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$p-$parabolic submanifolds in certain spacetimes: rigidity, uniqueness and non-existence results

por

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sob orientação de

Prof. Dr. Eraldo Almeida Lima Júnior

Tese apresentada ao Corpo Docente do Programa Associado de Pós-Graduação em Matemática - UFPB/UFCG, como requisito parcial para obtenção do título de Doutor em Matemática.

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Dedico este trabalho ao meu pai Guillermo

“Take these broken wings and learn to fly” (Paul McCartney)

Abstract

In this work we present rigidity and uniqueness results for parabolic and stable constant mean curvature hypersurfaces immersed in Generalized Robertson-Walker and Standard Static spacetimes. We obtained some conditions under which a hypersurface in these ambiences must be parabolic, as well as stable. In order to achieve the uniqueness results, we used some cut-off functions coming from the parabolicity jointly with the stability operator. Also, we introduced the concept of totally trapped submanifold and obtained some uniqueness and non-existence results when the submanifold is p-parabolic. We also presented a lemma of type Nishikawa in order to obtain Calabi-Berstein type results for surfaces in Robertson-Walker Generalized spacetimes.

Keywords: p-parabolic manifolds, GRW spacetimes, stable hypersurfaces, CMC hypersurfaces.

Resumo

Neste trabalho nós apresentamos resultados de rigidez e unicidade para hiperfíces de curvatura média constante parabólicas e estáveis imersas em espaços-tempo Robertson Walker e Standard Static. Nós obtivemos algumas condições sob as quais uma hiperfćie nestes ambientes deve ser parabólica, bem como estável. A fim de obter os resultados de unicidade, usamos algumas funções corte provenientes da parabolicidade juntamente com o operador estabilidade. Também, introduzimos o conceito de subvariedades totalmente presas e obtivemos alguns resultados de unicidade e não-existência quando a subvariedade é p-parabólica. Também apresentamos um lema do tipo Nishikawa a fim de obter resultados do tipo Calabi-Berstein para superfícies no Robertson Walker generalizado.

Palavras-chave: variedades p-parabólicas, GRW espaços-tempo, hiperfícies estáveis, hiperfícies CMC.

Contents

Introduction

The study of warped product spacetimes is important from both geometrical and physical point of view, specially in Lorentzian geometry and General Relativity, since they comprise a wide variety of exact solutions to Einstein's field equations

$$R_{ij} - \frac{1}{2}g_{ij}R + g_{ij}\Lambda = \frac{8\pi G}{c^4}T_{ij},$$

where R_{ij} is the Ricci curvature tensor, R is the scalar curvature, g_{ij} is the metric tensor, Λ is the cosmological constant, G is Newton's gravitational constant, c is the speed of light in vacuum, and T_{ij} is the stress-energy tensor.

In this thesis we focus on two classes of spacetimes, namelly the Generalized Robertson-Walker spacetimes (GRW) and the Standard Static spacetimes (SSST). The notion of GRW spacetimes as a family of cosmological models of the universe was firstly studied in [15] and [16] by Aledo, Romero and Salamanca. The SSST are a generalization of Einstein static universe which is the first relativistic cosmological model. Some classical and important examples of SSST are the *anti-de Sitter space*, the *exterior Schwarzschild spacetime*, the Minkowski spacetime, the *Einstein static universe* and some regions of the *Reissner-Nordstrom spacetime*, see for instance [21] and [38]. In [2] Allison made a deep approach about energy conditions in this ambient.

Spacelike constant mean curvature hypersurfaces are very important on General Relativity, since they are critical points of the area functional under certain volume constraint [16]. More reasons that justify the study of these hypersurfaces can be found on [48]. In particular, maximal hypersurfaces describes the transition between expanding and contracting phases of the universe. Also, they reflect properties of the ambient. The most expressive result about maximal hypersurfaces is the well known

Calabi-Bernstein theorem that asserts: The only complete maximal hypersurfaces in the Lorentz-Minkowski spacetime are the spacelike planes.

In Chapters 2 and 3 we deal with parabolic and stable CMC hypersurfaces and obtain some constraints under which they must have constant support function. In addition, we obtain some Calabi-Berstein type results. It is important to emphasize that when the ambient is a SSST, every CMC hypersurface immersed is stable (see Lemma 1.5.7). Our technique is based on the use of some cut-off functions directly related to the parabolicity jointly to the stability operator.

In Chapter 4 we introduce the concept of totally trapped submanifolds and obtain some uniqueness and non-existence results for $p-$parabolic totally trapped surfaces in SSST.

In Chapter 5, outside of the setting of the parabolic hypersurfaces, we prove a Nishikawa type principle in order to obtain Calabi-Berstein type results.

Chapter 1

Some Preliminaries

1.1 The Generalized Robertson Walker (GRW) spacetime

1.1.1 Some elements and basic results

Consider an n-dimensional connected manifold M^n, I an open interval of the real line $\mathbb{R}$ and $\rho : I \to \mathbb{R}$ a positive smooth function. The Generalized Robertson Walker (GRW) spacetime is denoted by $-I \times_\rho M^n$ and defined as the Lorentzian product manifold $-I \times_\rho M^n$ endowed with the metric

$$\langle \, , \, \rangle = -\pi_{\mathbb{R}}^*(dt^2) + \rho^2 \pi_M^*(\langle \, , \, \rangle_M).$$

A GRW spacetime is said to be a *Lorentzian product* if its warping function is constant. If the warping function is non-locally constant, the GRW spacetime is said to be *proper*.

An important tool used in GRW spacetimes is the Proposition 7.35 of [51], where we write $\overline{\nabla}$ for the Levi-Civita connection of $-I \times_\rho M^n$ and ∇^M for the Levi-Civita connection of M^n.

Proposition 1.1.1 *Let $-I \times_\rho M^n$ be a Lorentzian warped product, $\partial_t \in T\mathbb{R}$ and $X, Y \in TM$ then:*

(i) $\overline{\nabla}_{\partial_t} \partial_t = 0$

(ii) $\overline{\nabla}_{\partial_t} X = \overline{\nabla}_X \partial_t = \frac{\rho'}{\rho} X$

(iii) $\langle \overline{\nabla}_X Y, \partial_t \rangle = -\langle X, Y \rangle_M \rho\rho'$

(iv) $(\overline{\nabla}_X Y)^\top = \nabla^M_X Y$.

Let $\psi : \Sigma^n \to -I \times_\rho M^n$ be a spacelike hypersurface, i.e., a hypersurface whose metric induced by the immersion is Riemannian. Since every spacetime is time-oriented, let us say ∂_t the canonical timelike direction, we can take N as the only globally defined vector field normal to Σ^n in the same time-orientation of ∂_t, that is, $\langle N, \partial_t \rangle < 0$. This vector field will be called Gauss map.

Let X be a tangent vector of $-I \times_\rho M^n$ and

$$X^* = X + \langle X, \partial_t \rangle \partial_t \tag{1.1}$$

its tangential component to the fiber M. A direct application of Corollary 7.43 of [51] provides the following relation between the Ricci tensor of the fiber and the Ricci tensor of the ambient:

$$\overline{\mathrm{Ric}}(X) = \mathrm{Ric}^M(X^*) - (n-1)(\log \rho)''|X^*|^2 + n\frac{\rho''}{\rho}|X|^2. \tag{1.2}$$

The shape operator corresponding to N is given by

$$AX = -\overline{\nabla}_X N. \tag{1.3}$$

In the case of $A = 0$ we say that the immersion is *totally geodesic*.
The Gauss-Kronecker curvature of Σ^n is defined as $K_G = -\det A$ and the mean curvature of the immersion is defined by

$$H = -(1/n)\,\mathrm{tr}(A).$$

If H is constant, we say that Σ^n is a *CMC hypersurface*. If $H = 0$, we say that Σ^n is a maximal hypersurface, due to the fact that these hypersurfaces maximize locally the area functional.

1.1.2 The height and the suport functions

Let us define the height function of $\psi : \Sigma^n \to -I \times_\rho M^n$ by $h = \pi_{\mathbb{R}} \circ \psi$. From a simple calculation, we obtain

$$\overline{\nabla}\pi_{\mathbb{R}} = -\langle \overline{\nabla}\pi_{\mathbb{R}}, \partial_t \rangle \partial_t = -\partial_t.$$

Then setting ∇ the Levi-Civita conection of Σ^n we get:

$$\nabla h = -\partial_t{}^\top = -\partial_t - \langle N, \partial_t \rangle N. \tag{1.4}$$

Setting N^* the tangential component as (1.1) we have

$$\rho^2 |N^*|^2 = |\nabla h|^2 = -1 + \langle N, \partial_t \rangle^2. \tag{1.5}$$

Observe that $\langle N, \partial_t \rangle$ is the opposite of the hyperbolic angle between N and ∂_t. The function $\langle N, \partial_t \rangle$ is also called the *hyperbolic angle of* Σ^n. If h is constant we say that Σ^n is contained in a *slice* $\{t_0\} \times M$. In this case from (1.4) we must have $N = \partial_t$. From (1.3) and Proposition 1.1.1, a spacelike slice has shape operator

$$AX = -\overline{\nabla}_X \partial_t = -\frac{\rho'}{\rho} X, \tag{1.6}$$

therefore mean curvature $H = \frac{\rho'}{\rho}$.

Computing the divergence of (1.4), making use of the Gauss formula and Corollary 7.43 in [51] we have the following expression for the Laplacian of the height function:

$$\Delta h = -\frac{\rho'(h)}{\rho(h)}(n + |\nabla h|^2) - n\, H \langle N, \partial_t \rangle. \tag{1.7}$$

From a straightforward computation of (1.7) we obtain:

$$\Delta \rho(h) = -n \frac{\rho(h)'^2}{\rho(h)} + \rho(h)(\log \rho(h))'' |\nabla h|^2 - n\rho'(h) H \langle N, \partial_t \rangle. \tag{1.8}$$

The *support function* is defined as $\rho\langle N, \partial_t \rangle$.

Proceeding to the calculation of the gradient of the support function we have:

$$\begin{aligned} \langle \nabla \langle N, \rho\partial_t \rangle, X \rangle &= X \langle N, \rho\partial_t \rangle \\ &= \langle \overline{\nabla}_X N, \rho\partial_t \rangle + \langle N, \overline{\nabla}_X \rho\partial_t \rangle \\ &= \langle AX, -\rho\partial_t{}^\top \rangle \\ &= \langle \rho A \nabla h, X \rangle. \end{aligned}$$

Since X is arbitrary, we get

$$\nabla \langle N, \rho\partial_t \rangle = \rho A(\nabla h). \tag{1.9}$$

The following lemma is computed in [13] eq. (12). It gives us an expression for the Laplacian of the support function.

Lemma 1.1.2 *Let $\psi : \Sigma^n \to -I \times_\rho M^n$ be a spacelike hypersurface with Gauss map N. Suppose that the mean curvature H is constant. Then,*

$$\Delta\langle N, \rho\partial_t\rangle = \langle N, \rho\partial_t\rangle(\overline{\mathrm{Ric}}(N) + |A|^2) + n(H\rho' + \rho''\langle N, \partial_t\rangle), \tag{1.10}$$

where $\overline{\mathrm{Ric}}$ is the Ricci curvature of the ambient and $|A|$ is the Hilbert–Schmidt norm of the shape operator A of Σ^n.

Notice that since N and ∂_t are timelike,

$$\langle N, \partial_t\rangle = -\cosh\theta,$$

where θ is the hyperbolic angle between N and ∂_t. Then from (1.5) we get that $|\nabla h|^2 = \sinh^2\theta$. We have as corollary of a Lemma 1.1.2 version for the Laplacian of the hyperbolic angle of Σ in Lorentzian products:

Corollary 1.1.3 *Let $\psi : \Sigma^n \to -I \times M^n$ be a spacelike hypersurface with Gauss map N. Suppose that the mean curvature H is constant. Then,*

$$\Delta\langle N, \partial_t\rangle = \langle N, \partial_t\rangle(\overline{\mathrm{Ric}}(N) + |A|^2), \tag{1.11}$$

where $\overline{\mathrm{Ric}}$ is the Ricci curvature of the ambient and $|A|$ is the Hilbert–Schmidt norm of the shape operator A of Σ^n.

1.1.3 Vertical graphs in GRW spacetimes

Let $\Omega \subset M$ be a domain and $u \in C^\infty(\Omega)$ be a smooth function. A vertical graph over Ω is

$$\Sigma^n(u) = \{(u(x), x); x \in \Omega\} \subset -I \times_\rho M^n.$$

The metric induced on Ω from the Lorentzian metric on the ambient space via $\Sigma^n(u)$ is

$$\langle , \rangle = -du^2 + \rho(u)^2\langle , \rangle_M. \tag{1.12}$$

The graph is said to be entire if $\Omega = M$. If $\Sigma^n(u)$ is a hypersurface with constant mean curvature H we say that $\Sigma^n(u)$ is an H-graph.
With a straightforward computation we can see that the vector field

$$N(x) = \frac{1}{\rho(u)\sqrt{\rho(u)^2 - |Du|^2}}\left(\rho(u)^2\partial_t|_{(u(x),x)} + Du(x)\right),\ x \in M,$$

defines the future-pointing Gauss map of $\Sigma^n(u)$.

We can see that $\Sigma^n(u)$ is a spacelike surface if, and only if, $|Du| < \rho(u)$.

The height and angle functions are defined as above. From 1.5:

$$\rho^2|N^*|^2 = |\nabla h|^2 = \frac{|\nabla u|^2}{\rho(u)^2 - |\nabla u|^2}.$$

Observe that a graph is a slice if, and only if, u is constant. Notice also that in the case of Lorentzian products, a graph is of constant angle if, and only if, $|Du|$ is constant.

In counterpart of the Riemannian case, where any entire graph is necessarilly complete, it is not true for Lorentzian ambients. We can find some examples of non-complete entire maximal graphs in $-\mathbb{R} \times \mathbb{H}^2$ in [1]. Alfonso Romero got a reasonable assumption under which an entire spacelike graph must be complete.

Lemma 1.1.4 *Let $\Sigma^n(u)$ be an entire spacelike graph in $\overline{M}^{n+1} = -I \times_\rho M^n$. Suppose that for M^n is a complete Riemannian manifold such that $\rho \geq c > 0$, whre c is a constant. If the height function of $\Sigma^n(u)$ satisfies*

$$|\nabla h|^2 \ \leq \ \Gamma(r),$$

where r is the distance function and Γ a continuous function, then $\Sigma^n(u)$ is complete.

Proof. Suppose by contradiction that $\Sigma^n(u)$ is not complete, then there is a divergent curve $\gamma : [0, \infty) \to \Sigma(u)$ with finite length $L_\gamma = a < \infty$. Therefore

$$L_{\Gamma|\gamma} \leq L_\gamma = a,$$

that implies $|\nabla h|^2 \leq C < \infty$. Hence,

$$|Du|^2 = \frac{\rho^2|\nabla h|^2}{1 + |\nabla h|^2} \leq \frac{C}{1+C}\rho^2 = b\rho^2.$$

Without loss of generality we assume the projection γ_M of γ on M is parametrized by the arc length. By (1.12) we have,

$$\begin{aligned} L_\gamma &= \int_0^\infty \sqrt{\rho^2 - \langle Du, \gamma_M' \rangle^2} dt \\ &\geq \int_0^\infty \sqrt{\rho^2 - |Du|^2} dt \\ &\geq \sqrt{1-b} \int_0^\infty \rho dt \geq c\sqrt{1-b}\, L_{\gamma_M}. \end{aligned}$$

Therefore L_{γ_M} is finite that is an absurd since γ_M is also divergent in M^n because the composition of the projection π_M and the inclusion of $\Sigma^n(u)$ in $-\mathbb{R} \times M^n$ is a diffeomorphism. ■

1.2 The Standard Static Spacetime (SSST)

1.2.1 Some elements and basic results

Let $\overline{M}^{n+k+1}$ be an $(n+k+1)$-dimensional Lorentz manifold endowed with a timelike Killing vector field K. Suppose that the distribution $\mathcal{D}$ orthogonal to K is of rank constant and integrable. We denote by $\Psi : M^{n+k} \times \mathbb{I} \to \overline{M}^{n+k+1}$ the flow generated by K, where M^{n+k} is an arbitrarily fixed spacelike integral leaf of $\mathcal{D}$ labeled as $t = 0$, which we will suppose to be connected, and $\mathbb{I}$ is the maximal interval of definition. Without loss of generality, in what follows we will also consider $\mathbb{I} = \mathbb{R}$. In this setting, $\overline{M}^{n+k+1}$ can be regarded as the *standard static spacetime* $M^{n+k} \times_\rho \mathbb{R}_1$, that is, the product manifold $M^{n+k} \times \mathbb{R}_1$ endowed with the warping metric

$$\langle\ ,\ \rangle = \pi_M^*\left(\langle\ ,\ \rangle_M\right) - (\rho \circ \pi_M)^2 \pi_{\mathbb{R}}^*\left(dt^2\right), \tag{1.13}$$

where π_M and $\pi_{\mathbb{R}}$ denote the canonical projections from $M \times \mathbb{R}$ onto each factor, $\langle\ ,\ \rangle_M$ is the induced Riemannian metric on the Riemannian base M^{n+k}, $\mathbb{R}_1$ is the manifold $\mathbb{R}$ endowed with the metric $-dt^2$ and the warping function $\rho \in C^\infty(M)$ is given by $\rho = |K| = \sqrt{-\langle K, K\rangle}$. Let us consider a connected spacelike submanifold $\psi : \Sigma^n \to \overline{M}^{n+k+1}$ immersed in a standard static spacetime $\overline{M}^{n+k+1} = M^{n+k} \times_\rho \mathbb{R}_1$, which means that the metric induced on Σ^n via ψ is a Riemannian one. Let us denote by $\overline{\nabla}$ and ∇ the Levi-Civita connections in $\overline{M}^{n+k+1}$ and Σ^n, respectively. The Gauss formula for Σ in $\overline{M}^{n+k+1}$ is given by

$$\overline{\nabla}_X Y = \nabla_X Y - \alpha(X, Y)$$

for every tangent vector fields $X, Y \in \mathfrak{X}(\Sigma)$, where

$$\alpha : \mathfrak{X}(\Sigma) \times \mathfrak{X}(\Sigma) \to \mathfrak{X}^\perp(\Sigma)$$

denotes the vector valued second fundamental form of Σ; that is

$$\alpha(X, Y) = -(\overline{\nabla}_X Y)^\perp.$$

The *mean curvature vector field* $\vec{H}$ of Σ^n is defined by

$$\vec{H} = \frac{1}{n}\mathrm{tr}(\alpha) = \frac{1}{n}\sum_{i=1}^n \alpha(E_i, E_i),$$

where $\{E_i\}_{i=1}^n$ is a local frame on Σ.

