Theorem C.1. (Dickson's lemma). *Let $\mathcal{M}$ be a nonempty subset of $\mathrm{Mon}(R)$. Then $\mathcal{M}^{\min}$ is a finite set.*

A direct consequence of Dickson's lemma is that any monomial ideal is finitely generated. Furthermore, each monomial ideal has a unique minimal monomial set of generators. More precisely, let G denote the set of monomials in ideal monomial I which are minimal with respect to divisibility, then G is the unique minimal set of monomial generators. Denote the unique minimal set of monomial generators of the monomial ideal I by $G(I)$.

Given two monomials $f, g \in R$, denote by $\mathrm{lcm}(f, g)$ the least common multiple of f and g and by $\mathrm{ged}(f, g)$ the greatest common divisor of f and g.

The next two results will be important to present the combinatorial structure of the ideal Rees in the chapter 2.

Proposition C.2. *Let I and J be monomials ideals. Then $I \cap J$ is a monomial ideal and $\{\text{lcm}(f,g) : f \in G(I), g \in G(J)\}$ is a set of generators of $I \cap J$.*

Proposition C.3. *Let I and J be monomials ideals. Then $I : J$ is a monomial ideal and*

$$I : J = \bigcap_{g \in G(J)} I : (g).$$

Moreover, $\{\frac{f}{\text{gcd}(f,g)} : f \in G(I)\}$ is a set of generators of $I : (g)$.

Definition C.4. A *monomial order* on R is a total order $<$ on $\text{Mon}(R)$ such that

(a) $1 < u$ for all $1 \neq u \in \text{Mon}(R)$;

(b) if $u, v \in \text{Mon}(R)$ and $u < v$, then $uw < vw$ for all $w \in \text{Mon}(R)$.

Example C.5. Let $\mathbf{a} = (a_1, \dots, a_n)$ and $\mathbf{b} = (b_1, \dots, b_n)$ be vectors belonging to $\mathbb{Z}_+^n$, let $\mathbf{x}^\mathbf{a} = x_1^{a_1} \cdots x_n^{a_n}$ and $\mathbf{x}^\mathbf{b} = x_1^{b_1} \cdots x_n^{b_n}$ be monomials belonging to $\text{Mon}(R)$. We define the total order $<_{\text{lex}}$ on $\text{Mon}(R)$ by setting $\mathbf{x}^\mathbf{a} <_{\text{lex}} \mathbf{x}^\mathbf{b}$ if either $\sum_{i=1}^n a_i < \sum_{i=1}^n b_i$ or $\sum_{i=1}^n a_i = \sum_{i=1}^n b_i$ and the leftmost nonzero component of the vector $\mathbf{a} - \mathbf{b}$ is negative. It follows that $<_{\text{lex}}$ is a monomial order on R, which is called the *lexicographic order* on R induced by the ordering $x_1 > \cdots > x_n$.

Definition C.6. Let $<$ a monomial order in R and $f = \sum_{u \in \text{Mon}(R)} a_u u$ be a nonzero polynomial of R with each $a_u \in k$. The *initial monomial* of f with respect to $<$ is the biggest monomial with respect to $<$ among the monomials belonging to $\text{supp}(f)$. We write $\text{in}_<(f)$ for the initial monomial of f with respect to $<$. The *leading coefficient* of f is the coefficient of $\text{in}_<(f)$ in f.

Follows directly from the definition that if $f, g \in R \setminus \{0\}$ then $\text{in}_<(fg) = \text{in}_<(f)\text{in}_<(g)$ and $\text{in}_<(f+g) \leq \max\{\text{in}_<(f), \text{in}_<(g)\}$ with equality if $\text{in}_<(f) \neq \text{in}_<(g)$.

Let $I \subset R$ be a *monomial ideal*, i.e., it is generated by monomials. It follows that I is generated by a subset $\mathcal{N} \subset \text{Mon}(R)$ if only if $(I \cap \text{Mon}(R))^{\min} \subset \mathcal{N}$. Hence $(I \cap \text{Mon}(R))^{\min}$ is a unique minimal system of monomial generators of I. Dickson's Lemma guarantees that $(I \cap \text{Mon}(R))^{\min}$ is a finite set. Thus in particular every monomial ideal I of R is finitely generated.

