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Hörmander's theorem for stochastic evolution equations driven by fractional Brownian motion

por

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João Pessoa - PB
Fevereiro / 2019

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sob orientação do

Prof. Dr. Alberto Masayoshi Faria Ohashi

Tese apresentada ao Corpo Docente do Programa Associado de Pós-Graduação em Matemática - UFPB/UFCG, como requisito parcial para obtenção do título de Doutor em Matemática.

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Resumo

Nesta tese, nós provamos o teorema de Hörmander para uma equação de evolução estocástica dada por um movimento Browniano fracionário de classe traço com o expoente de Hurst $\frac{1}{2} < H < 1$ e um semigrupo analítico $\{S(t); t \geq 0\}$ em um espaço de Hilbert separável E. Ao contrário do caso clássico de dimensão finita, o operador Jacobiano em EDPs estocásticas parabólicas é tipicamente não invertível, o que causa uma grande dificuldade em expressar a matriz de Malliavin em termos de um processo adaptado. Através de uma condição de Hörmander sobre os colchetes de Lie aplicados aos campos da equação e uma suposição adicional de que $S(t)E$ é denso, provamos que a lei das projeções finito-dimensionais da EDP estocástica no tempo t admite uma densidade com respeito à medida de Lebesgue. O argumento baseia-se em técnicas de "rough path" no sentido de Gubinelli (*Controlling rough paths. J. Funct. Anal (2004)*) e uma análise do espaço Gaussiano do movimento Browniano fracionário.

Palavras-chave: Equação de evolução estocástica, Movimento Browniano fracionário, Cálculo de Malliavin, Teorema de Hörmander.

Abstract

In this thesis, we prove the Hörmander's theorem for a stochastic evolution equation driven by a trace-class fractional Brownian motion with Hurst exponent $\frac{1}{2} < H < 1$ and an analytical semigroup $\{S(t); t \geq 0\}$ on a given separable Hilbert space E. In contrast to the classical finite-dimensional case, the Jacobian operator in typical parabolic stochastic PDEs is not invertible which causes a severe difficulty in expressing the Malliavin matrix in terms of an adapted process. Under Hörmander's bracket condition on the vector fields of the stochastic PDE and the additional assumption that $S(t)E$ is dense, we prove the law of finite-dimensional projections of the stochastic PDE at time t has a density w.r.t Lebesgue measure. The argument is based on rough path techniques in the sense of Gubinelli (*Controlling rough paths. J. Funct. Anal (2004)*) and a suitable analysis on the Gaussian space of the fractional Brownian motion.

Keywords: Stochastic evolution equation, Fractional Brownian motion, Malliavin calculus, Hörmander's theorem.

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" O que impede de saber não são nem o tempo nem a inteligência, mas somente a falta de curiosidade."

Agostinho da Silva

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Introduction

Let $A_0, A_1, \ldots, A_n$ be C^1-vector fields on $\mathbb{R}^d$. Let $V : \mathbb{R}^d \to \mathbb{R}^d$ be a second-order differential operator of the following form

$$V = \frac{1}{2}\sum_{i=1}^{n} A_i^2 + A_0.$$

For a given function f, under elliptic conditions on the vector fields, it had been known for some time that

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = Vu(t,x); & \text{if } t > 0, x \in \mathbb{R}^d \\ u(0,x) = f(x); & \text{if } x \in \mathbb{R}^d \end{cases} \tag{1}$$

admits a smooth fundamental solution. In the celebrated 1967's paper, Lars Hörmander introduced a much weaker condition on the vector fields in such way that (1) admits a smooth C^∞-solution. This important result had an immediate and profound impact on Probability theory, more specifically, on the study of the infinitesimal behavior of Markov processes defined via the so-called stochastic differential equations (henceforth abbreviated by SDEs) previously introduced by Kiyoshi Itô in 1942. After Hörmander's fundamental work, probabilists made use of purely analytical arguments to infer smoothness of Feller's semigroups associated with strong Markov processes driven by second order differential operators.

This was the situation until Paul Malliavin has published his groundbreaking work [27] on what he called a stochastic calculus of variations and nowadays known as *Malliavin calculus*. His motivation was to investigate the existence of smooth densities for laws of Markov diffusions by using purely probabilistic techniques. In this direction, based on previous works by Leonard Gross [21] on the so-called Abstract Wiener spaces, Malliavin has introduced a differential structure on the Wiener space in such way that typical functionals of the Brownian motion are naturally smooth (in the sense of Malliavin calculus) although *not* Frechét differentiable like solutions of SDEs. The main insight was the observation that typical functionals of the Brownian motion are differentiable in certain directions (the Cameron-Martin space) whose shifts are equivalent to the Wiener measure. More importantly, the so-called Gross-Sobolev (Malliavin) derivative $\mathbf{D}$ admits an adjoint operator which lies at the heart of the success of the Malliavin calculus via integration by parts formula. His method was based on the infinite-dimensional Ornstein-Uhlenbeck semigroup and was rather elaborate. It has since been simplified and extended by many authors and has become a powerful tool in stochastic analysis.