On the other hand, the Weingarten formula is given by

$$A_\xi X = \nabla^\perp_X \xi - \overline{\nabla}_X \xi \tag{1.14}$$

for every tangent vector field $X \in \mathfrak{X}(\Sigma)$ and normal vector field $\xi \in \mathfrak{X}^\perp(\Sigma)$, where

$$A_\xi : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$$

denotes the shape operator (or Weingarten endomorphism) defined by

$$\langle A_\xi X, Y\rangle = \langle \alpha(X,Y), \xi\rangle, \ X, \ Y \in \mathfrak{X}(\Sigma). \tag{1.15}$$

From (1.14) and (1.15) we have that

$$\langle \overline{\nabla}_X K^\perp, Y\rangle = \langle \alpha(X,Y), K\rangle, \tag{1.16}$$

for $X, Y \in \mathfrak{X}(\Sigma)$.

A submanifold Σ is *totally umbilic with respect to a normal direction* ξ if A_ξ is a multiple of the identity operator. A submanifold Σ is *totally umbilic* if there exists $Z \in \mathfrak{X}(\Sigma)^\perp$ such that

$$\alpha(X,Y) = \langle X, Y\rangle Z \quad \forall X, Y \in \mathfrak{X}(\Sigma).$$

In this case we have that the submanifold is totally umbilic with respect to any normal direction $\xi \in \mathfrak{X}(\Sigma)^\perp$.

For the particular case of spacelike hypersurfaces $\psi : \Sigma^n \to M^n \times_\rho \mathbb{R}_1$, there exists a unique unitary timelike normal vector field N globally defined on Σ^n which is in the same time-orientation of K, that is, the *support function* $\Theta = \langle N, K\rangle$ is negative on Σ^n. Let us compute the gradient of the support function Θ. Let X be a tangent vector field in Σ^n. Then

$$\begin{aligned}
\langle \nabla\Theta, X\rangle &= X\langle N, K\rangle \\
&= \langle \overline{\nabla}_X N, K\rangle + \langle N, \overline{\nabla}_X K\rangle \\
&= \langle -AX, K^\top\rangle - \langle X, \overline{\nabla}_N K\rangle \\
&= \langle X, -AK^\top\rangle + \langle X, -(\overline{\nabla}_N K)^\top\rangle,
\end{aligned}$$

where $(\ )^\top$ denotes the tangential component of a vector field in $\mathfrak{X}(\overline{M})$ along Σ^n. Hence we conclude that $\nabla\Theta = -AK^\top - (\overline{\nabla}_N K)^\top$.

Computing the divergence of the gradient above, we obtain the Laplacian of the support function, as we see in Proposition 3.1 of [18]:

$$\Delta\Theta = (\overline{\mathrm{Ric}}(N) + |A|^2)\Theta, \tag{1.17}$$

where $|A| = \sqrt{\mathrm{tr}(A^2)}$ is the Hilbert-Schimidt norm of the shape operator A.

For our purposes, we will study the (vertical) height function $h = \pi_{\mathbb{R}} \circ \psi$. When the height function is constant, we say that the hypersurface is contained in a *slice* $\{t_0\} \times M^n$. From the decomposition $K = K^\top - \Theta N$ we obtain

$$\nabla h = -\frac{1}{\rho^2}K^\top \quad \text{and} \quad |\nabla h|^2 = \frac{\Theta^2 - \rho^2}{\rho^4}. \tag{1.18}$$

Let N^* be the orthogonal projection of N onto TM. Then

$$|\nabla h|^2 = \frac{1}{\rho^2}|N^*|^2 = \frac{1}{\rho^2}\left(\frac{\Theta^2}{\rho^2} - 1\right) \tag{1.19}$$

1.2.2 Killing graphs in SSST

We define the *entire Killing graph* $\Sigma^n(u)$ associated to a smooth function $u \in C^\infty(M)$ as the hypersurface given by

$$\Sigma^n(u) = \{(x, u(x)) : x \in M^n\} \subset M^n \times_\rho \mathbb{R}_1.$$

In this case we consider M^n endowed with the metric induced from the Lorentzian metric (1.13) in $\Sigma^n(u)$ which is given by

$$\langle\ ,\ \rangle_u = \langle\ ,\ \rangle_M - \rho^2 du^2. \tag{1.20}$$

We have that $\Sigma^n(u)$ is spacelike if, and only if, $|Du|_M^2 < \gamma$, where Du denotes the gradient of a function u with respect to the metric $\langle\ ,\ \rangle_M$ of M^n and $\gamma = \rho^{-2}$. Indeed, if $\Sigma^n(u)$ is spacelike, then

$$0 < \langle Du, Du\rangle_u = \langle Du, Du\rangle_M - \rho^2\langle Du, Du\rangle_M^2,$$

hence we conclude that $\rho^2|Du|_M^2 < 1$. Conversely, if $\rho^2|Du|_M^2 < 1$ and X is a vector field tangent to $\Sigma^n(u)$, we obtain from Cauchy-Schwarz inequality,

$$\langle X, X\rangle_u = \langle X^*, X^*\rangle_M - \rho^2\langle Du, X^*\rangle_M^2 \geq \langle X^*, X^*\rangle_M(1 - \rho^2|Du|_M^2),$$

where X^* is the orthogonal projection of X onto TM. Thus, $\langle X, X\rangle_u \geq 0$ and $\langle X, X\rangle_u = 0$ if, and only if, $X = 0$.

The function $g : M^n \times \mathbb{R}_1 \to \mathbb{R}$ given by $g(x,t) = u(x) - t$ is such that $\Sigma^n(u) = \psi(g^{-1}(0))$. Thus, for all vector field X tangent to $M^n \times_\rho \mathbb{R}_1$, we have

$$X(g) = X^*(g) - \frac{1}{\rho^2}\langle X, \partial_t\rangle \partial_t(g) = \langle \frac{1}{\rho^2}\partial_t + Du, X\rangle.$$

Thus,

$$\overline{\nabla} g = \frac{1}{\rho^2}\partial_t + Du$$

is a normal vector field on $g^{-1}(0)$ and, consequently,

$$N_0 = \psi_*(\overline{\nabla} g) = \frac{1}{\rho^2}K + \psi_*(Du)$$

is a normal timelike vector field on $\Sigma(u)$. Since,

$$|N_0| = \frac{(1-\rho^2|Du|_M^2)^{1/2}}{\rho},$$

it follows that

$$N = \frac{N_0}{|N_0|} = \frac{1}{\rho(1-\rho^2|Du|_M^2)^{1/2}}(K + \rho^2\psi_*(Du))$$

defines the future-pointing Gauss map of $\Sigma^n(u)$ such that its angle function is given by

$$\Theta = \langle N, K\rangle = -\frac{\rho}{(1-\rho^2|Du|_M^2)^{1/2}} < 0. \tag{1.21}$$

Moreover, for all vector field X tangent to M^n, the shape operator A of $\Sigma^n(u)$ with respect to N is given by

$$\begin{aligned} AX &= -\frac{\rho}{(1-\rho^2|Du|_M^2)^{1/2}} D_X Du - \frac{\rho^3\langle D_X Du, Du\rangle}{(1-\rho^2|Du|_M^2)^{3/2}} Du - \frac{\rho^2\langle D\rho, X\rangle |Du|_M^2}{(1-\rho^2|Du|_M^2)^{3/2}} Du \\ &\quad -\frac{\langle D\rho, X\rangle}{(1-\rho^2|Du|_M^2)^{1/2}} Du - \frac{\langle Du, X\rangle}{(1-\rho^2|Du|_M^2)^{1/2}} D\rho. \end{aligned}$$

So, it follows from above that the mean curvature H_u of a spacelike entire Killing graph $\Sigma^n(u)$ is given by

$$nH_u = \mathrm{Div}\left(\frac{\rho Du}{(1-\rho^2|Du|_M^2)^{1/2}}\right) + \frac{\langle Du, D\rho\rangle}{(1-\rho^2|Du|_M^2)^{1/2}},$$

where Div stands for the divergence operator on M^n with respect to the metric $\langle\ ,\ \rangle_M$. In particular, an entire Killing graph $\Sigma^n(u)$ is maximal if, and only if, the function $u \in$

$C^\infty(M^n)$ satisfies the following elliptic partial differential equation of the divergence form

$$\begin{cases} \operatorname{Div}\left(\dfrac{\rho Du}{(1-\rho^2|Du|_M^2)^{1/2}}\right) + \dfrac{\langle Du, D\rho\rangle}{(1-\rho^2|Du|_M^2)^{1/2}} = 0, \quad \text{in} \quad M^n \\ \\ \rho^2|Du|_M^2 < 1. \end{cases} \tag{1.22}$$

We also note that,

$$N^* = N - N^\perp = \frac{\rho\psi_*(Du)}{(1-\rho^2|Du|_M^2)^{1/2}},$$

since we have

$$|N^*|_M^2 = \frac{\rho^2|Du|_M^2}{1-\rho^2|Du|_M^2}, \tag{1.23}$$

and, consequently, we get from (1.18) and (1.23) the following relation

$$|\nabla h|^2 = \frac{|Du|_M^2}{1-\rho^2|Du|_M^2}. \tag{1.24}$$

Notice that if ρ and the support function Θ are constant in a Killing graph, then $|Du|$ is constant. A spacelike Killing graph is a slice if, and only if, u is constant. Also, using (1.21) $\eta = \frac{\Theta}{\rho}$ is automatically bounded, provided $\rho^2|Du|^2 < 1$.

1.3 Energy conditions in spacetimes

There are several references and physical interpretations about the energy conditions in spacetimes. See for instance [11], [57]. In General Relativity the Ricci tensor is related to the energy-momentum tensor via Einstein's field equations

$$\overline{\mathrm{Ric}}_{ij} - \frac{1}{2}g_{ij}R + g_{ij}\Lambda = \frac{8\pi G}{c^4}T_{ij},$$

where $\overline{\mathrm{Ric}}_{ij}$ is the Ricci curvature tensor of the ambient, R is the scalar curvature, g_{ij} is the metric tensor, Λ is the cosmological constant, G is Newton's gravitational constant, c is the speed of light in vacuum, and T_{ij} is the stress-energy tensor. Then the energy conditions may be guaranteed under some constraints on the Ricci tensor of the ambient. We will give a brief introduction about this theme. A spacetime obeys the Timelike Convergence Condition (TCC) if its Ricci tensor satisfies

$$\overline{\mathrm{Ric}}(Z) \geq 0,$$

for any timelike vector Z.

Physically, the TCC mean that the gravity, on average, attracts. Also, if a spacetime satisfies the Einstein equations with cosmological constant zero, then it obeys TCC.

A spacetime obeys the Null Convergence Condition (NCC) if its Ricci tensor satisfies

$$\overline{\mathrm{Ric}}(Z) \geq 0,$$

for any lightlike vector Z.

It is not difficult to check that TCC holds on a GRW if, and only if, NCC holds and $\rho'' \leq 0$.

When the ambient is a GRW spacetime, we can apply equation (1.2) to any lightlike vector in order to obtain that the GRW spacetime $-I \times_\rho M^n$ obeys NCC if the Ricci tensor of the fiber satisfies

$$\mathrm{Ric}^M \geq (n-1)\rho^2(\log \rho)''\langle\, , \,\rangle_M. \tag{1.25}$$

In particular, a GRW spacetime $\overline{M}^3$ with a 2-dimensional fiber obeys NCC if and only if

$$\frac{\kappa_M}{\rho^2} - (\log \rho)'' \geq 0, \tag{1.26}$$

where κ_M is the Gauss curvature of the fiber.

When the ambient is a SSST, Allison (see [2]) provided some conditions under which a SSST obeys the TCC. Namely: If $M^n \times_\rho \mathbb{R}_1$ $(n \geq 2)$ is a SSST such that the Ricci curvature of the base M^n is non-negative and the warping function satisfies:

$$g_M(w,w)\Delta\rho - \mathrm{Hess}\rho(w,w) \geq 0, \tag{1.27}$$

for all w tangent to M^n, then TCC holds on $M^n \times_\rho \mathbb{R}_1$.

A SSST obeys the *weak timelike convergence condition* if condition (1.27) holds true.

1.4 Parabolicity of manifolds

A manifold M is said to be $p-$*parabolic* provided there is no non-constant positive smooth function $u : M \to \mathbb{R}$ satisfying $\Delta_p u \leq 0$, where $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$. For

2−parabolic manifolds we say just *parabolic*. In fact, the p-Laplacian is the Euler-Lagrange operator associated to the energy functional (see [63], [24]). In [27] Colding and Minicozzi have proved that a complete surface M^2 satisfying quadratic area growth

$$Vol(B_r) \leq Cr^2 \tag{1.28}$$

must be parabolic, where B_r denote an intrinsic (geodesic) ball in M^2. Furthermore, it was showed in [30] by Cao and Zhou that in a schrinking gradient Ricci soliton there exists a uniform constant $C > 0$ that satisfies (1.28). Therefore all 2-dimensional schrinking gradient Ricci solitons M^2 must be parabolic. When a complete manifold is p-parabolic, it means that the growth of the volume of the geodesic balls is bounded by a polynomial of degree at most p in the distance function. Compact manifolds are trivially p-parabolics for all $1 \leq p < \infty$, as well complete with finite volume manifolds. The case $p = 2$ has been extensively studied linking several mathematical areas, namely geometry, analysis and probability ([34] provides a deep survey on this topic). In [62], Troyanov studied an invariant of Riemannian manifolds related to the non-linear potential theory of the p-Laplacian and which is called its parabolic or hyperbolic type.

Let M^n be a Riemannian manifold and B_r a geodesic ball of radius r about a fixed point a. For $0 < r < R$ let $\mathcal{A}_{r,R}$ be the geodesic annulus:

$$\mathcal{A}_{r,R} := B_R - \overline{B_r}.$$

Consider the boundary problem

$$\begin{cases} \Delta_p w & = & 0 \quad \text{in } \mathcal{A}_{r,R} \\ w & \equiv & 0 \quad \text{on } \partial B_r \\ w & \equiv & 1 \quad \text{on } \partial B_R, \end{cases} \tag{1.29}$$

and we denote $w = w_{r,R}$ the solution of (1.29) which is called *p-harmonic measure* of ∂B_R with respect to $\mathcal{A}_{r,R}$. The p−*capacity* of the annulus is defined by:

$$Cap_p \mathcal{A}_{r,R} := \int_{\mathcal{A}_{r,R}} |\nabla w_{r,R}|^p. \tag{1.30}$$

We say that M^n is p−*parabolic* if, and only if

$$\lim_{R\to\infty} Cap_p(r, R) = 0.$$

In what follows we enunciate Proposition 4.1 of [62], where Troyanov provides an equivalence to the $p-$parabolicity with the possibility of aproximate the function 1 by functions with compact support and small $p-$energy. This equivalence will be an important tool for our results.

Proposition 1.4.1 *A manifold M is p-parabolic if, and only if, there exists a sequence of functions $u_j \in C_0^1(M)$ such that $0 \leq u_j \leq 1$, $u_j \to 1$ uniformly on every compact subset of M and*

$$\int_M |\nabla u_j|^p dM \to 0.$$

In addition, in Theorem 4.2 of the same paper, Troyanov proved that a manifold is $p-$parabolic provided there is no non-constant positive $p-$superharmonic function. This last equivalence is the most known definition about $p-$parabolicity.

1.4.1 A Liouvile type result for $p-$parabolic manifolds

Results of parabolicity have been studied for many authors such as Grigoryan [34] and Pigola, Rigoli, and Setti [54]. In [61] Schoen and Yau proved that if M is a oriented, complete, non-compact manifold with non-negative Ricci curvature and $u : M \to \mathbb{R}$ is a non-negative, integrable, smooth function such that

$$u\Delta u \geq \frac{1}{2}|\nabla u|^2 \text{ on } M, \tag{1.31}$$

then u is identically zero. Recalling that the class of p-parabolic manifolds contain the compact ones, we proved a Schoen-Yau type theorem for the complementary part of the manifolds that satisfies (1.31) as we present below in Theorem 1.4.3. It is worth to emphasize that we could drop out some hipothesis in cmparison to the Schoen and Yau result. For the proof we were inspired in the ideas of Alías and Palmer [12].

Lemma 1.4.2 *Let M be a Riemannian manifold and let $u \in C^2(M)$ be a function that satisfies*

$$u\Delta_p u \geq 0$$

on M. Then for $0 < r < R$

$$\int_{B_r} u\Delta_p u \leq (p-1)^{p-1} Cap_p(r, R) \sup_{B_R} u^p,$$

where $1 < p < \infty$.

Proof. We consider $\xi \in C_0^\infty(B_R)$. Then,

$$\begin{aligned} 0 &= \int_{B_R} \mathrm{Div}(\xi^\alpha u|\nabla u|^{p-2}\nabla u)dB_R \\ &= \int_{B_R} \xi^\alpha u\Delta_p u + \xi^\alpha\langle\nabla u,\nabla u\rangle|\nabla u|^{p-2} + \alpha\xi^{\alpha-1}u|\nabla u|^{p-2}\langle\nabla\xi,\nabla u\rangle. \end{aligned}$$

This implies

$$\begin{aligned} \int_{B_R} \xi^\alpha u\Delta_p u + \xi^\alpha|\nabla u|^p &= -\alpha\int_{B_R}\xi^{\alpha-1}u\langle\nabla\xi,|\nabla u|^{p-2}\nabla u\rangle \\ &\leq \alpha\int_{B_R}|\xi^{\alpha-1}u||\nabla u|^{p-2}\langle\nabla\xi,\nabla u\rangle \\ &\leq \alpha\int_{B_R}|\xi^{\alpha-1}u|\nabla u|^{p-1}|\nabla\xi|. \end{aligned}$$

Using Yang inequality, we obtain

$$\int_{B_R}\xi^\alpha u\Delta_p u + \xi^\alpha|\nabla u|^p \leq \frac{\alpha}{p\beta^p}\int_{B_R}u^p|\nabla\xi|^p + \frac{\alpha\beta^q}{q}\int_{B_R}\xi^{(\alpha-1)q}|\nabla u|^p$$

Hence

$$\begin{aligned} \int_{B_R}\xi^\alpha u\Delta_p u &\leq \frac{\alpha}{p\beta^p}\int_{B_R}u^p|\nabla\xi|^p + \int_{B_R}|\nabla u|^p\left(\frac{\alpha\beta^q}{q}\xi^{(\alpha-1)q} - \xi^\alpha\right) \\ &= \frac{\alpha}{p\beta^p}\int_{B_R}u^p|\nabla\xi|^p + \int_{B_R}|\nabla u|^p\xi^\alpha\left(\frac{\alpha\beta^q}{q}\xi^{\alpha(q-1)-q} - 1\right). \quad (1.32) \end{aligned}$$

Taking $\alpha = p$ and $\beta = (p-1)^{\frac{1-p}{p}}$ the last term of (1.32) vanishes. Thus,

$$\int_{B_R}\xi^\alpha u\Delta_p u \leq (p-1)^{(p-1)}\int_{B_R}u^p|\nabla\xi|^p \leq (p-1)^{(p-1)}\sup_{B_R}u^p\int_{B_R}|\nabla\xi|^p.$$

Define ξ by

$$\xi(x) = \begin{cases} 1, & x \in \overline{B_r} \\ 1 - w_{r,R}, & x \in \mathcal{A}_{r,R} \end{cases}$$

Note that ξ is not smooth, but it can be approximated by a smooth function. So,

$$\int_{B_r} u\Delta_p u \leq (p-1)^{p-1}Cap_p(r,R)\sup_{B_R}u^p$$

as desired. ■

Theorem 1.4.3 *Let M be a $p-$parabolic Riemannian manifold. If $u \in C^2(M)$ is such that $\sup_{B_R} u^p < \infty$ satisfying*

$$u\Delta_p u \geq \alpha|\nabla u|^p$$

for some $\alpha > 0$ and $1 < p < \infty$, then u must be constant on B_r.