Let I be a nonzero ideal of R. The *initial ideal* of I with respect to $<$ is the monomial ideal of R which is generated by $\{\text{in}_<(f) : f \in I \setminus \{0\}\}$. We write $\text{in}_<(I)$ for the initial ideal of I. Thus

$$\text{in}_<(I) = (\{\text{in}_<(f) : f \in I \setminus \{0\}\}).$$

Since $(\text{in}_<(I) \cap \text{Mon}(R))^{\min}$ is the minimal system of monomial generators of $\text{in}_<(I)$ and since $(\text{in}_<(I) \cap \text{Mon}(R)) = \{\text{in}_<(f) : f \in I \setminus \{0\}\}$, there exist a finite number of nonzero polynomials $g_1, \dots, g_s$ belonging to I such that $\text{in}_<(I)$ is generated by their initial monomials $\text{in}_<(g_1), \dots, \text{in}_<(g_s)$.

Definition C.7. Let I be a nonzero ideal of R. A finite set of nonzero polynomials $\{g_1, \ldots, g_s\}$ with each $g_i \in I$ is said to be a *Gröbner basis* of I with respect to $<$ if the initial ideal $\text{in}_<(I)$ of I is generated by the monomials $\text{in}_<(g_1), \ldots, \text{in}_<(g_s)$.

A Gröbner basis of I with respect to $<$ exists. If $\mathcal{G}$ is a Gröbner basis of I with respect to $<$, then every finite set $\mathcal{G}'$ with $\mathcal{G}' \subset \mathcal{G} \subset I$ is also a Gröbner basis of I with respect to $<$. If $\mathcal{G} = \{g_1, \ldots, g_s\}$ is a Gröbner basis of I with respect to $<$ and if $f_1, \ldots, f_s$ are nonzero polynomials belonging to I with each $\text{in}_<(f_i) = \text{in}_<(g_i)$, then $\{f_1, \ldots, f_s\}$ is a Gröbner basis of I with respect to $<$.

Example C.8. Let $R = k[x_1, \ldots, x_7]$ and $<_{\text{lex}}$ the lexicographic order on R induced by $x_1 > \cdots > x_7$. Let $f = x_1x_4 - x_2x_3$ and $g = x_4x_7 - x_5x_6$ with their initial monomials $\text{in}_{<_{\text{lex}}}(f) = x_1x_4$ and $\text{in}_{<_{\text{lex}}}(g) = x_4x_7$. Let $I = (f, g)$. Then $\{f, g\}$ is not a Gröbner basis of I with respect to $<_{\text{lex}}$. In fact, the polynomial $h = x_7f - x_1g = x_1x_5x_6 - x_2x_3x_7$ belongs to I, but its initial monomial $\text{in}_{<_{\text{lex}}}(h) = x_1x_5x_6$ can be divided by neither $\text{in}_{<_{\text{lex}}}(f)$ nor $\text{in}_{<_{\text{lex}}}(g)$. Hence $\text{in}_{<_{\text{lex}}}(h) \notin (\text{in}_{<_{\text{lex}}}(f), \text{in}_{<_{\text{lex}}}(g))$. Thus $\text{in}_{<_{\text{lex}}}(I) \neq (\text{in}_{<_{\text{lex}}}(f), \text{in}_{<_{\text{lex}}}(g))$.

When introduced a monomial order and the previous notions, some classical results gain "new" version.

Theorem C.9. *Let I be a nonzero ideal of $R = k[x_1, \ldots, x_n]$ and $\mathcal{G} = \{g_1, \ldots, g_s\}$ a Gröbner basis of I with respect to a monomial order $<$ on R. Then $I = (g_1, \ldots, g_s)$.*

Corollary C.10. (Hilbert's basis theorem). *Every ideal of the polynomial ring is finitely generated.*

Theorem C.11. (The division algorithm). *Let $R = k[x_1, \ldots, x_n]$ and fix a monomial order $<$ on R. Let $g_1, \ldots, g_s$ be nonzero polynomials of R. Then, given a polynomial $f \in R \setminus \{0\}$, there exist polynomials $f_1, \ldots, f_s$ and f' of R with*

$$f = f_1g_1 + \cdots + f_sg_s + f', \tag{C.0}$$

such that the following conditions are satisfied:

(a) *if $f' \neq 0$ and if $u \in \text{supp}(f')$, then none of the initial monomials $\text{in}_<(g_1), \ldots, \text{in}_<(g_s)$ divides u;*

(b) *if $f_i \neq 0$, then*

$$\text{in}_<(f) \geq \text{in}_<(f_ig_i).$$

The right-hand side of equation (C.11) is said to be a *standard expression* for f with respect to $g_1, \ldots, g_s$, and the polynomial f' is said to be a *remainder* of f with respect to $g_1, \ldots, g_s$.