In order to illustrate the main argument based on Malliavin calculus and later highlight the main obstacles in dealing with stochastic partial differential equations (henceforth abbreviated by SPDEs) driven by the fractional Brownian motion, let us briefly recall the classical case: Let X be a finite-dimensional SDE written in Stratonovich form

$$dX_t = V_0(X_t)dt + \sum_{j=1}^{n} V_j(X_t) \circ dW_t^j \tag{2}$$

where $V_0, \ldots, V_n$ are smooth vector fields and $(W_i)_{i=1}^n$ is a standard n-dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Roughly speaking, the Hörmander's theorem for the SDE (2) is a statement on the relation between smoothness of the law $A \mapsto \mathbb{P}(X_t \in A)$ of X_t and a geometric condition on the vector fields $V_0, \ldots, V_n$ which ensures that the solution spreads over the entire space. The first central argument towards the Hörmander's theorem for the SDE (2) is to find weaker conditions on the vector fields (beyond ellipticity) to get invertibility of a certain random matrix involving $\mathbf{D}$. This is achieved by the so-called parabolic Hörmander's bracket condition: In the sequel, $[U, V]$ denotes the Lie bracket between two smooth vector fields U, V.

Definition 0.0.1 *Given an SDE (2), define a family of vector fields $\mathscr{V}_k$ by*

$$\mathscr{V}_0 = \{V_i; 1 \le i \le n\}, \quad \mathscr{V}_{k+1} = \mathscr{V}_k \cup \{[U, V_j]; U \in \mathscr{V}_k \text{ and } 0 \le j \le n\}.$$

Let us also define $\mathscr{V}_k(x) = span\{V(x); V \in \mathscr{V}_k\}$. We say that (2) satisfies the ***parabolic Hörmander condition*** *if $\cup_{k\geq 1}\mathscr{V}_k(x) = \mathbb{R}^d$ for every $x \in \mathbb{R}^d$.*

Let $\mathscr{M}_t$ be the Malliavin matrix

$$\mathscr{M}_t = \big(\langle \mathbf{D}^i X_t, \mathbf{D}^j X_t\rangle_{L^2([0,T];\mathbb{R}^d)}\big)_{1\leq i,j\leq n} \tag{3}$$

at a time $t > 0$, where $\mathbf{D}^j X_t$ is the Gross-Sobolev derivative of X_t w.r.t the j-th Brownian motion. The following result is the basis for the Hörmander's theorem.

Theorem 0.0.2 *Given $x_0 \in \mathbb{R}^d$ and $t \in (0,T]$, assume that X_t is smooth in Malliavin sense with integrable Gross-Sobolev derivatives of all orders and for every $p > 1$*

$$\mathbb{E}\Big(\frac{1}{|det\ \mathscr{M}_t|^p}\Big) < \infty. \tag{4}$$

Then, X_t has a C^∞-density w.r.t Lebesgue in $\mathbb{R}^d$.

The proof of Hörmander's Theorem 0.0.3 below is based on Theorem 0.0.2, a suitable linearization of the SDE (2) w.r.t its initial conditions and a quantitative version of Doob-Meyer decomposition, the so-called Norris's lemma ([34]). Denote by Φ_t the (random) solution map to (2) so that $X_t = \Phi_t(x_0)$. It is known that under Assumption 1 below, we do have a flow of smooth maps, namely a two parameter family of maps $\Phi_{s.t}$ such that $X_t = \Phi_{s,t}(X_s)$ for every $s \leq t$ and such that $\Phi_{t,u} \circ \Phi_{s,t} = \Phi_{s,u}$ and $\Phi_t = \Phi_{0,t}$. For a given initial condition x_0, we then denote by $J_{s,t}$ the derivative of $\Phi_{s,t}$ evaluated at X_s. The chain rule implies $J_{s,u} = J_{t,u}J_{s,t}$ and

$$dJ_{0,t} = DV_0(X_t)J_{0,t}dt + \sum_{j=1}^n DV_j(X_t)J_{0,t} \circ dW^j_t; \quad J_{0,0} = I$$

where I is the identity matrix. Higher order derivatives $J^{(k)}_{0,t}$ w.r.t initial conditions can be defined similarly.

By the composition property $J_{0,t} = J_{s,t}J_{0,s}$, we can write $J_{s,t} = J_{0,t}J^{-1}_{0,s}$, where the inverse $J^{-1}_{0,t}$ can be found by solving

$$dJ^{-1}_{0,t} = -J^{-1}_{0,t}DV_0(X_t)dt - \sum_{j=1}^n J^{-1}_{0,t}DV_j(X_t) \circ dW^j_t.$$

Assumption 1 *The vector fields* $V_0, \dots, V_n$ *are smooth and all their derivatives grow at most polynomially at infinity. Moreover,*

$$\mathbb{E} \sup_{0 \le t \le T} |X_t|^p < \infty, \quad \mathbb{E} \sup_{0 \le t \le T} |J_{0,t}^{-1}|^p < \infty \quad \textit{and}\ \mathbb{E} \sup_{0 \le t \le T} |J_{0,t}^{(k)}|^p < \infty$$

for every $k \ge 1$*, for every initial condition* x_0*, every terminal time* T *and every* $p \ge 1$.