Proof. We consider $0 < r < R$. Since $u\Delta_p u \geq \alpha|\nabla u|^p \geq 0$ we have by Lemma 1.4.2,

$$\alpha \int_{B_r} |\nabla u|^p \leq \int_{B_r} \leq \alpha(p-1)^{p-1} Cap_p(r, R) \sup_{B_R} u^p.$$

Taking $R \to \infty$ and using that M is $p-$parabolic and $\sup_{B_R} u^p < \infty$, we obtain

$$\int_{B_r} |\nabla u|^p = 0.$$

Therefore, u is constant on B_r. ■

1.4.2 A p-parabolicity criterium for hypersurfaces in GRW

A GRW spacetime is *spatially parabolic covered* if its universal Lorentzian covering is spatially parabolic. Notice that if $\tilde{M}$ is the universal Riemannian cover of the fiber M, then $-I \times_\rho \tilde{M}$ is the universal Lorentzian covering of of $-I \times_\rho M$.

The next result (Theorem 3 of [57]) provides some conditions that transmit the parabolicity of the fiber to the hypersurface.

Theorem 1.4.4 *Let $\psi : \Sigma^n \to -I \times_\rho M^n$ be a complete hypersurface in a spatially parabolic GRW spacetime. If the hyperbolic angle of Σ^n is bounded and the warping function on Σ^n satisfies:*

(i) $\sup \rho < \infty$ *and*

(ii) $\inf \rho > 0$;

then, Σ^n is parabolic.

Inspired in this result we were encouraged to make a $p-$parabolicity criterium for hypersurfaces immersed in GRW. For this let us present terminology of Rirmannian manifold approached in [62].

Definition 1.4.5 *A Riemannian manifold has bounded geometry if it is has a positive injectivity radius and its Ricci curvature is bounded from bellow.*

Example 1.4.6 *Compact manifolds, and any Riemann covering space of a manifold with bounded geometry have also bounded geometry. In particular, if M is a manifold with bounded geometry, then M is necessarily complete.*

Recall that given two Riemannian manifolds (M, g) and (M', g'), a difeomorphism $\varphi : M \to M'$ is a *quasi-isometry* if there exists a constant $c \geq 1$ such that

$$c^{-1}|v|_g \leq |d\varphi(v)|_{g'} \leq c|v|,$$

for all $v \in T_pM$, $p \in M$. Item (D) of Theorem 6.2 of [62] that claims that if two manifolds are quasi-isometric and have bounded geometry, then one of them is $p-$parabolic if so is the another one. Then it is sufficient to prove that Σ^n and M^n are quasi-isometric in order to obtain the below result. But this quasi-isometri is assured by hypothesis and Lemma 4.1 of [56].

Theorem 1.4.7 *Let $\psi : \Sigma \to -I \times_\rho M^n$ be a complete hypersurface immersed in a GRW spacetime whose fiber as bounded geometry. If the hyperbolic angle of Σ^n is bounded and the warping function on Σ^n satisfies:*

(i) $\sup \rho < \infty$ *and*

(ii) $\inf \rho > 0$;

then, Σ^n is $p-$parabolic if, and only if, so is M^n.

Corollary 1.4.8 *Let $\psi : \Sigma^n \to -I \times_\rho M^n$ be a spacelike compact hypersurface such that* $\inf \rho > 0$ *on it. Suppose M^n has bounded geometry. Then M^n is p-parabolic for every $(p \geq 1)$.*

Proof. Since compact manifolds are $p-$parabolic for $p \geq 1$ (see [62]) the result follows from Theorem 1.4.11. ■

Corollary 1.4.9 *Let $\psi : \Sigma^n \to \mathbb{L}^{n+1}$, where $\mathbb{L}^{n+1}$ is the Lorentz-Minkowski space, be a spacelike hypersurface with bounded geometry and bounded hyperbolic angle. Then Σ^n is $p-$parabolic for all $p \geq n$.*

Proof. Since $\mathbb{L}^{n+k+1} \equiv \mathbb{R}^{n+k} \times \mathbb{R}_1$ and $\mathbb{R}^{n+k}$ is p-parabolic for all $p \geq n+k$, it follows from Theorem 1.4.7. ■

1.4.3 A p-parabolicity criterium for submanifolds in SSST

In what follows we enunciate Theorem 1 of [31] that provides a criterium for a hypersurface immersed in a spatially parabolic SSST to be parabolic.

Theorem 1.4.10 *Let $M^n \times_\rho \mathbb{R}_1$ be a spatially parabolic SSST. If $\psi : \Sigma^n \to M^n \times_\rho \mathbb{R}_1$ is a complete spacelike hypersurface such that the function $\eta = \dfrac{\langle N, K\rangle}{\rho}$ is bounded on it, then M^n is complete and Σ^n is parabolic.*

As in Theorem 1.4.7, we just need to prove the existence of a quasi-isometry between Σ^n and M^{n+k} to obtain the following theorem:

Theorem 1.4.11 *Let $\psi : \Sigma^n \to M^{n+k} \times_\rho \mathbb{R}_1$ be a spacelike hypersurface such that the function $\rho^2|\nabla h|^2$ is bounded on it. Suppose that Σ^n and M^n have bounded geometry. Then Σ^n is p-parabolic if and only if M is p-parabolic.*

Proof. Consider $\pi : \pi_M \circ \psi : \Sigma^n \to M^n$, where $\pi_M : M^{n+k} \times_\rho \mathbb{R}_1 \to M$ is the canonical projection on the base M. For any tangent vector $v \in \mathcal{X}(\Sigma)$, using the Cauchy-Schwarz inequality we have:

$$\begin{aligned} \langle v, v\rangle &= \langle \pi_* v, \pi_* v\rangle_M - \rho^2 \langle h_* v, h_* v\rangle_{\mathbb{R}} \\ &\geq \langle \pi_* v, \pi_* v\rangle_M - \rho^2 |\nabla h|^2 \langle v, v\rangle, \end{aligned}$$

then

$$\langle v, v\rangle \geq \frac{1}{1+\rho^2|\nabla h|^2} \langle \pi_* v, \pi_* v\rangle_M \geq c^{-1} \langle \pi_* v, \pi_* v\rangle_M,$$

where $c = \sup_\Sigma (1 + \rho^2 |\nabla h|^2)$. On the other hand,

$$\langle v, v\rangle = \langle \pi_* v, \pi_* v\rangle_M - \rho^2 \langle h_* v, h_* v\rangle_{\mathbb{R}} \leq c \langle \pi_* v, \pi_* v\rangle_M.$$

Therefore Σ^n and M^n are quasi-isometric. ■ Similarlly to Corollary 1.4.8 we have:

Corollary 1.4.12 *Let $\psi : \Sigma^n \to M^{n+k} \times_\rho \mathbb{R}_1$ be a spacelike compact submanifold. Suppose M^n has bounded geometry. Then M^n is p-parabolic for every $(p \geq 1)$.*

Observe that Corollary 1.4.9 may also be obtained from Theorem 1.4.11.

1.5 Stability of hypersurfaces in spacetimes

Questions about stability were introduced in the Riemannian setting by Barbosa and do Carmo in [19] and [20]. In this last paper, they proved that spheres are the only stable critical points for the area functional. For the Lorentzian setting, the concept of stability was introduced by Barros, Brasil and Caminha in [18]. In [22] the authors proved that CMC spacelike hypersurfaces are critical points of volume-preserving variations.

Let $\overline{M}^{n+1}$ be a spacetime and $x : \Sigma^n \to \overline{M}^{n+1}$ be a spacelike hypersurface and N its future direct Gauss map.

A variation of x is a smooth map $X : \Sigma^n \times (-\epsilon, \epsilon) \to \overline{M}^{n+1}$ satisfying:

(i) For $t \in (-\epsilon, \epsilon)$, the map $X_t : \Sigma \to \overline{M}$ given by $X_t(p) = X(p,t)$ is a immersion such that $X_0 = x$;

(ii) If $\partial\Sigma \neq \emptyset$, $X_t|_{\partial\Sigma} = x|_{\partial\Sigma}$ for all $t \in (-\epsilon, \epsilon)$.

The variational vector field associated to the variation X is the vector field $\frac{\partial X}{\partial t} = X\left(\partial_t\right)$. Setting $g = -\langle \frac{\partial X}{\partial t}, N\rangle$, we get

$$\frac{\partial X}{\partial t}|_\Sigma = gN + \left(\frac{\partial X}{\partial t}\right)^\top .$$

We can associate the area functional for the variation X with

$$A(t) = A(X_t) = \int_\Sigma d\Sigma_t,$$

and the balance of the volume functional with

$$V(t) = \int_{\Sigma\times[0,t]} X^*(d\overline{M}),$$

where $d\Sigma_t$ is the volume element of Σ with the metric induced by X_t, and $d\overline{M}$ is the volume element of $\overline{M}^{n+1}$. We say that a variation is volume preserving if $V(t) = V(0)$ for all $t \in (-\epsilon, \epsilon)$. We have the well known first variation formulae for area and volume

$$A'(0) = n\int_\Sigma Hg d\Sigma \tag{1.33}$$

$$V'(0) = \int_\Sigma g d\Sigma. \tag{1.34}$$

It is well known that the condition

$$\int_\Sigma f d\Sigma = 0,\ f \in C^\infty(\Sigma),\ f = 0 \text{ on } \partial\Sigma$$

is necessary and sufficient for the existence of a volume preserving variation X whose variational field is $\xi = fN$. Also, it is known that Σ^n has constant mean curvature if, and only if, $A'(0) = 0$ for all volume preserving variations (see [22]).

We wish to extend our analysis about the critical points of the area functional for all variations (not just the ones which preserve volume). In this setting, we define the Jacobi functional of a variation X given by

$$J_\lambda(t) = A(t) - n\lambda V(t).$$

Proposition 2.3 in [20] claims that X has constant mean curvature H if and only if $J'_H(0) = 0$ for all variations.

Among all critical points of the Jacobi functional, we consider those which are local maximizing. In this case, the second derivative $J''_H(0)$ must be non-positive. The following Proposition found in [18] gives an expression for $J''_H(0)$.

Proposition 1.5.1 *Let $\overline{M}^{n+1}$ be a Lorentzian manifold and $x : \Sigma^n \to \overline{M}^{n+1}$ be a closed spacelike hypersurface having constant mean curvature H. If $X : \Sigma^n \times (-\epsilon, \epsilon) \to \overline{M}^{n+1}$ is a variation of x, then*

$$J''_H(0) = \int_\Sigma f\Delta f - (\overline{\mathrm{Ric}}(N) + |A|^2)f^2 d\Sigma, \tag{1.35}$$

where $\overline{\mathrm{Ric}}$ is the Ricci tensor of $\overline{M}^{n+1}$, A is the shape operator of the immersion and $|A|$ is the Hilbert-Schmidt norm of A and f is the normal component of the variational field related to the variation.

The previously stated results and definitions led us the stability definition:

Definition 1.5.2 *Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be an immersion with constant mean curvature H. We say that ψ is stable if for every function $f \in C_0^\infty(\Sigma)$ the following inequality holds:*

$$\int_\Sigma f\Delta f - (\overline{\mathrm{Ric}}(N) + |A|^2)f^2 d\Sigma \leq 0. \tag{1.36}$$

Remark 1.5.3 *Along the manuscript we will set*

$$Q = \overline{\mathrm{Ric}}(N) + |A|^2.$$

If $Q \geq 0$, then it is straightforward that the stability inequality (1.36) holds true.

The next lemma appears in [45] Lemma 1. For sake of completeness we will give a proof of it.

Lemma 1.5.4 *Let $\Sigma^n \to \overline{M}^{n+1}$ be a CMC spacelike hypersurface, where $\overline{M}^{n+1}$ is a Lorentzian manifold. If there exists a positive smooth function $u \in C^\infty(\Sigma)$ such that*

$$\Delta u - (\overline{\mathrm{Ric}}(N) + |A|^2)u \leq 0, \tag{1.37}$$

then Σ^n is stable.

Proof. Let $\varphi \in C_0^\infty(\Sigma)$ and set $Q = \overline{\mathrm{Ric}}(N) + |A|^2$. We can choose $\eta \in C_0^\infty(\Sigma)$ such that $\varphi = \eta u$. Hence,

$$\begin{aligned}
\int_\Sigma \varphi\Delta\varphi - Q\varphi^2 d\Sigma &= \int_\Sigma \eta u \Delta(\eta u) - Q\eta^2 u^2 d\Sigma \\
&= \int_\Sigma \eta u(\eta\Delta u + u\Delta\eta + 2\langle\nabla u, \nabla\eta\rangle) - Q\eta^2 u^2 d\Sigma \\
&= \int_\Sigma \eta^2 u(\Delta u - Qu) + \eta u^2\Delta\eta + 2\eta u\langle\nabla u, \nabla\eta\rangle d\Sigma \\
&= \int_\Sigma \eta^2 u(\Delta u - Qu) + \eta u^2\Delta\eta + \frac{1}{2}\langle\nabla u^2, \nabla\eta^2\rangle d\Sigma.
\end{aligned}$$

Observing that

$$\begin{aligned}
\mathrm{Div}(u^2\nabla\eta^2) &= \langle\nabla u^2, \nabla\eta^2\rangle + u^2\Delta\eta^2 \\
&= \langle\nabla u^2, \nabla\eta^2\rangle + 2\eta u^2\Delta\eta + 2u^2|\nabla\eta|^2,
\end{aligned}$$

and using the divergence Theorem, we can replace the last term of (1.41) and use hypothesis in order to get:

$$\int_\Sigma \varphi\Delta\varphi - Q\varphi^2 d\Sigma \leq \int_\Sigma \eta u^2\Delta\eta - \eta u^2\Delta\eta - u^2|\nabla\eta|^2 d\Sigma \leq 0.$$

Therefore, Σ^n is stable. ■

Proposition 1.5.5 *Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a CMC parabolic spacelike hypersurface, where $\overline{M}^{n+1}$ is a Lorentzian manifold. Then we can use any bounded function $f \in C^\infty(M)$ in the stability criteria (1.36).*

Proof. Let u_j be the sequence obtained from Proposition 1.4.1 and $f_j = fu_j$. Then we can apply f_j in the stability inequality,

$$\begin{aligned}
0 &\geq \int_\Sigma f_j\Delta(f_j) - Qf_j^2 d\Sigma \\
&= \int_\Sigma ff_j\Delta u_j + f_j u_j\Delta f + 2u_j f\langle\nabla u_j, \nabla f\rangle - Qf_j^2 d\Sigma.
\end{aligned} \tag{1.38}$$

By the Green Theorem:

$$\begin{aligned} 0 &= \int_\Sigma \mathrm{Div}(f f_j \nabla u_j) d\Sigma \\ &= \int_\Sigma f f_j \Delta u_j d\Sigma + \int_\Sigma \langle \nabla(f f_j), \nabla u_j\rangle d\Sigma. \end{aligned}$$

So, replacing the first term of (1.38),

$$\begin{aligned} 0 &\geq \int_\Sigma -\langle \nabla(f f_j), \nabla u_j\rangle + f_j u_j \Delta f + 2 f_j \langle \nabla u_j, \nabla f\rangle - Q f_j^2 d\Sigma \\ &= \int_\Sigma -f^2 \langle \nabla u_j, \nabla u_j\rangle + f_j u_j \Delta f - Q f_j^2 d\Sigma. \end{aligned} \tag{1.39}$$

Since f is bounded, $\int_\Sigma |\nabla u_j|^2 d\Sigma \to 0$ and $u_j \to 1$, taking the limit in (1.39), the first term goes to 0 and by the Dominated Convergence Theorem it provides:

$$\int_\Sigma f\Delta f - Q f^2 d\Sigma \leq 0,$$

as desired. ■

1.5.1 Stability criterium for CMC hypersurfaces in GRW

The following result is a generalization of Theorem 3.3 in [41], which give us a criteria for a CMC hypersurface in a GRW space to be stable.

Theorem 1.5.6 *Let $\Sigma^n \to -I \times_\rho M^n$ be a CMC hypersurface.*

(i) If $H\rho' + \rho''\langle N, \partial_t\rangle \geq 0$, then Σ^n is stable;

(ii) If Σ^n is parabolic, $\rho\langle N, \partial_t\rangle$ is bounded and

$$H\rho' + \rho''\langle N, \partial_t\rangle \leq 0,$$

then Σ^n is stable if, and only if, $H\rho' + \rho''\langle N, \partial_t\rangle = 0$;

(iii) If Σ^n is parabolic, $\rho\langle N, \partial_t\rangle$ is bounded and

$$H\rho' + \rho''\langle N, \partial_t\rangle < 0,$$

then Σ^n must not be stable.

Proof. For item 1, let $f = -\rho\langle N, \partial_t\rangle$. Using Lemma 1.1.2 and the hypothesis,

$$\Delta f - Q f = -n(H\rho' + \rho''\langle N, \partial_t\rangle) \leq 0.$$

Therefore, since f is positive, Lemma 1.5.4 provides that Σ^n is stable.

For item 2, since $\rho\langle N, \partial_t\rangle$ is bounded and Σ^n is parabolic, we can use $\rho\langle N, \partial_t\rangle$ in the stability criteria (Proposition 1.5.5). Making use of Lemma 1.1.2, we get

$$0 \geq \int_\Sigma f\Delta f - Qf^2 d\Sigma = \int_\Sigma \rho\langle N, \partial_t\rangle n(H\rho' + \rho''\langle N, \partial_t\rangle)d\Sigma. \tag{1.40}$$

From hypothesis, we conclude that $H\rho' + \rho''\langle N, \partial_t\rangle = 0$. The converse follows from item 1.