Given a set determine if it is a Gröbner basis is then equal to show the equality of two ideals, something not always feasible. Next, we will display an algorithm to determine when a set is a Gröbner basis.

Given nonzero polynomials f and g of R. Let c_f denote the coefficient on $\mathrm{in}_<(f)$ in f and c_g the coefficient of $\mathrm{in}_<(g)$ in g. The polynomial

$$S(f,g) = \frac{\mathrm{lcm}(\mathrm{in}_<(f), \mathrm{in}_<(g))}{c_f \mathrm{in}_<(f)} f - \frac{\mathrm{lcm}(\mathrm{in}_<(f), \mathrm{in}_<(g))}{c_g \mathrm{in}_<(g)} g$$

is called the *S-polynomial* of f and g.

We say that f *reduces to* 0 with respect to $g_1, \ldots, g_s$ if, in the division algorithm, there is a standard expression (C.11) of f with respect to $g_1, \ldots, g_s$ with $f' = 0$.

Proposition C.12. *Let f and g be nonzero polynomials and suppose that $\mathrm{in}_<(f)$ and $\mathrm{in}_<(g)$ are relatively prime. Then $S(f,g)$ reduces to 0 with respect to f, g.*

Proof. To simplify notation we will assume that each of the coefficients of $\mathrm{in}_<(f)$ in f and $\mathrm{in}_<(g)$ in g is equal to 1. Let $f = \mathrm{in}_<(f) + f_1$ and $g = \mathrm{in}_<(g) + g_1$. Since $\mathrm{in}_<(f)$ and $\mathrm{in}_<(g)$ are relatively prime, it follows that

$$S(f,g) = \mathrm{in}_<(g)f - \mathrm{in}_<(f)g = (g - g_1)f - (f - f_1)g = f_1 g - g_1 f.$$

We claim $\mathrm{in}_<(f_1)\mathrm{in}_<(g) = \mathrm{in}_<(f_1 g) \neq \mathrm{in}_<(g_1 f) = \mathrm{in}_<(g_1)\mathrm{in}_<(f)$. In fact, if $\mathrm{in}_<(f_1)\mathrm{in}_<(g) = \mathrm{in}_<(g_1)\mathrm{in}_<(f)$, then, since $\mathrm{in}_<(f)$ and $\mathrm{in}_<(g)$ are relatively prime, it follows that $\mathrm{in}_<(f)$ must divide $\mathrm{in}_<(f_1)$. However, since $\mathrm{in}_<(f_1) < \mathrm{in}_<(f)$, this is impossible. Let, say, $\mathrm{in}_<(f_1)\mathrm{in}_<(g) < \mathrm{in}_<(g_1)\mathrm{in}_<(f)$. Then $\mathrm{in}_<(S(f,g)) = \mathrm{in}_<(g_1 f)$ and $S(f,g) = f_1 g - g_1 f$ turns out to be a standard expression of $S(f,g)$ in terms of f and g. Hence $S(f,g)$ has remainder 0 with respect to f, g. □

Theorem C.13. (Buchberger's criterion). *Let I be a nonzero ideal of R and $\mathcal{G} = \{g_1, \ldots, g_s\}$ a system of generators of I. Then $\mathcal{G}$ is a Gröbner basis of I if and only if for all $i \neq j$, $S(i,j)$ reduces to 0 with respect to $g_1, \ldots, g_s$.*

This is the most important theorem in the theory of Gröbner bases. The Buchberger criterion supplies an algorithm to compute a Grobner basis starting from a system of generators of an ideal.

Let $\{g_1, \ldots, g_s\}$ be a system of generators of a nonzero ideal I of R. Compute the S-polynomials $S(g_i, g_j)$. If all $S(g_i, g_j)$ reduce to 0 with respect to $g_1, \ldots, g_s$, then, by the Buchberger criterion, $\{g_1, \ldots, g_s\}$ is a Gröbner basis. Otherwise one of the $S(g_i, g_j)$ has a nonzero remainder g_{s+1}. Then none of the monomials $\mathrm{in}_<(g_1), \ldots, \mathrm{in}_<(g_s)$ divides $\mathrm{in}_<(g_{s+1})$. In other words, the inclusion

$$(\mathrm{in}_<(g_1), \ldots, \mathrm{in}_<(g_s)) \subset (\mathrm{in}_<(g_1), \ldots, \mathrm{in}_<(g_s), \mathrm{in}_<(g_{s+1}))$$

is strict.