The Hörmander's theorem for the SDE (2) is given by the following result.

Theorem 0.0.3 *If the vector fields* $V_0, \dots, V_n$ *satisfy the Hörmander's bracket conditions and Assumption 1 is satisfied, then the law of* X_t *has a smooth density w.r.t Lebesgue for every* $t > 0$.

Let us now outline the classical proof of Theorem 0.0.3. The Malliavin matrix associated with the X_t is given by

$$\langle \xi, \mathscr{M}_t \xi \rangle = \sum_{j=1}^{n} \int_0^t \langle \xi, J_{s,t} V_j(X_s) \rangle^2 ds; \xi \in \mathbb{R}^d.$$

Let V be the $d \times n$-matrix-valued function obtained by concatenating the vector fields V_j for $j = 1, \dots, n$. One can check

$$\mathscr{M}_t = J_{0,t} \mathcal{C}_t J_{0,t}^* \tag{5}$$

where

$$\mathcal{C}_t = \int_0^t J_{0,s}^{-1} V(X_s) V^*(X_s) \big(J_{0,s}^{-1}\big)^* ds.$$

Representation (5) is due to the fundamental relation $J_{s,t} = J_{0,t} J_{0,s}^{-1}$ so that the invertibility of $\mathscr{M}_t$ is equivalent to the invertibility of the so-called *reduced* Malliavin matrix $\mathcal{C}_t$ given by the following quadratic form

$$\langle \mathcal{C}_t \xi, \xi \rangle = \sum_{j=1}^{n} \int_0^t \langle \xi, J_{0,s}^{-1} V_j(X_s) \rangle^2 ds; \xi \in \mathbb{R}^d.$$

At this point, a well-known trick (see e.g Lemma 2.3.1 in Nualart [31]) says that if

$$\sup_{\|\xi\|=1} \mathbb{P}\{\langle \xi, \mathcal{C}_t \xi \rangle \le \epsilon\} = O(\epsilon^p) \tag{6}$$

for every $p \geq 1$ and $\epsilon > 0$, then (4) holds true. In order to investigate (6) or even the simpler question of invertibility, it is important to notice that working with the reduced Malliavin matrix $\mathcal{C}_t$ is much simpler than $\mathscr{M}_t$. The reason is that the integrand in $\{\langle \xi, J_{0,s}^{-1} V_j(X_s)\rangle^2; 0 \leq s \leq t\}$ is adapted w.r.t driving Brownian motion noise along a given time interval $[0,t]$. In strong contrast, $\langle \xi, J_{s,t} V_j(X_s)\rangle^2$ is not adapted which prevents us to make use of standard stochastic calculus techniques. Working with $\mathcal{C}_t$ yields the following argument: For a given smooth vector field G, let us define $Z_G(t) = \langle \xi, J_{0,t}^{-1} G(X_t)\rangle$. In this case, Jensen's inequality yields

$$\langle \xi, \mathcal{C}_t \xi\rangle = \sum_{j=1}^{n} \int_0^t |Z_{V_j}(s)|^2 ds \geq C\Big(\int_0^t |Z_{V_j}(s)| ds\Big)^2 \tag{7}$$

for a constant C which only depends on t. By Itô's formula, the process Z_G has the nice property that it solves the SDE written in Itô's form

$$dZ_G(s) = \left(Z_{[G,V_0]}(s) + \sum_{j=1}^{n} \frac{1}{2} Z_{[[G,V_j],V_j]}(s) \right) ds + \sum_{j=1}^{n} Z_{[G,V_j]}(s) dW^j(s). \tag{8}$$

At this point, the standard argument is the following: If $\langle \xi, \mathcal{C}_t \xi\rangle$ is small, then (7) jointly with Norris's Lemma ([34]) and (8) ensure that $\{[V_j, V_0], [[V_j, V_k], V_k], [V_j, V_k]; 1 \leq k \leq n, 1 \leq j \leq n\}$ is small too. Since Hörmander's bracket condition ensures that these quantities cannot be small simultaneously, then (6) must follow.

Discussion of the literature

The goal of this thesis is to prove the Hörmander's theorem for a SPDE driven by a trace-class fractional Brownian motion with Hurst exponent $\frac{1}{2} < H < 1$. The novelty of our work is to handle the infinite-dimensional case jointly with the fractional case which requires a new set of ideas. For fractional Brownian motion driving noise with $H > \frac{1}{2}$ and under ellipticity assumptions on the vector fields $\{V_i; 0 \leq i \leq n\}$, the existence and smoothness of the density for SDEs are shown by Hu and Nualart [24] and Nualart and Saussereau [32]. The hypoelliptic case for $H > \frac{1}{2}$ is treated by Baudoin and Hairer [1] based on previous papers of Nualart and Saussereau [33] and the integrability of the Jacobian given by Hu and Nualart [24]. When $\frac{1}{4} < H < \frac{1}{2}$, integrability of the Jacobian given by Cass, Litterer and Lyons [7] yields smoothness of densities in the elliptic case. The hypoelliptic case was treated in a series of works by Cass and Friz [8], Cass, Friz and Victoir [9] and culminating with Cass, Hairer, Litterer and Tindel [7] who provide smoothness of densities for a wide class of Gaussian noises including FBM with $\frac{1}{4} < H < \frac{1}{2}$.