Finally, item 3 follows from (1.40). ■

1.5.2 Stability criterium for CMC hypersurfaces in SSST

The following Lemma provides us the stability as a tool to work with CMC hypersurfaces in standard static spacetimes. Notice that this fact was already proved in [45], Corollary 7.

Lemma 1.5.7 *Let $\psi : \Sigma^n \to M^n \times_\rho \mathbb{R}_1$ be a CMC hypersurface. Then Σ^n is stable.*

Proof. Let $\varphi \in C_0^\infty(\Sigma)$. We can choose $\eta \in C_0^\infty(\Sigma)$ such that $\varphi = \eta\Theta$. Hence,

$$\begin{aligned}
\int_\Sigma \varphi\Delta\varphi - Q\varphi^2 d\Sigma &= \int_\Sigma \eta\Theta\Delta(\eta\Theta) - Q\eta^2\Theta^2 d\Sigma \\
&= \int_\Sigma \eta\Theta(\eta\Delta\Theta + \Theta\Delta\eta + 2\langle\nabla\Theta, \nabla\eta\rangle) - Q\eta^2\Theta^2 d\Sigma \\
&= \int_\Sigma \eta^2\Theta(\Delta\Theta - Q\Theta) + \eta\Theta^2\Delta\eta + 2\eta\Theta\langle\nabla\Theta, \nabla\eta\rangle d\Sigma \\
&= \int_\Sigma \eta^2\Theta(\Delta\Theta - Q\Theta) + \eta\Theta^2\Delta\eta + \frac{1}{2}\langle\nabla\Theta^2, \nabla\eta^2\rangle d\Sigma. \quad (1.41)
\end{aligned}$$

Observing that

$$\begin{aligned}
\mathrm{Div}(\Theta^2\nabla\eta^2) &= \langle\nabla\Theta^2, \nabla\eta^2\rangle + \Theta^2\Delta\eta^2 \\
&= \langle\nabla\Theta^2, \nabla\eta^2\rangle + 2\eta\Theta^2\Delta\eta + 2\Theta^2|\nabla\eta|^2,
\end{aligned}$$

and using the Divergence Theorem, we can replace in (1.41) and use equation (1.1.3) in order to get:

$$\int_\Sigma \varphi\Delta\varphi - Q\varphi^2 d\Sigma \leq \int_\Sigma \eta\Theta^2\Delta\eta - \eta\Theta^2\Delta\eta - \Theta^2|\nabla\eta|^2 d\Sigma \leq 0.$$

Therefore, Σ is stable. ■

1.6 Some auxiliary lemmas

In what follows, we present an algebric lemma that will be helpfull for the proof of our results.

Lemma 1.6.1 *For a traceless symmetric $n \times n$ real matrix Φ and a vector v in the same dimension. We have that*

$$\langle \Phi v, v\rangle \leq \left(\frac{n-1}{n}\right)^{\frac{1}{2}} |\Phi||v|^2.$$

The equality holds if, and only if, $\Phi = 0$ or $v = 0$ or $v = \lambda e_1$ where e_1 is the eigenvector associated to the biggest eigenvalue, also in absolute value, α_1 of Φ. In particular, the eigenspace associated to this eigenvalue is unidimensional and we can write Φ as

$$\Phi = |\Phi| \begin{pmatrix} \sqrt{\frac{n-1}{n}} & 0 \\ 0 & -\sqrt{\frac{1}{n(n-1)}} I_{n-1} \end{pmatrix}.$$

Proof. Let us consider $\{e_j\}$ an orthonormal frame diagonalizing Φ, that is, $\Phi e_i = \alpha_i e_i$ for some $\alpha_i' s$ and $v = \sum_i \lambda_i e_i$. Initially let us assume without loss of generality that $|\Phi| = 1$ assume also that α_1 is the biggest eigenvalue of Φ in absolute value and positive when possible. Denote $|v|_1^2 = |v|^2 - v_1^2$ and $Q_1^2 = |\Phi|^2 - \alpha_1^2$.

Observe that using that Φ is traceless and the Cauchy-Schwarz inequality we obtain

$$|\alpha_1| = |\sum_{i=2}^{n} \alpha_i| \leq (n-1)^{\frac{1}{2}} Q_1 \tag{1.42}$$

Equivalently

$$\alpha_1^2 \leq (n-1)(1-\alpha_1^2) \Longleftrightarrow \alpha_1^2 \leq \frac{n-1}{n} \tag{1.43}$$

Since $\alpha_1^2 \geq \frac{1}{n}$ we get

$$Q_1^2 = 1 - \alpha_1^2 \leq 1 - \frac{1}{n} = \frac{n-1}{n}$$

Therefore

$$\begin{aligned} \langle \Phi v, v\rangle &= \sum_i \alpha_i v_i^2 \\ &= \alpha_1 v_1^2 + \sum_{i\neq 1} \alpha_i v_i^2 \\ &\leq \alpha_1 v_1^2 + Q_1 |v|_1^2 \\ &\leq \sqrt{\frac{n-1}{n}} v_1^2 + \sqrt{\frac{n-1}{n}} |v|_1^2 \qquad (1.44) \\ &\leq \sqrt{\frac{n-1}{n}} |v|^2. \end{aligned}$$

(1.45)

For the equality suppose $|\Phi| = 1$ and $n > 2$. Assume $\langle v, v_j\rangle \neq 0$, by the inequality (1.44) we must have $\alpha_1 = \sqrt{\frac{n-1}{n}}$ and therefore occurs the equality in (1.42) which occurs if, and only if, all $\alpha_i' s$ otherwise than α_j are equal. Therefore if $v \notin e_j^{\perp}$ then

$$\Phi = \pm \begin{pmatrix} \sqrt{\frac{n-1}{n}} & 0 \\ 0 & -\sqrt{\frac{1}{n(n-1)}} I_{n-1} \end{pmatrix}$$

In particular V_j is unidimensional. Moreover $v \in \langle e_1\rangle$, otherwise $|v|_1^2 > 0$ then $Q_1 = \sqrt{\frac{n-1}{n}}$ that means $\alpha_1^2 = \frac{1}{n}$ an absurd since $\alpha_1^2 = \frac{n-1}{n}$.

Affirmation: $v \in e_1^{\perp}$ does not occur unless $v = 0$. Notice that v is orthogonal to any space associated the biggest eigenvalue of Φ in absolute value, otherwise we would proceed as before. In this case we have $|v|_1^2 > 0$ and therefore $Q_1 = \sqrt{\frac{n-1}{n}}$ it means $\alpha_1^2 = \frac{1}{n}$ and as a consequence $\alpha_i^2 = \frac{1}{n}$ what is a contradiction since v is orthogonal to all eigenvectors associated to the eigenvalues with maximum absolute value. ■

The next result is an auxiliary lemma which is an extension of Hopf's theorem on a complete Riemannian manifold due to Yau in [65]. In what follows, $\mathcal{L}^1(\Sigma)$ denotes the space of Lebesgue integrable functions on Σ^n.

Lemma 1.6.2 *Let u be a smooth function defined on a complete Riemannian manifold Σ^n, such that Δu does not change sign on Σ^n. If $|\nabla u| \in \mathcal{L}^1(\Sigma)$, then Δu vanishes identically on Σ^n.*

The following Lemma was obtained in [53] by Pigola, Rigoli and Setti.

Lemma 1.6.3 *Let M be a $p-$ parabolic manifold. If f satisfies $\Delta_p f \geq 0$ and $|\nabla f| \in L^p(M)$, then f is constant.*

Chapter 2

Rigidity of parabolic and stable CMC hypersurfaces in GRW spacetimes

The goal of this chapter is to study the support function $\rho\langle N, \partial_t\rangle$ of a hypersurface immersed in GRW spacetime $-I \times_\rho M^n$, where ρ is the warping function, N is the normal vector field to the hypersurface and ∂_t is the natural unit timelike vector field. The vector fields N and ∂_t are two families of instantaneous observers and the quantity

$$v = \frac{N^*}{\cosh\theta},$$

where N^* is the projection on the fiber and θ is the hyperbolic angle between N and ∂_t, denotes the velocity that ∂_t measures for N. Along this chapter we will suppose the support function is bounded. Physically, this suposition means that the relative speed function $|v| = \tanh\theta$ does not approach to the light speed in the vacuum.

In order to make a study about the support function of parabolic and stable hypersurfaces, we use some cut-off functions obtained from the parabolicity (see [62]) joint with the stability operator.

2.1 Hypersurfaces in GRW spacetimes

In what follows, we present a result that gives us a criterium for a hypersurface immersed in $-I \times_\rho M^n$ to be a slice. This result is related with Theorem 1.1 in [7], where the authors proved that given a GRW spatially closed spacetime whose fiber has sectional curvature positive, $-\log\rho$ is convex and $H\rho' \leq 0$, then every hypersurface immersed must be a totally geodesic slice.

Proposition 2.1.1 *Let $\psi : \Sigma^n \to -I \times_\rho M^n$ be a CMC parabolic spacelike hypersurface. If $(\log\rho)'' \leq 0$ and $\rho' H \leq 0$, then ρ is constant on Σ^n. In addition, if $-I \times_\rho M^n$ is a proper GRW spacetime, then Σ^n is contained in a slice.*

Proof. From (1.8) and the hypothesis,

$$\Delta\rho = -n\frac{\rho'^2}{\rho} + \rho(\log\rho)''|\nabla h|^2 - n\rho' H\langle N, \partial_t\rangle \leq 0.$$

Since Σ^n is parabolic, we conclude that ρ is constant on Σ^n. If $-I \times_\rho M^n$ is proper, ρ' does not vanish in a non-degenerated interval, therefore Σ^n must be a slice. ■

Theorem 2.1.2 *Let $\psi : \Sigma^n \to -I \times_\rho M^n$ be a CMC parabolic spacelike hypersurface. Suppose that Σ^n is stable, the support function $\rho\langle N, \partial_t\rangle$ is bounded,*

$$\int_\Sigma (\overline{\mathrm{Ric}}(N) + |A|^2)\rho^4\langle N, \partial_t\rangle^4 d\Sigma \geq 0, \tag{2.1}$$

and

$$\rho' H + \rho''\langle N, \partial_t\rangle \leq 0. \tag{2.2}$$

Then the support function is constant. In addition, if $K_G \neq 0$, then Σ^n is contained in a slice $M \times \{t_0\}$ such that $\rho'(t_0) \neq 0$.

Proof. Item 2 of Theorem 1.5.6 yields $\rho' H + \rho''\langle N, \partial_t\rangle = 0$. Setting $f = \rho^2\langle N, \partial_t\rangle^2$ and using Lemma 1.1.2 we have that

$$\Delta f = 2fQ + 2f|\nabla\rho\langle N, \partial_t\rangle|^2. \tag{2.3}$$

Using Proposition 1.5.5 and equation (1.36) we obtain:

$$\begin{aligned} 0 &\geq \int_\Sigma f\Delta f - Qf^2 d\Sigma \\ &= \int_\Sigma f^2 Q + 2f|\nabla(\rho\langle N, \partial_t\rangle)|^2 d\Sigma. \end{aligned} \tag{2.4}$$

Using hypothesis (2.1), we conclude that

$$|\nabla(\rho\langle N, \partial_t\rangle)| \equiv 0.$$

Therefore, Σ^n has constant support function.

For the second claim of the theorem, suppose that $K_G \neq 0$. In this case, A is invertible. Using (1.9) we get

$$0 = \nabla(\rho\langle N, \partial_t\rangle) = \rho A \nabla h.$$

Then we conclude that $\nabla h = 0$, therefore h is constant and Σ^n is contained in a slice $M \times \{t_0\}$. Recalling that the shape operator of a slice in $-I \times_\rho M^n$ is given by $A = -\frac{\rho'}{\rho}\mathbb{I}$, we conclude that $\rho'(t_0) \neq 0$. ■

In Theorem 2.1.2 we assumed condition (2.1) which is weaker than TCC. In the next theorem we assume TCC and obtain a stronger result, since from equation (1.9) totally geodesic hypersurfaces have constant support function.

Theorem 2.1.3 *Let $\psi : \Sigma^n \to -I \times_\rho M^n$ be a CMC parabolic spacelike hypersurface. Suppose that TCC holds on the ambient, the support function $\rho\langle N, \partial_t\rangle$ is bounded and*

$$\rho' H + \rho''\langle N, \partial_t\rangle \leq 0.$$

Then Σ^n is totally geodesic and consequently the support function is constant.

Proof. Since TCC holds, from Remark 1.5.3, Σ^n is stable. Following the same reasoning as in Theorem 2.1.2, we obtain $\rho' H + \rho''\langle N, \partial_t\rangle = 0$ and from equation (2.4) we get

$$0 \geq \int_\Sigma |A|^2 f^2 + 2|\nabla(\rho\langle N, \partial_t\rangle)|^2 d\Sigma \geq 0.$$

Hence, Σ^n is totally geodesic. The second claim follows from (1.9). ■

Corollary 2.1.4 *Let $\psi : \Sigma^n \to -I \times_\rho M^n$ be a CMC spacelike hypersurface immersed into a spatially parabolic GRW spacetime. Suppose that TCC holds on the ambient, the hyperbolic angle between the normal N and ∂_t is bounded and*

(i) $\sup \rho < \infty$;

(ii) $\inf \rho > 0$ *and*

(iii) $\rho' H + \rho''\langle N, \partial_t\rangle \leq 0$,

then Σ^n is totally geodesic and consequently the support function is constant.

Proof. Items (1) and (2) jointly with the fact that the hyperbolic angle is bounded provides (Theorem 1.4.4) that Σ^n is parabolic. The result follows from Theorem 2.1.3. ■

2.2 Hypersurfaces in Lorentzian products

In this section, we present some corollaries from the results presented in the previous section for CMC hypersurfaces immersed in Lorentzian products $-I \times M^n$. Recall that from Remark 1.5.3, every CMC hypersurface in this ambient is stable. Thus, the stability condition can be dropped out of the hypothesis in this section. In addition, from Theorem 1.4.4 it is enough to assume that the hyperbolic angle is bounded and that the GRW is spatially parabolic to guarantee the parabolicity of the hypersurface.

Corollary 2.2.1 *Let $\psi : \Sigma^n \to -I \times M^n$ be a CMC spacelike hypersurface in a spatially parabolic Lorentzian product. Suppose that the hyperbolic angle between N and ∂_t is bounded and*

$$\int_{\Sigma} (\mathrm{Ric}^M(N^*) + |A|^2)\langle N, \partial_t\rangle^4 d\Sigma \geq 0. \tag{2.5}$$

Then Σ^n is a constant angle hypersurface and $K_G = 0$.

Proof. From Theorem 1.4.4 we get that Σ^n is parabolic. Equation (1.2) provides that

$$\overline{\mathrm{Ric}}(N) = \mathrm{Ric}^M(N^*).$$

Therefore the hypothesis of Theorem 2.1.2 are satisfied. Thus Σ^n is a constant angle hypersurface. If we had $K_G \neq 0$, the shape operator A would be invertible, and from (1.11) Σ^n would be a slice. But from (1.6), in the Lorentzian products the slices are totally geodesic, a contradiction with the assumption. Therefore $K_G = 0$. ■ A straightforward consequence of the above Theorem 2.1.3:

Corollary 2.2.2 *Let $\psi : \Sigma^2 \to \mathbb{L}^3$ be a CMC spacelike hypersurface with bounded hyperbolic angle. Then it must be a plane.*

The following corollary provides a complementary result related to item 2 of Theorem 3.7 of [40], where the authors studied complete CMC hypersurfaces in a Lorentzian product $-I \times M^n$ satisfying

$$|\nabla h|^2 \leq \frac{\alpha |A|^2}{(n-1)k^2},$$

where $0 < \alpha < 1$, and $K_M \geq -k^2$, where K_M is the setional curvature of the fiber. In this same work the authors exhibited an example that shows that this result cannot be extended for $\alpha = 1$. For this case we make an approach assuming that the hypersurface is parabolic, as follows:

Corollary 2.2.3 *Let $\psi : \Sigma^n \to -I \times M^n$ be a CMC parabolic spacelike hypersurface such that the sectional curvature of its Riemannian fiber M^n satisfies $K_M \geq -k^2$ for some constant k and*

$$|\nabla h|^2 \leq \frac{|A|^2}{(n-1)k^2}. \tag{2.6}$$

then Σ^n is a constant angle hypersurface and $K_G = 0$.

Proof. From a straightforward computation we have

$$\overline{\mathrm{Ric}}(N) = \mathrm{Ric}^M(N^*) \geq -k^2(n-1)|N^*|^2 = -k^2(n-1)|\nabla h|^2.$$

Using the hypothesis

$$\overline{\mathrm{Ric}}(N) + |A|^2 \geq -(n-1)k^2|\nabla h|^2 + |A|^2 \geq 0. \tag{2.7}$$

From Remark 1.5.3, Σ^n is stable and condition (2.5) is verified. Then the result follows from Corollary 2.2.1. ■

Corollary 2.2.4 *Let $\psi : \Sigma^n \to -I \times M^n$ be a CMC compact spacelike hypersurface. Suppose that*

$$\int_\Sigma (\mathrm{Ric}^M(N^*) + |A|^2)\langle N, \partial_t\rangle^4 d\Sigma \geq 0.$$

Then, Σ^n is contained in a slice of $-I \times M^n$.