Notice that $g_{s+1} \in I$. Now we replace $\{g_1, \ldots, g_s\}$ by $\{g_1, \ldots, g_s, g_{s+1}\}$ and compute all the S-polynomials for this new system of generators.

If all S-polynomials reduce to 0 with respect to $g_1, \ldots, g_s, g_{s+1}$, then $\{g_1, \ldots, g_s, g_{s+1}\}$ is a Gröbner basis. Otherwise there is a nonzero remainder g_{s+2} and we obtain the new system of generators $\{g_1, \ldots, g_{s+1}, g_{s+2}\}$, and the inclusion

$$(\mathrm{in}_<(g_1), \ldots, \mathrm{in}_<(g_s), \mathrm{in}_<(g_{s+1})) \subset (\mathrm{in}_<(g_1), \ldots, \mathrm{in}_<(g_s), \mathrm{in}_<(g_{s+1}), \mathrm{in}_<(g_{s+2}))$$

is strict.

By virtue of Dickson's lemma, it follows that these procedures will terminate after a finite number of steps, and a Gröbner basis can be obtained. In fact, if this were not the case, then a strictly increasing infinite sequence of monomial ideals

$$(\mathrm{in}_<(g_1), \ldots, \mathrm{in}_<(g_s)) \subset (\mathrm{in}_<(g_1), \ldots, \mathrm{in}_<(g_s), \mathrm{in}_<(g_{s+1}))$$

$$\subset \cdots \subset (\mathrm{in}_<(g_1), \ldots, \mathrm{in}_<(g_s), \ldots, \mathrm{in}_<(g_j)) \subset \cdots$$

would arise. However, if $\mathcal{M} = \{\mathrm{in}_<(g_1), \ldots, \mathrm{in}_<(g_s), \mathrm{in}_<(g_{s+1}), \ldots\}$ and if

$$\mathcal{M}^{\min} = \{\mathrm{in}_<(g_{i_1}), \ldots, \mathrm{in}_<(g_{i_q})\}, \qquad i_1 < \cdots < i_q,$$

then, for all $j > i_q$, one would have

$$(\mathrm{in}_<(g_{i_1}), \ldots, \mathrm{in}_<(g_{i_q})) = (\mathrm{in}_<(g_{i_1}), \ldots, \mathrm{in}_<(g_{i_q}), \mathrm{in}_<(g_{i_1+1}), \ldots, \mathrm{in}_<(g_j)),$$

which is a contradiction.

The above algorithm to find a Gröbner basis starting from a system of generators of I is said to be *Buchberger's algorithm.*

Example C.14. Recovering example C.8. We have

$$S(f, g) = x_7 f - x_1 g = x_1 x_5 x_6 - x_2 x_3 x_7$$

with respect to f and g, we choose $S(f, g)$ itself. Let $h = x_1x_5x_6 - x_2x_3x_7$ with $\mathrm{in}_{<_{\mathrm{lex}}}(h) = x_1x_5x_6$. Then $\mathrm{in}_{<_{\mathrm{lex}}}(g)$ and $\mathrm{in}_{<_{\mathrm{lex}}}(h)$ are relatively prime. On the other hand, $S(f, h) = x_2x_3(x_4x_7 - x_5x_6)$ reduces to 0 with respect to f, g, h. It follows from the Buchberger criterion that $\{f, g, h\}$ is a Gröbner basis of I with respect to $<_{\mathrm{lex}}$.

Appendix D

Additional preliminaries

Here is some independent results used throughout the text.