The main technical problem with the generalization of Hörmander's theorem to parabolic SPDEs is the fact that the Jacobian $J_{0,t}$ is typically *not* invertible regardless the type of noise. The existence of densities for images of SPDEs solutions through linear functionals and driven by Brownian motion was firstly tackled by Baudoin and Teichmann [2] where the linear part of the SPDE generates a group of bounded linear operators on a Hilbert space. In this case, the Jacobian becomes invertible. Shamarova [38] studies the existence of densities for a stochastic evolution equation driven by Brownian motion in 2-smooth Banach spaces. Recently, based on a pathwise Fubini theorem for rough path integrals, Gerasimovics and Hairer [20] overcome the lack of invertbility of the Jacobian for SPDEs driven by Brownian motion. They show that the Malliavin matrix is invertible on every finite-dimensional subspace and jointly with a purely pathwise Norris's lemma developed by Cass, Gerasimovics and Hairer [20], they prove that laws of finite-dimensional projections of SPDE solutions driven by Brownian motion admit smooth densities w.r.t Lebesgue measure. In contrast to [2], the authors are able to prove existence and smoothness of densities for truly parabolic systems generated by semigroups and SPDEs driven by Brownian motion under a priory integrability conditions on the Jacobian.

Main contributions

In this thesis, we investigate the existence of densities for finite-dimensional projections of SPDEs driven by fractional Brownian motion (henceforth abbreviated by FBM) with Hurst parameter $\frac{1}{2} < H < 1$. More precisely, let

$$dX_t = \big(A(X_t) + F(X_t)\big)dt + G(X_t)dB_t \tag{9}$$

be a SPDE taking values on a separable Hilbert space E, where $\big(A, \mathrm{dom}(A)\big)$ is the infinitesimal generator of an analytic semigroup $\{S(t); t \geq 0\}$ on E, B is a trace-class fractional Brownian motion taking values a separale Hilbert space U with Hurst parameter $\frac{1}{2} < H < 1$ and F, G are smooth coefficients. Let $\mathcal{T} : E \to \mathbb{R}^d$ be a bounded and surjective linear operator. The goal is to prove, under Hörmander's bracket conditions, that the law of

$$\mathcal{T}(X_t) \quad \text{has a density w.r.t Lebesgue}$$

for every $t > 0$. In this thesis, we obtain the proof of this result under the additional assumption that the analytical semigroup has a dense range in E at a given time $t > 0$. To the best of our knowledge, this is the first result of hypoellipticity for SPDEs driven by FBM. The result is build on a carefully analysis of the Itô map (solution map)

$$B \mapsto X(B)$$

defined on a suitable abstract Wiener space associated with a trace-class FBM B with parameter $\frac{1}{2} < H < 1$ and taking values on suitable space of increments. By means of rough path techniques, it is shown that $B \mapsto X(B)$ is Frechét differentiable and hence differentiable in sense of Malliavin calculus. Even though the noise B is more regular than Brownian motion (in the sense of Hölder regularity), the rough path formalism in the sense of Gubinelli [17, 18] allows us to obtain better estimates for the Itô map compared to the classical Riemann sum approach [39] or other more sophisticated frameworks based on fractional calculus [29].

Let us define

$$G_0(x) := Ax + F(x); x \in \mathrm{dom}(A^\infty).$$

where $\mathrm{dom}(A^\infty) = \cap_{n\geq 1}\mathrm{dom}(A^n)$ is equipped with the projective limit topology associated with the graph norm of $\mathrm{dom}(A)$. Given the SPDE (9), define a collection of vector fields $\mathcal{V}_k$ by

$$\mathcal{V}_0 = \{G_i; i \geq 1\}, \quad \mathcal{V}_{k+1} := \mathcal{V}_k \cup \big\{[G_j, U]; U \in \mathcal{V}_k \text{ and } j \geq 0\big\}.$$

where $G_i(x) = G(\eta_i)(x)$ for some orthonormal basis $(\eta_i)_{i=1}^\infty$ of $Q^{\frac{1}{2}}(U)$, where Q a trace-class linear operator on U. We also define the vector spaces $\mathcal{V}_k(x_0) := \mathrm{span}\{V(x_0); V \in$

$\mathcal{V}_k\}$ and we set

$$\mathcal{D}(x_0) := \cup_{k\geq 1}\mathcal{V}_k(x_0)$$

for each $x_0 \in \mathrm{dom}(A^\infty)$. Let us now state the main result of this work.