Proof. From [62] compact manifolds are parabolic. Since Σ^n is compact, the hyperbolic angle between N and ∂_t is bounded. By Corollary 3.1.2 we conclude that Σ^n is a constant angle hypersurface. Furthermore, there exists a point $x \in \Sigma^n$ where h attains a maximum. Hence,

$$\nabla h(x) = 0.$$

Since

$$|\nabla h|^2 = -1 + \langle N, \partial_t\rangle^2,$$

and $\langle N, \partial_t\rangle$ is constant, we conclude that $|\nabla h| \equiv 0$, and Σ^n is contained in a slice. ■

Remark 2.2.5 *Example 6.0.6 shows that the compacity cannot be dropped out of the hypothesis in the above corollary.*

Corollary 2.2.6 *Let $\psi : \Sigma^n \to -I \times M^n$ be a CMC spacelike hypersurface immersed into a spatially parabolic GRW spacetime. Suppose that TCC holds on the ambient and the hyperbolic angle between N and ∂_t is bounded. Then Σ^n is totally geodesic and has constant angle. In addition, if there is a point $x_0 \in \Sigma^n$ such that $\mathrm{Ric}^M(x_0) > 0$, then Σ^n is contained in a slice.*

Proof. The first claim of the corollary above is a straight consequence of Theorem 2.1.3. For the second claim we observe that from (1.11)

$$\nabla\langle N, \partial_t\rangle = A\nabla h = 0.$$

Then, using Lemma 1.1.2 and equation (1.2),

$$\begin{aligned} 0 &= \Delta\langle N, \partial_t\rangle \\ &= \overline{\mathrm{Ric}}(N)\langle N, \partial_t\rangle \\ &= \mathrm{Ric}^M(N^*)\langle N, \partial_t\rangle. \end{aligned} \tag{2.8}$$

Since by hypothesis there is a point $x_0 \in \Sigma^n$ such that $\mathrm{Ric}^M(x_0) > 0$, then there is a neighborhood U of x_0 such that $\mathrm{Ric}^M > 0$ in U. Equation (2.8) yields $N^* \equiv 0$ in U. By the Unique Continuation Principle (see Theorem 1.8 of [39]) we conclude that $N^* \equiv 0$ in Σ^n, therefore Σ^n is contained in a slice. ∎

Corollary 2.2.7 *Let $\psi : \Sigma^2 \to \mathbb{L}^3$ be a complete spacelike surface with bounded geometry and bounded hyperbolic angle. Then Σ^2 must be a plane.*

Proof. Since Σ^2 has bounded geometry, from Corollary 1.4.9 we get that Σ^2 is parabolic. From Theorem 2.1.3 we conclude that Σ^2 must be totally geodesic, therefore maximal. From the well known Calabi-Berstein Theorem [55], it must be a plane. ∎

2.3 Surfaces in $-I \times M^2$

In this section, we discuss about CMC surfaces immersed in a spatially parabolic Lorentzian product space $-I \times M^2$. Before we present the corollaries, let us make a brief discussion about Gauss equation for surfaces in $-I \times M^2$.

The Gauss curvature of a spacelike surface Σ^2 in $-I \times M^2$ is described in terms of the shape operator A and the curvature of $-I \times M^2$ by the Gauss equation, which is given by

$$K_\Sigma = \overline{K} + K_G, \tag{2.9}$$

where $\overline{K}(p)$, $p \in \Sigma^2$ denotes the sectional curvature in $-I \times M^2$ of the tangent plane $d\psi_p(T_p(\Sigma))$, K_Σ stands for the Gauss curvature of Σ and K_G the Gauss-Kronecker curvature.

We also have the well known equation

$$|A|^2 = 4H^2 - 2K_G. \tag{2.10}$$

This equality joint with Corollary 2.2.1 provides this straight corollary:

Corollary 2.3.1 *Let $\psi : \Sigma^2 \to -I \times M^2$ be a maximal surface immersed into a spatially parabolic Lorentzian product. Suppose that the hyperbolic angle between N and ∂_t is bounded and*

$$\int_\Sigma (\overline{K}(N) + |A|^2)\langle N, \partial_t\rangle^4 d\Sigma \geq 0. \tag{2.11}$$

then Σ^2 is a totally geodesic surface and consequently the hyperbolic angle between N and ∂_t is constant.

In [3] A. L. Albujer and L. J. Alías proved that when the Riemannian surface M^2 has non-negative Gauss curvature, any complete maximal surface in $-I \times M^2$ must be totally geodesic. Besides, if M^2 is non-flat, the authors concluded that such a surface must be a slice. In what follows, we obtained a similar result for CMC complete surfaces. We can replace the maximality by boundeness on the hyperbolic angle and obtain the same thesis.

Corollary 2.3.2 *Let $\psi : \Sigma^2 \to -I \times M^2$ be a CMC complete surface. Suppose that $K_M \geq 0$ and the hyperbolic angle between N and ∂_t is bounded. Then Σ^2 is a totally geodesic surface and the hyperbolic angle is constant. In addition, if M^2 is non-flat, then Σ^2 is contained in a slice.*

Proof. From [36] complete surfaces with non-negative Gauss curvature are parabolic, then M^2 is parabolic. Since $\langle N, \partial_t\rangle$ is bounded, Corollary 2.2.6 provides the result. ∎

The following corollary presents an alternative result to Theorem 4.2 in [44] for the parabolic setting.

Corollary 2.3.3 *Let $\psi : \Sigma^2 \to -I \times M^2$ be a maximal parabolic surface such that the Gauss curvature of its Riemannian fiber M^2 satisfies $K_M \geq -k^2$ for some constant k and*

$$|\nabla h|^2 \leq \frac{|A|^2}{k^2}. \tag{2.12}$$

then Σ^2 is contained in a slice of $-I \times M^2$.

Proof. Since (2.12) implies in condition (2.11), we have from Corollary 2.3.1 that Σ^2 is a totally geodesic surface. From condition (2.12) we have that $\nabla h = 0$, hence Σ^2 is contained in a slice. ∎

2.4 Non-parametric and Calabi-Berstein type results for hypersurfaces in GRW

The discussion in Section 1.1.3 allows us to get the following non-parametric versions of some results presented in the previous section.

Corollary 2.4.1 *Let $\Sigma^n(u)$ be a spacelike parabolic H-graph over M. Suppose that $\Sigma^n(u)$ is stable, the support function $\rho\langle N, \partial_t\rangle$ is bounded,*

$$\int_\Sigma (\overline{\mathrm{Ric}}(N) + |A|^2)\rho^4\langle N, \partial_t\rangle^4 d\Sigma \geq 0, \tag{2.13}$$

and

$$\rho' H + \rho''\langle N, \partial_t\rangle \leq 0.$$

Then $\Sigma^n(u)$ has constant support function. In addition, if $K_G \neq 0$, then u is constant.

Corollary 2.4.2 *Let $\Sigma^n(u)$ be parabolic spacelike H-graph over M^n. Suppose that TCC holds on $-I \times_\rho M$, the support function $\rho\langle N, \partial_t\rangle$ is bounded and*

$$\rho' H + \rho''\langle N, \partial_t\rangle \leq 0.$$

Then $\Sigma^n(u)$ is totally geodesic and consequently the support function is constant.

Corollary 2.4.3 *Let $\Sigma^n(u)$ be parabolic spacelike H-graph over M^n immersed into a spatially parabolic GRW spacetime. Suppose that TCC holds on $-I \times_\rho M$, the hyperbolic angle between the normal N and ∂_t is bounded and*

(i) $\sup \rho < \infty$;

(ii) $\inf \rho > 0$ *and*

(iii) $\rho' H + \rho''\langle N, \partial_t\rangle \leq 0$.

Then $\Sigma^n(u)$ is totally geodesic and consequently the support function is constant.

Corollary 2.4.4 *Let $\Sigma^n(u)$ be a parabolic spacelike $H-$graph over M^n. Let us suppose that M^n satisfies $K_M \geq -k^2$ for some constant k and*

$$|\nabla u|^2 \leq \frac{|A|^2}{k^2(n-1) + |A|^2}$$

then $|\nabla u|$ is constant.

Corollary 2.4.5 *Let $\Sigma^n(u)$ be a compact spacelike $H-$graph over M^n. If $\Sigma^n(u)$ is stable and*

$$\int_\Sigma (\overline{\mathrm{Ric}}(N) + |A|^2)\langle N, \partial_t\rangle^4 d\Sigma \geq 0.$$

Then, u is constant.

Now we consider the following versions for graphs in $-I \times M^2$.

Corollary 2.4.6 *Let $\Sigma^2(u)$ be a parabolic maximal graph over M^2. Suppose that the hyperbolic angle between N and ∂_t is bounded and*

$$\int_\Sigma \left(K_M \frac{|\nabla u|^2}{1 - |\nabla u|^2} + |A|^2 \right) \langle N, \partial_t \rangle^4 d\Sigma \geq 0. \tag{2.14}$$

then $\Sigma^2(u)$ is a totally geodesic hypersurface.

Corollary 2.4.7 *Let $\Sigma^2(u)$ be a parabolic maximal graph over M^2. Suppose that the Gauss curvature of its Riemannian fiber M^2 satisfies $K_M \geq -k$ for some positive constant k and*

$$\frac{|\nabla u|^2}{1 - |\nabla u|^2} \leq \frac{|A|^2}{k}. \tag{2.15}$$

then u is constant.

Chapter 3

Hypersurfaces in SSST

In this chapter we make a study of the support function $\langle N, K\rangle$ related to a hypersurface $\psi : \Sigma^n \to M^{n+k} \times_\rho \mathbb{R}_1$ using a similar technique to the one of Chapter 2. It is important to emphasize that as we saw in Lemma 1.5.7, in SSST every CMC hypersurface is stable. Then, this condition can be dropped out of the hypothesis in this chapter.

3.1 Rigidity of parabolic hypersurfaces in SSST

In Theorem 3.1 of [26] the authors dealt with parabolic hypersurfaces immersed in Riemannian Killing warped product satisfying

$$|\nabla h|^2 \leq \frac{\alpha}{k^2(n-1)\rho^2}|A|^2,$$

with $0 \leq \alpha < 1$. We obtained a counterpart of this result for hypersurfaces immersed in SSST.

Theorem 3.1.1 *Let $\psi : \Sigma^n \to M^n \times_\rho \mathbb{R}_1$ be a CMC parabolic hypersurface immersed in a SSST that obeys the TCC. Suppose that the sectional curvature of the base M^n is bounded from below by $-k^2$, where k is a nonzero constant. If*

$$|\nabla h|^2 \leq \frac{\alpha}{k^2(n-1)\rho^2}|A|^2 \tag{3.1}$$

for some constant $0 \leq \alpha < 1$, then Σ^n is contained in a totally geodesic slice of $M^n \times_\rho \mathbb{R}_1$.

Proof. Taking $w = N^*$ in the weak TCC condition (1.27) and (1.19) we obtain

$$\begin{aligned} 0 &\leq |N^*|^2\Delta\rho - \text{Hess}\rho(N^*) \\ &= \left(\frac{\Theta^2}{\rho^2} - 1\right)\Delta\rho - \text{Hess}\rho(N^*). \end{aligned}$$

Using that from Proposition 3.1 of [2] $\Delta\rho \geq 0$ we obtain:

$$-\frac{1}{\rho}\text{Hess}(\rho)(N^*) + \Theta^2\frac{\Delta\rho}{\rho^3} \geq \frac{\Delta\rho}{\rho} \tag{3.2}$$

From Corollary 7.43 in [51] we have that

$$\overline{\text{Ric}}(N) = \text{Ric}^M(N^*) - \frac{1}{\rho}\text{Hess}(\rho)(N^*) + \Theta^2\frac{\Delta\rho}{\rho^3} \geq 0.$$

Using (3.2) in equation above we get

$$\overline{\text{Ric}}(N) \geq \text{Ric}^M(N^*).$$

Replacing equation above on (1.1.3), making a straightforward computation and using hypothesis we obtain

$$\begin{aligned} \Delta\Theta &\geq (-k^2(n-1)\rho^2|\nabla h|^2 + |A|^2)\Theta \\ &\geq (1-\alpha)|A|^2\Theta. \\ &\geq 0. \end{aligned} \tag{3.3}$$

Since $\Theta < 0$ and Σ^n is parabolic, it follows that Θ must be constant, therefore its Laplacian must vanish. Coming back to (3.3) we have that Σ^n is totally geodesic. Hypothesis (3.1) yields that Σ^n is a slice. ■

Example 4.4 of [32] shows that this theorem cannot be extended if we set $\alpha = 1$ in the constraint (3.1). For this case we obtained a corollary with a weaker thesis, that the support function is constant, as follows we see in Corollary 3.1.4 below.

Theorem 3.1.2 *Let $\psi : \Sigma^n \to M^n \times_\rho \mathbb{R}_1$ be a CMC parabolic spacelike hypersurface. Suppose that Θ is bounded and*

$$\int_\Sigma (\overline{\text{Ric}}(N) + |A|^2)\Theta^4 d\Sigma \geq 0. \tag{3.4}$$

Then Θ is constant.

Proof. Set $f = \Theta^2$. Using (1.1.3) we obtain

$$\Delta f = 2Qf + 2|\nabla\Theta|^2 \tag{3.5}$$

Since f is bounded and Σ^n is parabolic, from Proposition 1.5.5 we can set f in the stability criteria (1.36). Recalling (3.5) we get

$$\begin{aligned} 0 &\geq \int_\Sigma f\Delta f - Qf^2 d\Sigma \\ &= \int_\Sigma Qf^2 + 2f|\nabla\Theta|^2 d\Sigma. \end{aligned} \tag{3.6}$$

Using hypothesis, we conclude that $\nabla\Theta = 0$, therefore Θ is constant. ■

Remark 3.1.3 *Observe that condition* (3.4) *is weaker than TCC.*

Corollary 3.1.4 *Let $\psi : \Sigma^n \to M^n \times_\rho \mathbb{R}_1$ be a CMC parabolic spacelike hypersurface imersed in a SSST that obeys TCC. Suppose that the sectional curvature of the base M^n is bounded from below by $-k^2$, where k is a nonzero constant. If*

$$|\nabla h|^2 \leq \frac{|A|^2}{k^2(n-1)\rho^2}, \tag{3.7}$$

then Θ is constant.

Proof. Following the initial steps of the proof of Theorem 3.1.1 and using hypothesis and (1.19) we obtain

$$\begin{aligned} Q &= \overline{\mathrm{Ric}}(N) + |A|^2 \\ &= \mathrm{Ric}^M(N^*) - \frac{1}{\rho}\mathrm{Hess}(\rho)(N^*) + \Theta^2\frac{\Delta\rho}{\rho^3} + |A|^2 \\ &\geq -k^2(n-1)|N^*|^2 - \frac{1}{\rho}\mathrm{Hess}(\rho)(N^*) + \Theta^2\frac{\Delta\rho}{\rho^3} + |A|^2 \\ &\geq -k^2\rho^2(n-1)|\nabla h|^2 - \frac{1}{\rho}\mathrm{Hess}(\rho)(N^*) + \Theta^2\frac{\Delta\rho}{\rho^3} + |A|^2 \\ &\geq 0. \end{aligned}$$

From Theorem 3.1.2, Θ is constant. ■

Corollary 3.1.5 *Let $\psi : \Sigma^n \to M^n \times_\rho \mathbb{R}_1$ be a CMC complete spacelike hypersurface immersed into a SSST whose Riemannian base M^n is spatially parabolic. Suppose that $\eta = \frac{\Theta}{\rho}$, ρ are bounded and*

$$\int_\Sigma (\overline{\mathrm{Ric}}(N) + |A|^2)\Theta^4 d\Sigma \geq 0.$$

Then M^n is complete and Θ is constant.

Proof. From Theorem 1 of [31] we have that Σ^n is parabolic and M^n is complete. Since η and ρ are bounded, it follows that Θ is bounded. Theorem 3.1.2 implies that Θ is constant. ■

Lemma 3.1.6 *Let $\psi : \Sigma^n \to M^n \times_\rho \mathbb{R}_1$ be a CMC spacelike hypersurface. If Θ is constant and Σ^n is totally geodesic, then ρ and the hyperbolic angle between N and K are constant.*

Proof. Let X be a vector field in Σ. Since $\rho^2 = -\langle K, K\rangle$ and using that K is a Killing vector field we get

$$\begin{aligned}\langle X, \overline{\nabla}\rho\rangle &= -\frac{1}{\rho}\langle \overline{\nabla}_X K, K\rangle \\ &= \left\langle \frac{\overline{\nabla}_K K}{\rho}, X\right\rangle.\end{aligned}$$

Hence

$$\overline{\nabla}\rho = \frac{\overline{\nabla}_K K}{\rho}. \tag{3.8}$$

Since Σ is totally geodesic, from Proposition 1.1.1 we get

$$\begin{aligned}X(\Theta) &= \langle N, \overline{\nabla}_X K\rangle \\ &= \langle N, \overline{\nabla}_{X^*} K\rangle - \frac{\langle X, K\rangle}{\rho^2}\langle N, \overline{\nabla}_K K\rangle \\ &= \frac{1}{\rho}\langle X, \overline{\nabla}\rho\rangle\langle N, K\rangle - \frac{1}{\rho}\langle X, K\rangle\langle N, \overline{\nabla}\rho\rangle\end{aligned}$$

Therefore

$$\nabla\Theta = \frac{1}{\rho}(\Theta\overline{\nabla}\rho - \langle N, \overline{\nabla}\rho\rangle K). \tag{3.9}$$

Multiplying (3.9) by K and using that Θ is constant we obtain

$$0 = \Theta\left\langle \frac{\overline{\nabla}_K K}{\rho}, K\right\rangle + \left\langle N, \frac{\overline{\nabla}_K K}{\rho}\right\rangle \rho^2.$$

Since K is a killing vector field, $\langle \overline{\nabla}_K K, K\rangle = 0$, then we get $\langle N, \overline{\nabla}_K K\rangle = 0$. Therefore recalling (3.9) we obtain

$$0 = \nabla\Theta = \frac{1}{\rho}\Theta\overline{\nabla}\rho.$$

Hence $\rho = \sqrt{-\langle K, K\rangle}$ is constant. Since $\Theta = \langle N, K\rangle$ is constant, from the relation:

$$\langle N, K\rangle = |N||K|\cosh\theta,$$

where θ is the hyperbolic angle between N and K, we conclude that θ is constant. ■

The following theorem is similar to Theorem 2 of [31] for the Riemannian setting.

Theorem 3.1.7 *Let $\psi : \Sigma^n \to M^n \times_\rho \mathbb{R}_1$ be a CMC parabolic spacelike hypersurface. If TCC holds on the ambient and Θ is bounded, then Σ^n is totally geodesic. In addition, ρ and the hyperbolic angle between N and K are constant. Furthermore, if there is a point $p \in \Sigma^n$ such that $\mathrm{Ric}^M(p) > 0$, then Σ^n is contained in a slice of $M^n \times_\rho \mathbb{R}_1$.*

Proof. From Theorem 3.1.2, Θ is constant. Since $\overline{\mathrm{Ric}}$ is non-negative, equation (3.6) provides $|A| = 0$, therefore Σ^n is totally geodesic. From Lemma 3.1.6 we have that ρ and the hyperbolic angle between N and K are constant. From Corollary 7.43 in [51] we have that $\overline{\mathrm{Ric}}(N) = \mathrm{Ric}^M(N^*)$. Hence, if there is a point $p \in \Sigma$ such that $\mathrm{Ric}^M(p) > 0$, then

$$0 = \Delta\Theta = \mathrm{Ric}^M(N^*(p))\Theta,$$

it yields $N^*(p) = 0$. Therefore using (1.19), since $|\nabla h|$ is constant,

$$|\nabla h|^2 = \frac{1}{\rho^2}|N^*|^2 = 0,$$

then Σ^n is contained in a slice of $M^n \times_\rho \mathbb{R}_1$. ■

Corollary 3.1.8 *Let $\psi : \Sigma^n \to M^n \times_\rho \mathbb{R}_1$ be a CMC spacelike hypersurface immersed in a spatially parabolic SSST. If TCC holds on the ambient, the functions η and ρ are bounded, then Σ^n is totally geodesic and M^n is complete. In addition, ρ and the hyperbolic angle between N and K are constant. Furthermore, if there is a point $p \in \Sigma^n$ such that $\mathrm{Ric}^M(p) > 0$, then Σ^n is contained in a slice of $M^n \times_\rho \mathbb{R}_1$.*

Proof. From Theorem 1 of [31] it follows that Σ^n is parabolic and M^n is complete. Since η and ρ are bounded, it follows that Θ is bounded. Theorem 3.1.7 provides the further claims. ■

3.2 Non-parametric and Calabi-Berstein type results for hypersurfaces in SSST

The discussion in Section 1.2.2 allows us to get the following non-parametric versions of some results presented in the previous section.