Theorem D.15. ([19, Corollary 10.2]) *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension positive and let I be an $\mathfrak{m}$-primary ideal. The following are equivalent:*

a) *I is an almost complete intersection;*

b) $\mathrm{Sym}(I)$ *is Cohen-Macaulay.*

Proposition D.16. ([22, Proposition 3.12]) *Let $(\mathbf{R}, \mathfrak{m})$ be a Gorenstein local ring of dimension $d > 0$, $J = (a_1, \ldots, a_d)$ a parameter ideal and $I = (J, a)$ and $a \in \mathfrak{m}$. Then:*

$$\begin{aligned}\lambda(I^2/JI) &= \lambda(I/J) - \lambda(\mathbf{R}/I_1(\varphi))\\ &= \lambda(\mathbf{R}/J : a) - \lambda(\mathbf{R}/I_1(\varphi))\\ &= \lambda(\mathbf{R}/J : a) - \lambda(\mathrm{Hom}(\mathbf{R}/I_1(\varphi)), \mathbf{R}/J : a)\\ &= \lambda(\mathbf{R}/J : a) - \lambda((J : a) : I_1(\varphi))/J : a)\\ &= \lambda(\mathbf{R}/(J : a) : I_1(\varphi)).\end{aligned}$$

Proposition D.17. ([21, Proposition 3.3]) *Let $\mathbf{R} = k[x_1, \ldots, x_d]$ and let $I = (f_1, \ldots, f_{d+1})$ be an ideal of finite colength, generated by forms of degree n. Denote by $\mathcal{F}$ and $\mathcal{F}'$ the special fibers of $\mathcal{R}(I)$ and $\mathcal{R}(\mathfrak{m}^n)$, respectively. The following conditions are equivalent:*

(i) $[\mathcal{F}' : \mathcal{F}] = 1$, *that is, the rational mapping*

$$\Psi_I = [f_1 : f_2 : \cdots : f_{d+1}] : \mathbb{P}^{d-1} \dashrightarrow \mathbb{P}^d$$

is birational onto its image;

(ii) $\deg(\mathcal{F}) = n^{d-1}$;

(iii) $e_1(I) = \frac{d-1}{2}(n^d - n^{d-1})$;

(iv) *$\mathcal{R}(I)$ is non-singular in codimension one.*

References

[1] P. Aluffi, Log canonical threshold and Segre classes of monomial schemes. Manu. Math. **146** (2015) 1–6.

[2] D. Bayer and M. Stillman, Macaulay: a computer algebra system for algebraic geometry, Macaulay version 3.0 1994 (Macaulay for Windows by Bernd Johannes Wuebben, 1996).

[3] W. Bruns and J. Herzog, Cohen–Macaulay Rings, Cambridge University Press, Cambridge, 1993.

[4] R. Burity, A. Simis and S. O. Tohăneanu, On a conjecture of Vasconcelos, arXiv:submit/1410.1210v2 [math.AC] 7 Mar 2015.

[5] T. B. Cortadellas and C. D'Andrea, Rational plane curves parameterizable by conics. J. Algebra **373** (2013) 453–480.

[6] T. B. Cortadellas and C. D'Andrea, Minimal generators of the defining ideal of the Rees Algebra associated to a rational plane parameterization with $\mu = 2$, Canad. J. Math. **66** (2014), 1225–1249.

[7] T. B. Cortadellas and C. D'Andrea, The Rees Algebra of a monomial plane parametrization, J. Symbolic Comput. **70** (2015), 71–105.

[8] D. A. Cox, Four conjectures: Two for the moving curve ideal and two for the Bezoutian, Proceedings of Commutative Algebra and its Interactions with Algebraic Geometry", CIRM, Luminy, France, May 2006.

[9] D. Cox, The moving curve ideal and the Rees algebra, Theoretical Computer Science, **392** (2008), 23–36.

[10] D. Cox, J. W. Hoffman and H. Wang, Syzygies and the Rees algebra, J. Pure Appl. Algebra, **212** (2008), 1787–1796.

[11] D. Eisenbud and B. Sturmfels, Binomial ideals, Duke Math. J. **84** (1996), 1–45.

[12] R. Fröberg, An inequality for Hilbert series of graded algebras, Math. Scand. **56** (1985), 117–144.

[13] L. Ghezzi, S. Goto, J. Hong and W. Vasconcelos, Variation of Hilbert coefficients, Proc. Amer. Math. Soc. **141** (2013), 3037–3048.

[14] P. Gordan, Les invariantes des formes binaires, J. Math. Pures et Appliqués, **6**, (1900), 141–156.

[15] S. H. Hassanzadeh and A. Simis, Plane Cremona maps: saturation, regularity and fat ideals, J. Algebra, **371** (2012), 620–652.