Theorem 0.0.4 *Fix $x_0 \in dom(A^\infty)$ and assume that $\mathcal{D}(x_0)$ is a dense subset of E and $S(t)E$ is a dense subset of E for a given $t \in (0,T]$. Under H1-A1-A2-A3-B1-B2-C1-C2-C3, if $\mathcal{T} : E \to \mathbb{R}^d$ is a bounded linear surjective operator, then the law of $\mathcal{T}(X_t^{x_0})$ has a density w.r.t Lebesgue measure in $\mathbb{R}^d$.*

Outline of the thesis: In chapter 1, we establish some preliminary results on the Gaussian space of trace-class FBM and the associated Malliavin calculus. Chapter 2 presents the main technical results concerning the Malliavin (actually Frechét regularity) of the Itô map and the existence of the right-inverse of the Jacobian. Chapter 3 presents the proof of Theorem 0.0.4.

Chapter 1

Preliminaries on the Gaussian space of fractional Brownian motion

1.1 The fractional Brownian motion

The fractional Brownian motion (henceforth abbreviated by FBM) with Hurst parameter $0 < H < 1$ is a centered Gaussian process with covariance

$$R_H(t,s) := \frac{1}{2}\big(s^{2H} + t^{2H} - |t-s|^{2H}\big).$$

Throughout this paper, we fix $\frac{1}{2} < H < 1$. Let $\beta = \{\beta_t; 0 \le t \le T\}$ be a FBM defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\mathcal{E}$ be the set of all step functions on $[0,T]$ equipped with the inner product

$$\langle \mathbb{1}_{[0,t]}, \mathbb{1}_{[0,s]} \rangle_{\mathcal{H}} := R_H(t,s).$$

One can check (see e.g Chapter 5 in [31] or Chapter 1 in [30]) for every $\varphi, \psi \in \mathcal{E}$, we have

$$\langle \varphi, \psi \rangle_{\mathcal{H}} = \alpha_H \int_0^T \int_0^T |r-u|^{2H-2} \varphi(r)\psi(u) du dr \tag{1.1}$$

where $\alpha_H := H(2H-1)$. Let $\mathcal{H}$ be the reproducing kernel Hilbert space associated with FBM, i.e., the closure of $\mathcal{E}$ w.r.t (1.1). The mapping $\mathbb{1}_{[0,t]} \to \beta_t$ can be extended to an isometry between $\mathcal{H}$ and the first chaos $\{\beta(\varphi); \varphi \in \mathcal{H}\}$. We shall write this isometry as $\beta(\varphi)$.

Let us define the following kernel

$$K_H(t,s) := c_H s^{\frac{1}{2}-H}\int_s^t (u-s)^{H-\frac{3}{2}}u^{H-\frac{1}{2}}du; s<t, \tag{1.2}$$

where $c_H = \left(\frac{H(2H-1)}{\text{beta}(2-2H,H-\frac{1}{2})}\right)^{\frac{1}{2}}$ and beta denotes the Beta function. We set $K_H(t,s)=0$ for $s\ge t$. From (1.2), we have

$$\frac{\partial K_H}{\partial t}(t,s) = c_H\Big(\frac{t}{s}\Big)^{H-\frac{1}{2}}(t-s)^{H-\frac{3}{2}}.$$

Consider the linear operator $K^*_H : \mathcal{E}\to L^2([0,T];\mathbb{R})$ defined by

$$(K^*_H\varphi)(s) := \int_s^T \varphi(t)\frac{\partial K_H}{\partial t}(t,s)dt; 0\le s\le T.$$

We observe $(K^*_H\mathbb{1}_{[0,t]})(s) = K_H(t,s)\mathbb{1}_{[0,t]}(s)$. It is well-known (see e.g [31]) that K^*_H can be extended to an isometric isomorphism between $\mathcal{H}$ and $L^2([0,T];\mathbb{R})$. Moreover,

$$\beta(\varphi) = \int_0^T (K^*_H\varphi)(t)dw_t; \varphi\in\mathcal{H}, \tag{1.3}$$

where

$$w_t := \beta\big((K^*_H)^{-1}(\mathbb{1}_{[0,t]})\big) \tag{1.4}$$

is a real-valued Brownian motion. From (1.3),

$$\beta_t = \int_0^t K_H(t,s)dw_s; 0\le t\le T,$$

and (1.4) implies both β and w generate the same filtration. Lastly, we recall that $\mathcal{H}$ is a linear space of distributions of negative order. In order to obtain a space of functions contained in $\mathcal{H}$, we consider the linear space $|\mathcal{H}|$ as the space of measurable functions $f:[0,T]\to\mathbb{R}$ such that

$$\|f\|^2_{|\mathcal{H}|} := \alpha_H\int_0^T\int_0^T |f(t)||f(s)||t-s|^{2H-2}dsdt<\infty, \tag{1.5}$$

for a constant $\alpha_H>0$. The space $|\mathcal{H}|$ is a Banach space with the norm (1.5) and isometric to a subspace of $\mathcal{H}$ which is not complete under the inner product (1.1). Moreover, $\mathcal{E}$ is dense in $|\mathcal{H}|$. The following inclusions hold true