Corollary 3.2.1 *Let $\Sigma^n(u)$ be a CMC parabolic spacelike Killing graph in a SSST that obeys the weak TCC. Suppose that the sectional curvature of the base M^n is bounded from below by $-k^2$, where k is a nonzero constant. If*

$$|Du|^2 \le \frac{\alpha|A|^2}{\rho^2(k^2(n-1) + \alpha|A|^2)}$$

for some constant $0 \le \alpha < 1$, then u is constant.

Corollary 3.2.2 *Let $\Sigma^n(u)$ be a CMC spacelike Killing graph in a spatially parabolic SSST. If TCC holds on the ambient and ρ is bounded, then $\Sigma^n(u)$ is totally geodesic and M^n is complete. In addition, $|Du|$ is constant. Furthermore, if there is a point $p \in \Sigma^n(u)$ such that $\mathrm{Ric}^M(p) > 0$, then u is constant.*

Chapter 4

Submanifolds in SSST

Based on the causal orientation of the mean curvature vector field, Penrose [52] firstly introduced the concept of codimension 2 trapped surfaces immersed in spacetimes, that plays an important role on General Relativity in the theory of cosmic black holes. In [8] the authors made a generalization of this concept for codimension 2 n-submanifolds. In what follows, we will introduce a generalization for high codimenion submanifolds, as well the notion of totally trapped submanifolds. This last concept is based on the causal orientation of the image of the shape operator α.

Recall that a vector field X tangent to a spacetime is *causal* if it is lightlike ($X \neq 0$ and $\langle X, X\rangle = 0$) or timelike ($\langle X, X\rangle < 0$). The timelike Killing vector field K defines a time orientation as follows: a causal vector field X tangent to $M^{n+k} \times_\rho \mathbb{R}_1$ is said to be *future direct* if $\langle X, K\rangle < 0$, and X is said to be *past direct* if $\langle X, K\rangle > 0$.

Namelly, a spacelike submanifold Σ of $M^{n+k} \times_\rho \mathbb{R}_1$ is said to be *future (resp. past) trapped* if $\vec{H}$ is causal and future (resp. past) direct.

A spacelike submanifold Σ of $M^{n+k} \times_\rho \mathbb{R}_1$ is said to be *totally future (resp. past) trapped* if $\alpha(X, Y)$ is causal and future (resp. past) direct for all X, Y tangent to Σ.

A spacelike submanifold Σ of $M^{n+k} \times_\rho \mathbb{R}_1$ is said to be *totally trapped* if it is totally future trapped or totally past trapped.

For the particular case when $\vec{H} \equiv 0$ we say that Σ is *minimal*.

Accordingly to the terminology of [5] we say that a submanifold immersed in a SSST is said to be *bounded from the future infinity* provided its height function is upper bounded. Analogously, a submanifold immersed in a SSST is said to be *bounded from the past infinity* provided its height function is lower bounded. We say that a submanifold in a SSST *lies in a slab* provided its height function is bounded. The purpose of this chapter is to obtain uniqueness and non-existence results for $p-$parabolic totally trapped hypersurfaces immersed in SSST by means of the study of the heght function.

4.1 Trapped submanifolds contained in slices

Let $\psi : \Sigma^n \to M^{n+k} \times_\rho \mathbb{R}_1$ be an immersed submanifold of codimension $k+1$. The height function of Σ^n defined by $h = \pi_I|_\Sigma = \pi_I \circ \psi$ as above.

Observe that the gradient on $M^{n+k} \times_\rho \mathbb{R}_1$ of the projection $\pi_I(t,q) = t$ is given by

$$\overline{\nabla}\pi_I = -\left\langle \overline{\nabla}\pi_I, \frac{K}{|K|} \right\rangle \frac{K}{|K|} = -\frac{1}{\rho^2}\langle \overline{\nabla}\pi_I, K\rangle K = -\frac{1}{\rho^2}K.$$

Then, the gradient of h on Σ^n is given by

$$\nabla h = -\frac{1}{\rho^2}K^\top. \tag{4.1}$$

When the height function is constant, we say that the submanifold is contained in a *slice* $M^{n+k} \times \{t_0\}$. Let us analyse the shape operator of submanifolds contained in slices.

Let $\varphi : \Sigma^n \to M^{n+k}$ be a submanifold immersed into M^{n+k}. Define $\varphi_0 : \Sigma^n \to M^{n+k} \times \{t_0\}$ by $\varphi_0(p) = (\varphi(p), t_0)$. The metric induced in Σ via φ_0 is the same as the one induced by φ.

On the other hand, given a spacelike submanifold $\psi : \Sigma^n \to M^{n+k} \times_\rho \mathbb{R}_1$ contained in a slice $M^{n+k} \times \{t_0\}$, we have that $\varphi = \pi_M \circ \psi : \Sigma^n \to M^{n+k}$, where $\pi_M : M^{n+k} \times_\rho \mathbb{R}_1 \to M^{n+k}$ is the canonical projection on M, is such that $\psi(p) = (\varphi(p), t_0) = \varphi_0(p)$.

Let α be the shape operator of φ. Setting $V = \frac{K}{|K|}$, the shape operator α_0 of φ_0 is given by:

$$\begin{aligned} \alpha_0(X,Y) &= \alpha_0(X,Y)^M - \langle \alpha_0(X,Y), V\rangle V \\ &= \alpha(X,Y) - \langle A_V X, Y\rangle V, \end{aligned} \tag{4.2}$$

where $\alpha_0(X,Y)^M$ stands for the tangent part to M.

Since $\nabla^{\perp}_X V = 0$, using the Weingarten formula (1.14) and Proposition 7.35 of [51] we achieve:

$$A_V X = -\overline{\nabla}_X V = -\frac{X(\rho)}{\rho} V.$$

Coming back to (4.2), we obtain:

$$\alpha_0(X, Y) = \alpha(X, Y) + \left\langle \frac{X(\rho)}{\rho} V, Y \right\rangle V.$$

Taking a local ortonormal frame $\{E_1, \cdots, E_n\}$ on Σ, we have that the mean curvature of Σ with respect to φ_0 is

$$\begin{aligned}
\vec{H}_0 &= \frac{1}{n}\mathrm{tr}\alpha_0 \\
&= \frac{1}{n}\sum_{i=1}^{n} \alpha(E_i, E_i) + \frac{1}{n\rho}\sum_{i=1}^{n} \langle E_i(\rho)V, E_i\rangle V \\
&= \frac{1}{n}\mathrm{tr}\alpha + \frac{1}{n\rho}\sum_{i=1}^{n} \langle \nabla\rho, E_i\rangle\langle V, E_i\rangle V \\
&= \vec{H} + \frac{1}{n\rho^3}\langle \nabla\rho, K\rangle K
\end{aligned}$$

where $\vec{H}$ is the mean curvature of Σ with respect to φ and ∇ stands for the Levi-Civita connection of Σ.

Then we have that

$$\langle \vec{H}_0, \vec{H}_0\rangle = |\vec{H}|^2 - \frac{1}{n^2\rho^4}\langle \nabla\rho, K\rangle^2.$$

Therefore a submanifold $\psi : \Sigma^n \to M^{n+k} \times_\rho \mathbb{R}_1$ contained in a slice is trapped if, and only if, the mean curvature of $\pi_M \circ \psi : \Sigma^n \to M^{n+k}$ satisfies:

$$|\vec{H}|^2 \leq \frac{1}{n^2\rho^4}\langle \nabla\rho, K\rangle^2 = \frac{1}{n^2}\langle \nabla\rho, \nabla h\rangle^2. \tag{4.3}$$

In addition, since

$$\begin{aligned}
\langle \vec{H}_0, K\rangle &= -\frac{1}{n\rho}\langle \nabla\rho, K\rangle, \\
&= \frac{\rho}{n}\langle \nabla\rho, \nabla h\rangle,
\end{aligned} \tag{4.4}$$

we have that Σ^n is future (resp. past) trapped if, and only if, (4.3) holds and $\langle \nabla\rho, \nabla h\rangle < 0$ (resp. $\langle \nabla\rho, \nabla h\rangle > 0$).

In particular, when ρ is constant, we obtain that a trapped submanifold contained in a slice must be minimal.

4.2 Uniqueness and non-existence results for p-parabolic submanifolds in SSST

Our purpose is to obtain uniqueness and non-existence results for trapped $p-$parabolic submanifolds immersed in $M^{n+k} \times_\rho \mathbb{R}_1$ by means of the study of the height function associated to the submanifold. Let us compute the $p-$laplacian of the height function.

Lemma 4.2.1 *Let $\psi : \Sigma^n \to M^{n+k} \times_\rho \mathbb{R}_1$ be a spacelike submanifold. Then for $p > 2$:*

$$\begin{aligned}\Delta_p h = & -\frac{1}{\rho^2}(p-2)|\nabla h|^{p-2}\left\{\frac{2\rho(p-1)}{(p-2)}\langle\nabla\rho, \nabla h\rangle\right. \\ & \left. -\left\langle\frac{\alpha(\nabla h, \nabla h)}{|\nabla h|^2} + \frac{n}{p-2}\vec{H}, K\right\rangle\right\}. \end{aligned} \tag{4.5}$$

Proof. By definition of p-Laplacian

$$\begin{aligned}\Delta_p h &= \operatorname{div}(|\nabla h|^{p-2}\nabla h) \\ &= (p-2)|\nabla h|^{p-2}\left\{\frac{1}{|\nabla h|^2}\text{Hess}h(\nabla h, \nabla h) + \frac{1}{p-2}\Delta h\right\}. \end{aligned} \tag{4.6}$$

Let us compute $\text{Hess}h(\nabla h, \nabla h)$ and Δh. Using the Gauss formula we have

$$\nabla_X K^\top = \overline{\nabla}_X K^\top + \alpha(X, K^\top).$$

Since $K^\top = -\rho^2 \nabla h$, equation above provides:

$$\begin{aligned}\nabla_X \nabla h &= X\left(-\frac{1}{\rho^2}\right)K^\top - \frac{1}{\rho^2}\overline{\nabla}_X K^\top - \frac{1}{\rho^2}\alpha(X, K^\top) \\ &= \frac{2}{\rho^3}\langle\nabla\rho, X\rangle K^\top - \frac{1}{\rho^2}\overline{\nabla}_X K^\top - \frac{1}{\rho^2}\alpha(X, K^\top). \end{aligned} \tag{4.7}$$

Taking a local orthonormal tangent frame $\{E_1, \cdots, E_n\}$ on Σ^n, applying in equation (4.7), using that K is a Killing vector field and equation (1.16) we get:

$$\begin{aligned}\Delta h &= \sum_{j=1}^n \langle\nabla_{E_j}\nabla h, E_j\rangle \\ &= \frac{2}{\rho^3}\sum_{j=1}^n\langle\nabla\rho, E_j\rangle\langle K^\top, E_j\rangle + \frac{1}{\rho^2}\sum_{j=1}^n\langle\alpha(E_j, E_j), K\rangle \\ &= \frac{2}{\rho^3}\langle\nabla\rho, K^\top\rangle + \frac{n}{\rho^2}\langle\vec{H}, K\rangle \\ &= -\frac{2}{\rho}\langle\nabla\rho, \nabla h\rangle + \frac{n}{\rho^2}\langle\vec{H}, K\rangle. \end{aligned} \tag{4.8}$$

Taking $X = \nabla h$ in the equation (4.7) and again using (1.16) we obtain

$$\begin{aligned}
\mathrm{Hess}h(\nabla h, \nabla h) &= \langle \nabla_{\nabla h} \nabla h, \nabla h \rangle \\
&= \frac{2}{\rho^3} \langle \nabla \rho, \nabla h \rangle \langle K^\top, \nabla h \rangle + \frac{1}{\rho^2} \langle \overline{\nabla}_{\nabla h} K^\top, \nabla h \rangle \\
&= -\frac{2}{\rho} \langle \nabla \rho, \nabla h \rangle |\nabla h|^2 + \frac{1}{\rho^2} \langle \alpha(\nabla h, \nabla h), K \rangle. \qquad (4.9)
\end{aligned}$$

Thus, replacing (4.8) and (4.9) in (4.6) we obtain the desired expression (4.5). ■

In [31] the authors make a study of parabolic spacelike hypersurfaces in SSST. Inspired in their ideas we obtained the below uniqueness results for $p-$parabolic ($p \geq 2$) submanifolds with high codimension immersed in SSST.

Theorem 4.2.2 *Let $\psi : \Sigma^n \to M^{n+k} \times_\rho \mathbb{R}_1$ be a complete spacelike trapped submanifold. Suppose that $\langle \nabla \rho, \nabla h \rangle$ and $\langle \vec{H}, K \rangle$ have opposite sign. If $|\nabla h| \in \mathcal{L}^1(\Sigma)$, then Σ^n is a minimal submanifold. In addition, if Σ^n is parabolic and bounded from the future or from the past infinity, then Σ^n is a submanifold of a slice $M^{n+k} \times \{t_0\}$.*

Proof. Since $\langle \nabla \rho, \nabla h \rangle$ and $\langle \vec{H}, K \rangle$ have opposite sign, we have from equation (4.8) that

$$\Delta h = -\frac{2}{\rho} \langle \nabla \rho, \nabla h \rangle + \frac{n}{\rho^2} \langle \vec{H}, K \rangle$$

does not change sign. Since $|\nabla h| \in \mathcal{L}^1(\Sigma)$, Lemma 1.6.2 provides that Δu must be in fact harmonic, therefore $\langle \vec{H}, K \rangle = 0$. Since $\vec{H}$ is causal or zero, it must be zero, and Σ^n is minimal.

In addition, if Σ^n is parabolic and h is bounded from above or from below, it must be constant, therefore Σ^n is contained in a slice. ■

Theorem 1 of [47] asserts that a spacetime endowed with a timelike Killing vector field does not admit closed trapped imbedded submanifolds. Another proof of this result for the setting of SSST is in a preprint of H. F. de Lima, A. Freitas, E. A. Lima and M. Santos. In what follows we obtained an extension to this non-existence result for $p-$parabolic submanifolds in SSST.

Theorem 4.2.3 *There exists no p-parabolic spacelike submanifold Σ^n immersed in $M^{n+k} \times_\rho \mathbb{R}_1$ such that $\langle \nabla \rho, \nabla h \rangle \geq 0$ (resp. $\langle \nabla \rho, \nabla h \rangle \leq 0$), Σ^n is bounded away from the past (resp. future) infinity and totally future (resp. past) trapped.*

Proof. Let us assume the first case. Since Σ^n is totally future trapped, $\alpha(\nabla h, \nabla h)$ is causal future direct and so is $\vec{H} = \frac{1}{n} \operatorname{tr} \alpha$. Then equation (4.5) provides $\Delta_p h \geq 0$.

Since Σ^n is bounded from the past infinity and p-parabolic, it follows that h must be constant, therefore a slice. But (4.4) and hypothesis provide

$$0 > \langle \vec{H}, K\rangle = \frac{\rho}{n}\langle \nabla\rho, \nabla h\rangle \geq 0,$$

and achieve a contradiction. ■

In the next results we will work with the shape operator $A_{K^\perp}$. Recall that from (1.15)

$$\langle A_{K^\perp}X, Y\rangle = \langle \alpha(X,Y), K\rangle. \tag{4.10}$$

Then $\mathrm{tr}(A_{K^\perp}) = n\langle \vec{H}, K\rangle$.

We can compare the two theorems below with the first case of Theorem 4.2.3. In this last one we assumed Σ to be totally future trapped. In the below cases we assumed Σ to be future trapped, a weaker hypothesis.

Theorem 4.2.4 *There exists no p-parabolic spacelike submanifold Σ^n immersed in $M^{n+k}\times_\rho\mathbb{R}_1$ bounded away from the past infinity and future trapped such that $\langle\nabla\rho, \nabla h\rangle \geq 0$ and*

$$\left(\frac{n-1}{n}\right)^{\frac{1}{2}}|\Phi| \leq -\frac{n+p-2}{p-2}\langle \vec{H}, K\rangle, \tag{4.11}$$

where $\Phi = A_{K^\perp} - \langle \vec{H}, K\rangle I$ is the traceless shape operator of Σ^n related to $K^\perp$ and $|\Phi| = \sqrt{\mathrm{tr}(\Phi^2)}$ is the Hilbert-Schimidt norm.

Proof. We will prove that $\Delta_p h \leq 0$. Using Lemma 1.6.1 and hypothesis, we obtain

$$\begin{aligned}\left\langle \frac{\alpha(\nabla h, \nabla h)}{|\nabla h|^2} + \frac{n}{p-2}\vec{H}, K\right\rangle &= \frac{\langle A_{K^\perp}\nabla h, \nabla h\rangle}{|\nabla h|^2} + \frac{n}{p-2}\langle \vec{H}, K\rangle \\ &= \frac{\langle \Phi\nabla h, \nabla h\rangle}{|\nabla h|^2} + \frac{n+p-2}{p-2}\langle \vec{H}, K\rangle \\ &\leq \left(\frac{n-1}{n}\right)^{\frac{1}{2}}|\Phi| + \frac{n+p-2}{p-2}\langle \vec{H}, K\rangle \\ &\leq 0.\end{aligned} \tag{4.12}$$

From (4.5) we get that $\Delta_p h \leq 0$. Taking into account that Σ^n is bounded away from the past infinity (h is lower bounnded) and Σ^n is p-parabolic, we conclude that h must be constant, therefore Σ^n must be a submanifold of a slice of $M^{n+k}\times_\rho\mathbb{R}_1$. We conclude the proof as in Theorem 4.2.3. ■

Remark 4.2.5 *Notice that in the theorem above we just need* $\langle \vec{H}, K\rangle \leq 0$*, independly of* $\vec{H}$ *to be causal.*

If equality holds in (4.11) *for a spacelike submanifold* $\psi : \Sigma^n \to M^{n+k} \times_\rho \mathbb{R}_1$ *immersed into* $M^{n+k} \times_\rho \mathbb{R}_1$ *whose height function is non-constant,* $p-$*harmonic and* $\langle \nabla\rho, \nabla h\rangle \geq 0$*, then from Lemma 1.6.1 we have that* Σ^n *is totally umbilic or* ∇h *is eigenvector of* Φ *associated to the biggest eigenvalue of* Φ*.*

The following corollary is a straightforward consequence of Theorem 4.2.4.