[16] S. H. Hassanzadeh and A. Simis, Implicitization of de Jonquières parametrizations, J. Commut. Algebra, **6** (2014) 149–172.

[17] W. Heinzer, D. Lantz and K. Shah, The Ratliff-Rush ideals in a Noetherian ring, Comm. Algebra, **20** (1992), 591–622.

[18] J. Herzog and T. Hiby, Monomial Ideals, Graduate Texts in Mathematics, Vol. 260, Springer, 2011.

[19] J. Herzog, A. Simis and W. V. Vasconcelos, Koszul homology and blowing-up rings, in *Commutative Algebra*, Proceedings, Trento (S. Greco and G. Valla, Eds.). Lecture Notes in Pure and Applied Mathematics **84**, Marcel-Dekker, 1983, 79–169.

[20] J. Hong, A. Simis, W. V. Vasconcelos, On the homology of two-dimensional elimination, J. Symb. Comp. **43** (2008) 275–292.

[21] J. Hong, A. Simis, W. V. Vasconcelos, The equations of almost complete intersections, Bull. Braz. Math. Soc. (New Series), **43** (2012), 171–199.

[22] J. Hong, A. Simis and W. V. Vasconcelos, Extremal Rees algebras, J. Comm. Algebra, **5** (2013), 231–267.

[23] S. Huckaba, Reduction numbers for ideals of higher analytic spread, Math. Proc. of the Cambridge Philosophical Soc., **102** (1987), 49–57.

[24] S. Huckaba and T. Marley, Hilbert coefficients and the depths of associated graded rings, J. London Math. Soc. **56** (1997), 64–76.

[25] C. Huneke, Hilbert functions and symbolic powers, Michigan Math. J. **34** (1987), 293–318.

[26] A. Kustin, C. Polini and B. Ulrich, Rational normal scrolls and the defining equations of Rees algebras, J. Reine Angew. Math. **650** (2011), 23–65.

[27] J. Migliore and R. Miró-Roig, Ideals of General Forms and the Ubiquity of the Weak Lefschetz Property, J. Pure Appl. Algebra **182** (2003), 79–107.

[28] J. Migliore and R. Miró-Roig, On the Minimal Free Resolution of $n+1$ General Forms, Trans. Amer. Math. Soc. **355** (2003), 1–36.

[29] J. Migliore, R. Miró-Roig and U. Nagel, Monomial ideals, almost complete intersections and the Weak Lefschetz Property, Trans. Amer. Math. Soc. **363** (2011), 229–257.

[30] M. Nagata, Local Rings, Kreiger, Huntington/New York, 1975.

[31] D. G. Northcott, A note on the coefficients of the abstract Hilbert function, J. London Math. Soc. **35** (1960), 209–214.

[32] D. G., Northcott, and D., Rees, Reductions of ideals in local rings. Proc. Cambridge Phil. Soc. 50 (1954), 145–158.

[33] A. Ooishi, Δ-genera and sectional genera of commutative rings, Hiroshima Math. J. **17** (1987), 361–372.

[34] R. J. Ratliff and D. E. Rush, Two notes on reductions of ideals, Indiana Univ. Math. J., **27** (1978), 929–934.

[35] D. Rees, $\mathcal{A}$-transforms of local rings and a theorem on multiplicities of ideal, Proc. Cambridge Phil. Soc. **57** (1961), 8–17.

[36] M. E. Rossi and I. Swanson, Notes on the behavior of the Ratliff-Rush filtration, *in* Contemp. Math., 331, Amer. Math. Soc., Providence, RI, 2003.

[37] P. Samuel, La notion de multiplicité en algébre et en géométrie algébrique, J. Math. Pures et Appl., **30** (1951), 159–274.

[38] A. Simis and S. Tohăneanu, The ubiquity of Sylvester forms in almost complete intersections, Collectanea Math, **66** (2015), 1–31.

[39] I. Swanson and C., Huneke, Integral Closure of Ideals, Rings, and Modules, Cambridge University Press, 2006.

[40] W. Vasconcelos, Computational Methods in Commutative Algebra and Algebraic Geometry, Springer-Verlag, New York, 2004.

[41] J. K. Verma, Hilbert Coefficients and depth of the associated graded ring of an ideal, Math. Student, Special Centenary Volume, **76** (2007), 237–264.

[42] H. Wiebe, Über homologische invarianten lokaler ringe, Math. Ann. **179** (1969), 257–274.