$$L^{\frac{1}{H}}([0,T];\mathbb{R}) \hookrightarrow |\mathcal{H}| \hookrightarrow \mathcal{H}. \tag{1.6}$$

and

$$\langle f, g\rangle_{\mathcal{H}} = \alpha_H \int_0^T \int_0^T |u-v|^{2H-2} f(u)g(v)dudv \tag{1.7}$$

for $f, g \in L^{\frac{1}{H}}([0,T];\mathbb{R})$. Moreover, there exists a constant C such that

$$\|f\|^2_{\mathcal{H}} = C\int_0^T |I^{H-\frac{1}{2}}_{T-} f(s)|^2 ds \tag{1.8}$$

where $I^{H-\frac{1}{2}}_{T-}$ is the right-sided fractional integral given by

$$I^{H-\frac{1}{2}}_{T-} f(x) := \frac{1}{\Gamma(H-\frac{1}{2})}\int_x^T f(s)(s-x)^{H-\frac{3}{2}} ds; 0 \le x \le T.$$

See Lemma 1.6.6 and (1.6.14) in [30].

1.2 Malliavin Calculus on Hilbert spaces

Throughout this thesis, we fix a self-adjoint, non-negative and trace-class operator $Q: U \to U$ defined on a separable Hilbert space U. Then, there exists an orthonotmal basis $\{e_i; i \ge 1\}$ of U and eigenvalues $\{\lambda_i; i \ge 1\}$ such that

$$Qe_i = \lambda_i e_i; i \ge 1$$

and trace $Q = \sum_{k=1}^\infty \lambda_k < +\infty$ such that $\lambda_k > 0$ for every $k \ge 1$. Let $U_0 := Q^{\frac{1}{2}}(U)$ be the linear space equipped with the inner product

$$\langle u_0, v_0\rangle_0 := \langle Q^{-\frac{1}{2}}u_0, Q^{-\frac{1}{2}}v_0, \rangle_U; u_0, v_0 \in U_0$$

where $Q^{-\frac{1}{2}}$ is the inverse of $Q^{\frac{1}{2}}$. Then, $(U_0, \langle\cdot,\cdot\rangle_0)$ is a separable Hilbert space with an orthonormal basis $\{\sqrt{\lambda_k}e_k; k \ge 1\}$.

Let W be a Q-Brownian motion given by

$$W_t := \sum_{k\geq 1} \sqrt{\lambda_k} e_k w_t^k; t \geq 0$$

where $(w^k)_{k\geq 1}$ is a sequence of independent real-valued Brownian motions. Let $(\beta^k)_{k\geq 1}$ be a sequence of independent FBM where β^k is associated with w^k via (1.3), i.e.,

$$\beta_t^k = \int_0^t K_H(t,s) dw_s^k; 0 \leq t \leq T.$$

We then set

$$B_t := \sum_{k=1}^{\infty} \sqrt{\lambda_k} e_k \beta_t^k; 0 \leq t \leq T. \tag{1.9}$$

For separable Hilbert spaces E_1 and E_2, let us denote $\mathcal{L}_2(E_1; E_2)$ as the space of all Hilbert-Schmidt operators from E_1 to E_2 equipped with the usual inner product. Let $\mathcal{F}$ be the sigma-field generated by $\{B(\varphi); \varphi \in \mathcal{H} \otimes \mathcal{L}_2(U_0, \mathbb{R})\}$ where $B : \mathcal{H} \otimes \mathcal{L}_2(U_0, \mathbb{R}) \to L^2(\Omega, \mathcal{F}, \mathbb{P})$ is the linear operator defined by

$$B(\Phi) := \int_0^T \Phi(t) dB_t := \sum_{k=1}^{\infty} \int_0^T (K_H^* \Phi^k)(t) dw_t^k; \Phi \in \mathcal{H} \otimes \mathcal{L}_2(U_0, \mathbb{R}),$$

where

$$\Phi^i := \Phi(\sqrt{\lambda_i} e_i); i \geq 1.$$

We recall that $\mathcal{H} \otimes \mathcal{L}_2(U_0, \mathbb{R})$ is isomorphic to $\mathcal{L}_2(U_0, \mathcal{H})$. The elements of $\mathcal{H} \otimes \mathcal{L}_2(U_0, \mathbb{R})$ are described by

$$\sum_{m,j=1}^{\infty} a_{mj} \sqrt{\lambda_m} e_m \otimes h_j$$

where $(a_{mj})_{m,j} \in \ell^2(\mathbb{N}^2)$, (h_j) is an orthonormal basis for $\mathcal{H}$ and we denote

$$e \otimes h : y \in U_0 \mapsto \langle e, y \rangle_{U_0} h.$$

It is easy to check that $\mathbb{E}\big[B(\Phi)B(\Psi)\big] = \langle \Phi, \Psi \rangle_{\mathcal{L}_2(U_0, \mathcal{H})}$ for every $\Phi, \Psi \in \mathcal{L}_2(U_0, \mathcal{H})$. In this case, $\Big(\Omega, \mathcal{F}, \mathbb{P}; \mathcal{L}_2(U_0, \mathcal{H})\Big)$ is the Gaussian space associated with the isonormal Guassian process B.