Corollary 4.2.6 *There exists no p-parabolic spacelike submanifold* Σ^n *immersed in* $M^{n+k}\times_\rho\mathbb{R}_1$ *bounded away from the past infinity and future trapped such that* $\langle \nabla\rho, \nabla h\rangle \geq 0$ *and totally umbilic with respect to the direction* $K^\perp$*.*

Remark 4.2.7 *Notice that in equation* (4.12) *we could use the Cauchy-Schwarz inequality and put the condition*

$$|A_{K^\perp}| \leq -\frac{n}{p-2}\langle \vec{H}, K\rangle,$$

in Theorem 4.2.4; But observing that $|\Phi|^2 = |A_{K^\perp}|^2 - n\langle \vec{H}, K\rangle^2$ *it is not difficult to verify that this constraint is more restrictive than* (2.3.1).

Taking Lemma 1.6.3 into accout, the boundedness of h in the above results of this chapter can be replaced by $|\nabla h| \in L^p(M)$. We can see the integrability on $|\nabla h|$ as an extension to the compact case, as well the $p-$parabolicity.

Chapter 5

Nishikawa approach to uniqueness of maximal surfaces in GRW

Albujer and Alías established Calabi-Bernstein results for complete maximal surfaces in a Lorentzian product spacetime $-\mathbb{R} \times M^2$ see [3] and [4]. In particular, when the Riemannian surface M^2 has non-negative Gauss curvature, they proved that any complete maximal surface must be totally geodesic. Besides, if M^2 is non-flat, the authors concluded that it must be a slice $\{t\} \times M^2$. The necessity of the assumption on the Gauss curvature can be observed from the examples of maximal surfaces in $-\mathbb{R} \times \mathbb{H}^2$, where $\mathbb{H}^2$ is the hyperbolic plane, constructed in [1]. In [46], G. Li and I. Salavessa generalized such results of [4] to higher dimension and codimension.

Eraldo A. Lima Jr and H.F. de Lima exhibit in [40] an example of a (non totally geodesic) complete spacelike surface of constant mean curvature (CMC) in $-\mathbb{R} \times \mathbb{H}^2$ whose hyperbolic angle function is constant.

More recently, in [29], M. Caballero, A. Romero and M. Rubio worked in 3-dimensional Generalized Robertson-Walker (GRW) spacetimes considering maximal surfaces with uniqueness results for the case the fibre has non-negative Gauss curvature generalizing results from Albujer and Alías.

They proved uniqueness results for surfaces in GRW spaces using an estimative involving the capacity of anulus in the surfaces assuming physical conditions like the Timelike Convergence Condiction (TCC).

Furthermore Albujer, de Lima and Camargo proved uniqueness results for CMC spacelike hypersurfaces in a GRW spacetime of arbitrary dimension considering *a priori* growth estimates for the height function. They actually considered Robertson-Walker spacetimes, i.e., $-\mathbb{R} \times_\rho M^3$ whose fibre M^3 has constant sectional curvature κ. See [10].

Albujer in [1] has shown that there are complete maximal surfaces which are not totally geodesic in $-\mathbb{R} \times \mathbb{H}^2$, nevertheless in [40] and [44] the authors presented suitable conditions in order to guarantee that such surfaces are trivial slices. The natural question:

"there exist maximal surfaces in the ambient $-\mathbb{R} \times_\rho \mathbb{H}^2$ for some non-trivial function ρ and what are the needed assumptions to conclude that a complete maximal surface in $-\mathbb{R} \times_\rho M^2$, where $K_M \geq -\kappa$, is totally geodesic or a slice?"

encouraged us to make a survey on this topic.

Our technique is based on a proper extension of a result due to Nishikawa in [49] and relies within the applications of the generalized maximum principle due to Yau [64] to complete Riemannian manifolds. In fact, we use an extension of Lemma 2 of [49] to the case the Ricci curvature is no longer bounded by a constant but by a more general function $\Gamma(r)$ of the distance r from a fixed point on the manifold.

5.1 A Nishikawa-Omori-Yau type lemma

We start with result that is a generalization of a lemma due to Nishikawa in [49].

Lemma 5.1.1 *Let M $n(\geq 2)$ be a complete Riemannian manifold such that* $\mathrm{Ric} \geq -\Gamma(r)$*, where r is the distance function, and Γ such that*

$$\Gamma(0) \geq 1, \;\; \Gamma' \geq 0 \;\; \text{and} \;\; \Gamma^{-1/2} \notin L^1[0,\infty).$$

If $u \in C^\infty(M)$ is a non-negative function satisfying

$$\Delta u \geq \beta\, u^{1+\alpha}, \; \text{for constants } \alpha, \beta > 0, \tag{5.1}$$

then $u = 0$.

Proof. Under these assumptions M satisfies the Omori-Yau-Borbély generalized maximum principle [17]. Since $u \in C^\infty(M)$ is non negative, consider the following smooth function on M,

$$F = \frac{1}{(1+u)^{\theta}},$$

where $\theta > 0$ will be chosen latter, notice that satisfies $F > 0$ and $\inf(F) \geq 0$ whenever $\theta > 0$. Therefore

$$\nabla u = -\frac{\theta^{-1}}{F^{\frac{\theta+1}{\theta}}} \nabla F$$

and

$$\Delta F = -\theta F^{\frac{\theta+1}{\theta}} \Delta u + \theta(\theta+1) F^{\frac{\theta+2}{\theta}} |\nabla u|^2.$$

Then

$$\Delta F = -\theta F^{\frac{\theta+1}{\theta}} \Delta u + \frac{\theta+1}{\theta} \frac{|\nabla F|^2}{F},$$

therefore

$$F\Delta F = -\theta F^{\frac{2\theta+1}{\theta}} \Delta u + \frac{\theta+1}{\theta} |\nabla F|^2. \tag{5.2}$$

Thus using an Omori-Yau sequence we have

$$|\nabla F|(p_m) < \frac{1}{m},$$
$$\Delta F(p_m) > -\frac{1}{m}$$

and

$$0 \leq \inf F \leq F(p_m) < \inf F + \frac{1}{m}.$$

By definition of F, whenever $\theta > 0$ we have

$$\lim_{m\to\infty} F(p_m) = \inf F \iff \lim_{m\to\infty} u(p_m) = \sup u. \tag{5.3}$$

Combining it with (5.2) and (5.1) we obtain

$$-\frac{1}{m} F(p_m) + \theta\beta u^{1+\alpha}(p_m) F^{\frac{2\theta+1}{\theta}}(p_m) < \frac{\theta+1}{\theta} \frac{1}{m^2},$$

that is,

$$-\frac{1}{m} F(p_m) + \theta\beta \frac{u^{1+\alpha}(p_m)}{(1+u(p_m))^{2\theta+1}} < \frac{\theta+1}{\theta} \frac{1}{m^2},$$

then choosing $2\theta = \alpha$ we get $\lim_{m\to\infty} u(p_m) = 0$ and therefore $u = 0$. ■

5.2 Gauss equation

The Gauss curvature K_Σ of a spacelike surface Σ^2 in $\overline{M}^3$ is described in terms of A and the curvature of $\overline{M}^3$ by the Gauss equation, which is given by

$$K_\Sigma = \overline{K} + K_G, \tag{5.4}$$

where $\overline{K}(p)$, $p \in \Sigma$ denotes the sectional curvature in $\overline{M}^3$ of the tangent plane $d\psi_p\left(T_p(\Sigma)\right)$ and $K_G = -\det A$ is the Gauss-Kronecker curvature of Σ. We can also write $\overline{K}$ in terms of the Gauss curvature of M as

$$\overline{K} = \frac{\kappa_M + \rho'^2}{\rho^2}(1+|\nabla h|^2) - \frac{\rho''}{\rho}|\nabla h|^2. \tag{5.5}$$

Combining equations (5.4) and (5.5) we obtain

$$K_\Sigma = \frac{\kappa_M + \rho'^2}{\rho^2}\cosh^2\theta - \frac{\rho''}{\rho}\sinh^2\theta + K_G, \tag{5.6}$$

or equivalently

$$K_\Sigma = \frac{\kappa_M + \rho'^2}{\rho^2} + \frac{\kappa_M}{\rho^2}\sinh^2\theta - (\log\rho)''\sinh^2\theta + K_G, \tag{5.7}$$

where θ is the hyperbolic angle between N and ∂_t. We also have the well-known relation

$$|A|^2 = 2H^2 + 2(H^2 + K_G). \tag{5.8}$$

If we assume $\kappa_M \geq -\kappa$ for some positive constant κ, then (5.7) gives the following inequality

$$K_\Sigma \geq \frac{-\kappa + \rho'^2}{\rho^2}\cosh^2\theta - \frac{\rho''}{\rho}\sinh^2\theta + K_G. \tag{5.9}$$

5.3 Uniqueness of surfaces in GRW

We will make a study of the hyperbolic cossine of the angle θ between N and ∂_t. Recall that $\cosh\theta = -\langle N, \partial_t\rangle$. And then, from (1.5) we get that $|\nabla h|^2 = \sinh^2\theta$. Let us compute the Laplacian of $\cosh\theta$ for a hypersurface immersed in a GRW spacetime.

Lemma 5.3.1 *Let $\psi : \Sigma^n \to -I \times_\rho M^n$ be a spacelike maximal surface. Then*

$$\begin{aligned} \frac{1}{2}\Delta\cosh^2\theta \;=\; & \left[|A|^2 + \mathrm{Ric}_M(N^*, N^*) - n\left(\frac{\rho''}{\rho} - \frac{\rho'^2}{\rho^2}\right)\sinh^2\theta\right]\cosh^2\theta \qquad (5.10)\\ & + |A\partial_t^\top|^2 - 4\frac{\rho'}{\rho}\cosh\theta\langle A\partial_t^\top, \partial_t^\top\rangle \\ & + \left(n + 3\cosh^2\theta\right)\frac{\rho'^2}{\rho^2}\sinh^2\theta + n\frac{\rho'^2}{\rho^2}. \end{aligned}$$

Proof. Firstly we develop a formula for $\Delta\langle N,\xi\rangle$, where $\xi=\rho\partial_t$. Notice initially that $\xi=\rho\partial_t$ is a conformal vector field in $-I\times_\rho M^2$, (see [18]). More precisely,

$$\overline{\nabla}_X(\rho\partial_t)=\rho' X.$$

So as obtained in (1.9):

$$\nabla\langle N,\xi\rangle=-A\xi^\top,$$

using Codazzi equation as in [15] we get

$$\mathrm{div}(-A\xi^\top)=n\langle\nabla H,\xi\rangle+\overline{\mathrm{Ric}}(\xi^\top,N)+n\rho' H+\langle\xi,N\rangle|A|^2,$$

that is,

$$\Delta\langle N,\xi\rangle=n\langle\nabla H,\xi\rangle+\overline{\mathrm{Ric}}(\xi^\top,N)+n\rho' H+\langle\xi,N\rangle|A|^2.$$

We also have the following

$$\overline{\mathrm{Ric}}(\xi^\top,N)=\langle\xi,N\rangle\left(\overline{\mathrm{Ric}}(N^*,N^*)-\frac{1}{\rho^2}\left(1-\langle N,\partial_t\rangle^2\right)\overline{\mathrm{Ric}}(\xi,\xi)\right),$$

from Proposition 7.42 in [51] we obtain

$$\begin{aligned}\overline{\mathrm{Ric}}(\xi^\top,N) &= \langle N,\xi\rangle\left(\mathrm{Ric}_M(N^*,N^*)+|N^*|^2\left(\frac{\rho''}{\rho}+(n-1)\frac{\rho'^2}{\rho^2}\right)\right.\\ &\quad\left.+\frac{n}{\rho}\left(1-\langle N,\partial_t\rangle^2\right)\rho''\right).\end{aligned}$$

Since $|N^*|^2=-1+\langle N,\partial_t\rangle^2$ we obtain

$$\overline{\mathrm{Ric}}(\xi^\top,N)=\langle N,\xi\rangle\left(\mathrm{Ric}_M(N^*,N^*)-(n-1)\left(\frac{\rho''}{\rho}-\frac{\rho'^2}{\rho^2}\right)|N^*|^2\right),$$

that is,

$$\begin{aligned}\Delta\langle N,\xi\rangle &= \langle N,\xi\rangle\left(\mathrm{Ric}_M(N^*,N^*)-(n-1)(\log\rho)''|N^*|^2\right)\\ &\quad+n\langle\nabla H,\xi\rangle+n\rho' H+\langle\xi,N\rangle|A|^2.\end{aligned}$$

For the maximal case we obtain

$$\begin{aligned}\frac{1}{2}\Delta\langle N,\xi\rangle^2 &= \langle N,\xi\rangle^2\left(|A|^2+\mathrm{Ric}_M(N^*,N^*)-(n-1)\left(\log\rho\right)''|N^*|^2\right)\\ &\quad+|A\xi^\top|^2.\end{aligned}$$

Therefore observing that $\cosh^2\theta=\langle N,\partial_t\rangle^2$ and $\sinh^2\theta=\cosh^2\theta-1$ through (1.8) we get

$$\begin{aligned}\frac{1}{2}\Delta\rho^2\sinh^2\theta &= |A\xi^\top|^2+n\rho'^2-\rho^2\left(\log\rho\right)''\sinh^2\theta-\rho'^2\sinh^2\theta\\ &\quad+\langle N,\xi\rangle^2\left(|A|^2+\mathrm{Ric}_M(N^*,N^*)-(n-1)\left(\log\rho\right)''|N^*|^2\right).\end{aligned}$$

Now we evaluate the following Laplacian

$$\frac{1}{2}\Delta(\rho^2\cosh^2\theta) = \frac{1}{2}\rho^2\Delta\cosh^2\theta + \frac{1}{2}\cosh^2\theta\Delta\rho^2 + \langle\nabla\rho^2, \nabla\cosh^2\theta\rangle,$$

then

$$\begin{aligned}\frac{1}{2}\Delta\cosh^2\theta &= \frac{1}{2\rho^2}\Delta(\rho^2\cosh^2\theta) - \frac{1}{2\rho^2}\cosh^2\theta\Delta\rho^2 - \frac{1}{\rho^2}\langle\nabla\rho^2, \nabla\cosh^2\theta\rangle \\ &= \cosh^2\theta\left(|A|^2 + \mathrm{Ric}_M(N^*, N^*) - (n-1)(\log\rho)''|N^*|^2\right) \\ &\quad + \left[n\frac{\rho'^2}{\rho^2} - (\log\rho)''\sinh^2\theta - \frac{\rho'^2}{\rho^2}\sinh^2\theta\right]\cosh^2\theta \\ &\quad - \frac{1}{\rho^2}\langle\nabla\rho^2, \nabla\cosh^2\theta\rangle + |A\partial_t^\top|^2.\end{aligned}$$

Since

$$\nabla\cosh^2\theta = \frac{1}{\rho^2}\nabla\langle N, \xi\rangle^2 - \frac{1}{\rho^2}\cosh^2\theta\nabla\rho^2,$$

we obtain

$$\begin{aligned}\langle\nabla\rho^2, \nabla\cosh^2\theta\rangle &= \frac{1}{f^2}\langle\nabla\rho^2, \nabla\langle N, \xi\rangle^2\rangle - \frac{1}{\rho^2}\cosh^2\theta|\nabla\rho^2|^2 \\ &= 4\rho\rho'\langle N, \partial_t\rangle\langle A\partial_t^\top, \partial_t^\top\rangle - 4\rho'^2\sinh^2\theta\cosh^2\theta.\end{aligned}$$

Thus we get

$$\begin{aligned}\frac{1}{2}\Delta\cosh^2\theta &= \left[n\frac{\rho'^2}{\rho^2} - (\log\rho)''\sinh^2\theta - \frac{\rho'^2}{\rho^2}\sinh^2\theta\right]\cosh^2\theta \\ &\quad + \cosh^2\theta\left(|A|^2 + \mathrm{Ric}_M(N^*, N^*) - (n-1)(\log\rho)''|N^*|^2\right) \\ &\quad + 4\left(-\frac{\rho'}{\rho}\cosh\theta\langle A\partial_t^\top, \partial_t^\top\rangle + \frac{\rho'^2}{\rho^2}\sinh^2\theta\cosh^2\theta\right) + |A\partial_t^\top|^2.\end{aligned}$$

Rearranging it

$$\begin{aligned}\frac{1}{2}\Delta\cosh^2\theta &= |A\partial_t^\top|^2 - 4\frac{\rho'}{\rho}\cosh\theta\langle A\partial_t^\top, \partial_t^\top\rangle + \Big[|A|^2 + \mathrm{Ric}_M(N^*, N^*) \\ &\quad + n\frac{\rho'^2}{\rho^2} - n(\log\rho)''\sinh^2\theta + 3\frac{\rho'^2}{\rho^2}\sinh^2\theta\Big]\cosh^2\theta.\end{aligned}$$

From which it follows that the desired formula (5.10). ■

From now on, $\Gamma(r)$, will denote the function described in Lemma 5.1.1.

Theorem 5.3.2 *Let $\overline{M}^3 = -I\times_\rho M^2$ be a GRW spacetime whose fiber has Gauss curvature K_M satisfying $K_M \geq -\kappa$, for some positive constant κ. Consider $\psi : \Sigma^2 \to -I\times_\rho M^2$ be a maximal complete surface such that*

$$|\nabla h|^2 \leq \frac{5}{6}\alpha\frac{|A|^2}{C_\rho}, \tag{5.11}$$

where $0 \leq \alpha < 1$ is a constant, and $C_\rho = \sup\left(\frac{\kappa}{\rho^2} + 2(\log\rho)''\right) > 0$. If ρ is bounded away from zero, $(\log\rho)''$ is bounded and $K_G \leq \Gamma(r)$, then Σ^2 is a totally geodesic slice.