For Hilbert spaces E_1 and E_2, let $C_p^k(E_1; E_2)$ be the space of all $f : E_1 \to E_2$ such that f and all its derivatives has polynomial growth. Let $\mathcal{P}$ be the set of all cylindrical random variables of the form

$$F = f(B(\varphi_1), \ldots, B(\varphi_m)) \tag{1.10}$$

where $f \in C_p^\infty(\mathbb{R}^m; \mathbb{R})$ an $\varphi_i \in \mathcal{L}_2(U_0, \mathcal{H})$. The Malliavin derivative of an element of $F \in \mathcal{P}$ of the form (1.10) over the Gaussian space $\Big(\Omega, \mathcal{F}, \mathbb{P}; \mathcal{L}_2(U_0, \mathcal{H})\Big)$ is defined by

$$\mathbf{D}F := \sum_{k=1}^m \frac{\partial}{\partial x_k} f(B(\varphi_1), \ldots, B(\varphi_m))\varphi_k.$$

We observe

$$\begin{aligned}
\langle \mathbf{D}F, h\rangle_{\mathcal{L}_2(U_0,\mathcal{H})} &= \sum_{k=1}^m \frac{\partial}{\partial x_k} f(B(\varphi_1), \ldots, B(\varphi_m))\langle \varphi_k, h\rangle_{\mathcal{L}_2(U_0,\mathcal{H})} \\
&= \frac{d}{d\epsilon} f\big(B(\varphi_1) + \epsilon\langle\varphi_1, h\rangle_{\mathcal{L}_2(U_0,\mathcal{H})}, \ldots, B(\varphi_m) + \epsilon\langle\varphi_m, h\rangle_{\mathcal{L}_2(U_0,\mathcal{H})}\big)|_{\epsilon=0}.
\end{aligned}$$

The k-th derivative is naturally defined as the iterated derivative $\mathbf{D}^k F$ for $F \in \mathcal{P}$ as a random variable with values in $(\mathcal{L}_2(U_0, \mathcal{H}))^{\otimes k}$. For a given separable Hilbert space E, let $\mathcal{P}(E)$ be the set of all cylindrical E-valued random variables of the form

$$F = \sum_{j=1}^n F_j h_j$$

where $F_j \in \mathcal{P}$ and $h_j \in E$ for $j = 1, \ldots, n$ and $n \geq 1$. We then define

$$\mathbf{D}^k F := \sum_{j=1}^n \mathbf{D}^k F_j \otimes h_j; k \geq 1$$

A routine exercise yields the following result.

Lemma 1.2.1 *The operator $\mathbf{D}^k : \mathcal{P}(E) \subset L^p(\Omega; E) \to L^p\big(\Omega; (\mathcal{L}_2(U_0, \mathcal{H}))^{\otimes k} \otimes E\big)$ is closable and densely defined for every $p \geq 1$.*

For an integer $k \geq 1$ and $p \geq 1$, let $\mathbb{D}^{k,p}(E)$ be the completion of $\mathcal{P}(E)$ w.r.t the semi-norm

$$\|F\|_{\mathbb{D}^{k,p}(E)} := \left[\mathbb{E}\|F\|_E^p + \sum_{j=1}^k \mathbb{E}\|\mathbf{D}^j F\|^p_{\mathcal{L}_2(U_0,\mathcal{H})^{\otimes j}\otimes E}\right]^{1/p}.$$

One can check the family of seminorms satisfies the properties of monotonicity and compatibility (see Section 1.2, Chapter 2 in [31]). Moreover, $\mathbb{D}^{k+1,p}(E) \subset \mathbb{D}^{k,q}(E)$ for $p > q$ and $k \geq 0$.

Let us now devote our attention to some criteria for checking when a given functional $F : \Omega \to E$ belongs to the Sobolev spaces $\mathbb{D}^{k,p}(E)$ for $p > 1$ and $k \geq 1$.

Lemma 1.2.2 *Let $p > 1$ and $F \in L^p_{loc}(\Omega; E)$ be such that for every $x \in E$ one has $\langle F, x\rangle_E \in \mathbb{D}^{1,p}_{loc}(\mathbb{R})$. If there exists*[1] *$\xi \in L^p_{loc}\big(\Omega; \mathcal{L}_2(U_0; \mathcal{H}) \otimes E\big)$ such that*

$$\Big\langle \mathbf{D}\langle F, u\rangle_E, h\Big\rangle_{\mathcal{L}_2(U_0;\mathcal{H})} = \langle \xi(u), h\rangle_{\mathcal{L}_2(U_0;\mathcal{H})} \text{ locally} \tag{1.11}$$

for every $u \in E, h \in \mathcal{L}_2(U_0; \mathcal{H})$, then $F \in \mathbb{D}^{1,p}_{loc}(E)$ and $\mathbf{D}F = \xi$.