Proof. Consider equation (5.10) to the case of $n=2$. Using that since Σ^2 is maximal, $|A\partial_t^\top|^2 = \frac{1}{2}|A|^2|\partial_t^\top|^2$. we achieve:

$$\begin{aligned}\frac{1}{2}\Delta\cosh^2\theta &\geq \left[\frac{5}{6}|A|^2 + \mathrm{Ric}_M(N^*,N^*) - 2(\log\rho)''\sinh^2\theta\right]\cosh^2\theta \\ &+\left(\frac{2}{3}|A|^2 - \frac{4}{\sqrt{2}}\frac{|\rho'|}{\rho}|A|\cosh\theta + \left(2+3\cosh^2\theta\right)\frac{\rho'^2}{\rho^2}\right)\sinh^2\theta \\ &+2\frac{\rho'^2}{\rho^2}.\end{aligned}$$

Completing the square we accomplish:

$$\begin{aligned}\frac{1}{2}\Delta\cosh^2\theta &\geq \left[\frac{5}{6}|A|^2 + \kappa_M\frac{|N^*|^2}{\rho^2} - 2(\log\rho)''\sinh^2\theta\right]\cosh^2\theta \\ &+\left[\left(\sqrt{\frac{2}{3}}|A| - \sqrt{3}\frac{|\rho'|}{\rho}\cosh\theta\right)^2 + 2\frac{\rho'^2}{\rho^2}\right]\sinh^2\theta \\ &+2\frac{\rho'^2}{\rho^2}.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\frac{1}{2}\Delta\cosh^2\theta &\geq \left[\frac{5}{6}(1-\alpha)|A|^2 + \frac{5}{6}\alpha|A|^2 + \kappa_M\frac{|N^*|^2}{\rho^2} - 2(\log\rho)''\sinh^2\theta\right]\cosh^2\theta \\ &\geq \left[\frac{5}{6}(1-\alpha)|A|^2 + C_\rho|\nabla h|^2 + \kappa_M\frac{|N^*|^2}{\rho^2} - 2(\log\rho)''\sinh^2\theta\right]\cosh^2\theta \quad (5.12) \\ &\geq \frac{1-\alpha}{\alpha}C_\rho\sinh^2\theta\cosh^2\theta \\ &\geq c\sinh^4\theta.\end{aligned}$$

Since $\Delta\cosh^2\theta = \Delta\sinh^2\theta$,

$$\Delta\sinh^2\theta \geq 2c\sinh^4\theta.$$

In order to apply the Lemma 5.1.1, observe that from (5.9) and (5.11), we get

$$\begin{aligned}K_\Sigma &\geq \frac{-\kappa+\rho'^2}{\rho^2}\cosh^2\theta - \frac{\rho''}{\rho}\sinh^2\theta + K_G \\ &\geq -\frac{\kappa}{\rho^2} - \left[\frac{\kappa}{\rho^2} + (\log\rho)''\right]\sinh^2\theta \qquad (5.13)\end{aligned}$$

If $\frac{\kappa}{\rho^2} + (\log\rho)''$ is upper bounded by a non-positive constant, then (5.13) provides K_Σ is lower bounded by a constant. On the other hand, if $\frac{\kappa}{\rho^2} + (\log\rho)''$ is upper bounded by a positive constant C_2, then using hypothesis we get

$$\begin{aligned} K_\Sigma &\geq C_1 - C_2 \sinh^2 \theta \\ &\geq C_1 - C_2 \frac{5\alpha}{3C_\rho} K_G \\ &\geq C_1 - C_2 \frac{5\alpha}{3C_\rho} \Gamma(r). \end{aligned}$$

Calling the Lemma 5.1.1, we conclude that Σ^2 is a slice. In order to see that Σ^2 is totally geodesic, it is enough to recall that the shape operator of the slices are given by $A = -\frac{\rho'}{\rho}\mathbb{I}$ and by hypothesis Σ^2 is maximal. ■

Remark 5.3.3 *Example 6.0.4 below shows that condition* (5.11) *of Theorem 5.3.2 cannot be withdraw.*

Remark 5.3.4 *In [7], they deal with CMC hypersurfaces in n-dimensional spatially closed GRW spacetimes. In the main result they assume that the sectional curvature of the fiber is non-negative and a logaritmic convexity on the warping function. Notice that in our assumption it is possible K_M to be negative.*

Also, we can compare Theorem 5.3.2 with Theorem 4.2 of [29]. In this last reference the fiber is allowed to have negative mean curvature. But notice that they do not comprise the case when the ambient is $-\mathbb{R} \times_{\cosh\theta} \mathbb{H}^2$, for example.

Remark 5.3.5 *In the proof of Theorem 5.3.2 the assumption $K_G \leq \Gamma(r)$ is not necessary if $\frac{k}{\rho^2} + (\log\rho)''$ is upper bounded by a non-positive constant. Otherwise, the assumption on K_G may not be removed, as evidenced in Example* (6.0.5)*. In Example 3.3 of [1] it is showed that this graph is not complete. Taking Lemma 1.1.4 into account, we see that condition $K_G \leq \Gamma(r)$ must not hold for this example.*

When $\frac{k}{\rho^2} + (\log\rho)''$ is upper bounded by a positive constant, the constraint on K_G can be replaced by a upper bound on the hyperbolic angle, as we enunciate below.

Corollary 5.3.6 *Let $\overline{M}^3 = -I \times_\rho M^2$ be a GRW spacetime whose fiber has Gauss curvature satisfying $K_M \geq -\kappa$, for some positive constant κ. Let Σ^2 be a complete maximal surface in $\overline{M}^3$ such that the hyperbolic angle between N and ∂_t is bounded. If* (5.11) *holds, then Σ must be a slice.*

Proof. From equation (5.13) we have that the Gauss curvature of Σ^2 is bounded from below, and then Lemma 5.1.1 can be called. ■

Using Physical interpretation as well equation (1.26) we obtain

Theorem 5.3.7 *Let $\Sigma^2 \to -I \times_\rho M^2$ be a complete maximal surface. Suppose that $-I \times_\rho M^2$ satisfies the TCC and*

$$|\nabla h| \leq \beta |A|^\alpha \tag{5.14}$$

for constants $\alpha, \beta > 0$. If $-\frac{\rho''}{\rho} \leq \Gamma(r)$, then Σ^2 is a slice.

Proof. Applying it in (5.7) and hypothesis $-\frac{\rho''}{\rho} \leq \Gamma(r)$ we obtain the condition of the Lemma 5.1.1

$$K_\Sigma \geq \frac{\rho''}{\rho} \geq -\Gamma(r),$$

which ensures the possibility of using Lemma 5.1.1.

Now, notice that the equality in Lemma 5.3.1 becomes the following:

$$\begin{aligned} \frac{1}{2}\Delta \cosh^2\theta &\geq \left[|A|^2 + (\log\rho)''\sinh^2\theta - 2(\log\rho)''\sinh^2\theta\right]\cosh^2\theta \\ &\quad + |A\partial_t^\top|^2 - 4\frac{\rho'}{\rho}\cosh\theta\langle A\partial_t^\top, \partial_t^\top\rangle \\ &\quad + \left(2 + 3\cosh^2\theta\right)\frac{\rho'^2}{\rho^2}\sinh^2\theta + 2\frac{\rho'^2}{\rho^2} \\ &= \left[|A|^2 - (\log\rho)''\sinh^2\theta\right]\cosh^2\theta \\ &\quad + |A\partial_t^\top|^2 - 4\frac{\rho'}{\rho}\cosh\theta\langle A\partial_t^\top, \partial_t^\top\rangle \\ &\quad + \left(2 + 3\cosh^2\theta\right)\frac{\rho'^2}{\rho^2}\sinh^2\theta + 2\frac{\rho'^2}{\rho^2}. \end{aligned}$$

Proceeding as in the proof of the previous theorem we obtain an inequality similar to (5.12).

$$\frac{1}{2}\Delta \cosh^2\theta \geq \left[\frac{5}{6}|A|^2 - (\log\rho)''\sinh^2\theta\right]\cosh^2\theta.$$

Again using the TCC we have $-(\log\rho)'' \geq 0$, therefore

$$\frac{1}{2}\Delta \cosh^2\theta \geq \frac{5}{6}|A|^2\cosh^2\theta.$$

Then by hypothesis (5.14) and $\Delta\cosh^2\theta = \Delta\sinh^2\theta$ we obtain

$$\frac{1}{2}\Delta \sinh^2\theta \geq \frac{5}{6}\frac{1}{\beta^{\frac{2}{\alpha}}}\sinh^{2+\frac{2}{\alpha}}\theta.$$

Therefore by Lemma 5.1.1 we get the desired result. ■

The next result we assume a weaker energy condition that is the Null Convergence Condition (NCC).

Theorem 5.3.8 *Let $\overline{M}^3 = -I \times_\rho M^2$ be a GRW spacetime satisfying the NCC. Consider $\psi : \Sigma^2 \to -I \times_\rho M^2$ be a maximal complete hypersurface. If $-\frac{\rho''}{\rho} \leq \Gamma(r)$ and*

$$|\nabla h|^2 \leq \frac{5}{6}\alpha \frac{|A|^2}{C_\rho}, \tag{5.15}$$

where $0 \leq \alpha < 1$ is constant, $C_\rho = \sup\{\rho''/\rho\} > 0$, then Σ^2 is a totally geodesic slice.

Proof. Analogouslly to Theorem 5.3.7 we get $K_\Sigma \geq -\Gamma(r)$. Now following the proof of Theorem 5.3.2, from (5.12) and using the NCC condition

$$\begin{aligned}
\frac{1}{2}\Delta \cosh^2\theta &\geq \left[\frac{5}{6}(1-\alpha)|A|^2 + C_\rho|\nabla h|^2 + \kappa_M \frac{|N^*|^2}{\rho^2} - 2\,(\log\rho)''\sinh^2\theta\right]\cosh^2\theta \\
&\geq \left[\frac{5}{6}(1-\alpha)|A|^2 + C_\rho|\nabla h|^2 - (\log\rho)''\sinh^2\theta\right]\cosh^2\theta \\
&\geq \frac{1-\alpha}{\alpha} C_\rho |\nabla h|^2 \cosh^2\theta \\
&\geq c\sinh^2\theta.
\end{aligned}$$

Since $\Delta\cosh^2\theta = \Delta\sinh^2\theta$, we finish the proof using the Lemma 5.1.1. ■

5.4 Calabi-Bersntein type results for maximal surfaces in GRW

The discussion in Section 1.1.3 led us in position to obtain some Calabi-Berstein type results as corollaries of the results obtained in the previous section.

Corollary 5.4.1 *Let $\overline{M}^3 = -I \times_\rho M^2$ be a GRW spacetime whose fiber has Gauss curvature satisfying $K_M \geq -\kappa$, for some positive constant κ. Let $\Sigma(u)$ be an entire graph in $\overline{M}^3$ such that $K_G \leq \Gamma(r)$. If*

$$|Du|^2 \leq \frac{C\rho^2|A|^2}{1 + C|A|^2},$$

where $C = \frac{5\alpha}{6C_\rho}$, then u must be constant.

Corollary 5.4.2 *Let $u : M \to -I \times_\rho M^2$ be a maximal entire graph. Suppose that $-I \times_\rho M^2$ satisfies the TCC and*

$$|Du|^2 \leq \frac{\beta^2|A|^{2\alpha} f^2}{1 + \beta^2|A|^{2\alpha}}$$

for constants $\alpha, \beta > 0$. If $-\frac{\rho''}{\rho} \leq \Gamma(r)$, then u is constant.

Corollary 5.4.3 *Let $u : M \to -I \times_\rho M^2$ be a maximal entire graph. Suppose that $-I \times_\rho M^2$ satisfies the NCC. If $-\frac{\rho''}{\rho} \leq \Gamma(r)$ and*

$$|Du|^2 \leq \frac{C\rho^2|A|^2}{1 + C|A|^2},$$

where $C = \frac{5\alpha}{6C_\rho}$, $0 \leq \alpha < 1$ is constant, $C_\rho = \sup\{\rho''/\rho\} > 0$, then u is constant.

Chapter 6

Examples

It follows two examples given by A. Albujer in [1] (see also [3]).

Example 6.0.4 *Consider the upper half-plane model for the two-dimensional hyperbolic space* $\mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2; y > 0\}$ *endowed with the complete metric*

$$\langle\ ,\ \rangle_{\mathbb{H}^2} = \frac{1}{y^2}(dx^2 + dy^2).$$

In this example the function $u : \mathbb{H}^2 \to \mathbb{R}$ *given by* $u(x, y) = a\,\mathrm{Log}(x^2 + y^2)$*, and its corresponding entire graph*

$$\Sigma(u) = \{(a\,\mathrm{Log}(x^2 + y^2), x, y)\ :\ y > 0\} \subset -\mathbb{R} \times \mathbb{H}^2.$$

We have that $Du(x, y) = 2a\,\frac{y^2}{x^2+y^2}(x, y)$ *and, hence,*

$$|Du(x, y)|^2 = 4a^2\,\frac{y^2}{x^2 + y^2}.$$

If we take $0 < |a| < \frac{1}{2}$*, we have that* $\Sigma(u)$ *will be a complete spacelike surface in* $-\mathbb{R} \times \mathbb{H}^2$*. This spacetime graph is maximal [1].*

Notice that

$$|\nabla h|^2 = \frac{|Du(x, y)|^2}{1 - |Du(x, y)|^2}$$

is bounded. A direct computation as made in [44] gives us $A_{(0,y)} \equiv 0$ *and*

$$|\nabla h|^2_{(0,y)} = \frac{4a^2}{1 - 4a^2} > 0.$$

Therefore, inequality (2.12) *does not hold for this example. Meanwhile, this graph is not parabolic although it is stable, see [41], therefore it does not satisfy the hypothesis of our results.*

The following example presented here is maximal but it is not complete [1].

Example 6.0.5 *Here the function u is given by $u(x,y) = \mathrm{Log}(y + \sqrt{a + y^2})$, for a a positive constant. Consider these two tangent vector fields*

$$X_x = \partial_x + \langle \partial_x, N\rangle N,$$

$$X_y = \partial_y + \langle \partial_y, N\rangle N.$$

Observing that u depends only on y, in [44] the authors obtained that $\langle AX_x, X_y\rangle = 0$, as well $\langle X_x, X_y\rangle = 0$. Since the graph of u is maximal, the norm of A is given by

$$|A|^2 = 2|X_x|^{-4}\langle AX_x, X_x\rangle^2 = \frac{2}{W^2}y^2u_y^2\,. \tag{6.1}$$

Since

$$|\nabla h|^2 = \frac{|\nabla u|^2}{W^2} = \frac{1}{W^2}y^2u_y^2,$$

we obtain

$$|\nabla h|^2 = \frac{1}{2}|A|^2.$$

Therefore inequality (2.12) holds for this example, in fact it is valid for any maximal graph in $-\mathbb{R}\times\mathbb{H}^2$ such that u depends only on y. Since this graph is not a slice, we see that parabolicity cannot be dropped out of the hypothesis in Corollary 2.3.3.

Example 6.0.6 *Consider the smooth function $u : \mathbb{H}^2 \to \mathbb{R}$ given by $u(x,y) = a\,\mathrm{Log}\,y$, $a \in \mathbb{R}$, $a < 1$. The graph of u:*

$$\Sigma(u) = \{a\,\mathrm{Log}\,y, x, y); y > 0\} \subset -\mathbb{R}\times\mathbb{H}^2$$

is the Abresch-Rosenberg surface which is detailed in [58]. We have that $\nabla u(x,y) = (0, ay)$ and hence

$$|\nabla u(x,y)|^2 = |a|^2 < 1.$$

Then $\Sigma(u)$ is a complete spacelike surface in $-\mathbb{R}\times\mathbb{H}^2$. Moreover, the height function satisfies:

$$|\nabla h|^2 = \frac{|\nabla u|^2}{1 - |\nabla u|^2} = \frac{|a|^2}{1 - |a|^2}.$$

Consequently,

$$\langle N, \partial_t\rangle = -\frac{1}{\sqrt{1 - |a|^2}}.$$

On the other hand, the mean curvature H of $\Sigma(u)$ is given by

$$2H = \mathrm{Div}\left(\frac{\nabla u}{\sqrt{1 - |\nabla u|^2}}\right),$$

where Div *is the divergence on* $\mathbb{H}^2$*. So, as* $\mathrm{Div} = \mathrm{Div}_0 - \frac{2}{y}dy$*, where* Div_0 *denotes the usual divergence on* $\mathbb{R}^2$*, we get*

$$2H^2r^3 = r^2y^2\Delta_0 u + y^3(yQ(u) + u_y|\nabla_0 u|_0^2), \tag{6.2}$$

where

$$r = \sqrt{1 - |\nabla u|^2} = \sqrt{1 - a^2}$$

and

$$Q(u) = u_x^2 u_{xx} + 2u_x u_y u_{xy} + u_y^2 u_{yy}$$

Furthermore, Δ_0*,* ∇_0 *and* $|\cdot|_0$ *respectively are the Laplacian, the gradient and the norm in the Euclidean metric. Replacing* $u(x, y) = a \operatorname{Log} y$ *in equation (6.2), we obtain*

$$H = -\frac{a}{2\sqrt{1-a^2}},$$

and since

$$\langle N, \partial_t \rangle$$

is constant, from Corollary (1.1.3) we get

$$0 = \Delta\langle N, \partial_t\rangle = (|A|^2 - |\nabla h|^2)\langle N, \partial_t\rangle.$$

Hence,

$$|\nabla h|^2 = |A|^2. \tag{6.3}$$

Furthermore, by the well known equality

$$|A|^2 = 4H^2 - 2K_G,$$

we see that $0 = K_G = k_1k_2$*, where* k_1*,* k_2 *are the eigenvalues of* A*. Therefore, supposing* $k_2 = 0$ *and since*

$$H = -\frac{k_1 + k_2}{2} = -\frac{k_1}{2},$$

we obtain

$$k_1 = \frac{a}{\sqrt{1+a^2}}.$$

The last example sheds light on the study of the uniqueness of stable CMC (non-maximal) hypersurfaces without the parabolicity assumption, which is strictly necessary in the maximal case.

Example 6.0.7 *We can obtain an example of constant angle graph immersed in* $-\mathbb{R} \times \mathbb{H}^2$ *of the type*

$$u(x, y) = f\left(\frac{x}{y}\right).$$

Denoting $r = \frac{x}{y}$ *and* $C = |\nabla u|$*, since* $|\nabla u|^2 = y^2 |\nabla_0 u|_0^2$ *we get:*

$$C^2 = f'(r)^2(1 + r^2).$$

Integrating we obtain

$$f(r) = C \ln(\sqrt{1 + r^2} + r) + k$$

where k *is a constant. In order to check if the mean curvature of the graph of* u *is constant we make use of the following equation that can be found in [40]:*

$$2HR^3 = R^2 y^2 \Delta_0 u + y^3 (yQ(u) + u_y |\nabla_0 u|_0 2)$$

where $R = \sqrt{1 - |\nabla u|^2} = \sqrt{1 - C^2}$ *and* $Q(u) = u_x^2 u_{xx} + 2u_x u_y u_{xy} + u_y^2 u_{yy}$*. Doing the computations we obtain that*

$$H = \frac{rC(R^2 - C^2 r^4)}{2R^3 \sqrt{1 + r^2}}.$$

that is clearly non constant. For this it is enough evaluate in points of kind $(0, y)$ *and* (x, y)*. Also, observe that from [57], the graph is not parabolic.*

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