Proof. Consider the Gaussian space $\Big(\Omega, \mathcal{F}, \mathbb{P}; \mathcal{L}_2(U_0, \mathcal{H})\Big)$, take a localizing sequence $(\Omega_n, F_n) \in \mathcal{F} \times \mathbb{D}^{1,2}(\mathbb{R})$ such that $F_n = \langle F, u\rangle_E$ on Ω_n and $\Omega_n \uparrow \Omega$ as $n \to +\infty$. Then, apply Theorem 3.3 given by [36]. ■

In view of the Hölder path regularity of the underlying noise, it will be useful to play with Fréchet and Malliavin derivatives. In this case, it is convenient to realize $\mathbb{P}$ as a Gaussian probability measure defined on a suitable Hölder-type separable Banach space equipped with a Cameron-Martin space which supports infinitely many independent FBMs. Let $\mathcal{C}^\infty_0(\mathbb{R}_+)$ be the space of smooth functions $w : [0, \infty) \to \mathbb{R}$ satisfying $w(0) = 0$ and having compact support. Given $\gamma \in (0, 1)$ and $\delta \in (0, 1)$, we define for every $w \in \mathcal{C}^\infty_0(\mathbb{R}_+)$, the norm

$$\|w\|_{\mathcal{W}^{\gamma,\delta}} := \sup_{t,s\in\mathbb{R}_+} \frac{|w(t) - w(s)|}{|t - s|^\gamma (1 + |t| + |s|)^\delta}.$$

Let $\mathcal{W}^{\gamma,\delta}$ be the completion of $\mathcal{C}^\infty_0(\mathbb{R}_+)$ w.r.t $\|\cdot\|_{\mathcal{W}^{\gamma,\delta}}$. We also write $\mathcal{W}^{\gamma,\delta}_T$ when we restrict the arguments to the interval $[0, T]$. It should be noted that $\|\cdot\|_{\mathcal{W}^{\gamma,\delta}_T}$ is equivalent to the γ-Hölder norm on $[0, T]$ given by

$$|f|_0 + |f|_\gamma,$$

[1] We observe $\mathcal{L}_2(U_0, \mathcal{H} \otimes E) \equiv \mathcal{L}_2(U_0, \mathcal{H}) \otimes E \equiv E \otimes U_0 \otimes \mathcal{H} \equiv \mathcal{L}_2(E; \mathcal{L}_2(U_0; \mathcal{H}))$

where

$$|f|_\gamma := \sup_{0\le s<t\le T} \frac{|f(t)-f(s)|}{|t-s|^\gamma}, \ \frac{1}{2}<\gamma<1.$$

Moreover, $\mathcal{W}_T^{\gamma,\delta}$ is a separable Banach space. Let $\lambda = (\lambda_i)_{i=1}^\infty$ be the sequence of strictly positive eigenvalues of Q. In addition to trace $Q = \sum_{i\ge 1}\lambda_i < \infty$, let us assume $\sum_{i\ge1}\sqrt{\lambda_i}<\infty$. Let $\mathcal{W}_{\lambda,T}^{\gamma,\delta,\infty}$ be the vector space of functions $g:\mathbb{N}\to\mathcal{W}_T^{\gamma,\delta}$ such that

$$\|g\|_{\mathcal{W}_{\lambda,T}^{\gamma,\delta,\infty}} := \sum_{i=1}^\infty \sqrt{\lambda_i}\|g^i\|_{\mathcal{W}_T^{\gamma,\delta}} < \infty.$$

Clearly, $\mathcal{W}_{\lambda,T}^{\gamma,\delta,\infty}$ is a normed space.

Lemma 1.2.3 *$\mathcal{W}_{\lambda,T}^{\gamma,\delta,\infty}$ is a separable Banach space equipped with the norm $\|\cdot\|_{\mathcal{W}_{\lambda,T}^{\gamma,\delta,\infty}}$.*

Proof. Let $\|g_n - g_m\|_{\mathcal{W}_{\lambda,T}^{\gamma,\delta,\infty}} \to 0$ as $n,m\to+\infty$. Then, for $\epsilon>0$, there exists $N(\epsilon)$ such that

$$\sum_{i=1}^\infty \sqrt{\lambda_i}\|g_n^i - g_m^i\|_{\mathcal{W}_T^{\gamma,\delta}} < \epsilon$$

for every $n,m>N(\epsilon)$. Since $\mathcal{W}_T^{\gamma,\delta}$ is complete, then there exists $g:\mathbb{N}\to\mathcal{W}_T^{\gamma,\delta}$ defined by $g^i := \lim_{n\to\infty} g_n^i$ in $\mathcal{W}_T^{\gamma,\delta}$ for each $i\ge1$. By construction, we observe that

$$\begin{aligned}
\sum_{i=1}^\infty \sqrt{\lambda_i}\|g^i\|_{\mathcal{W}_T^{\gamma,\delta}} &\le \sum_{i=1}^\infty \sqrt{\lambda_i}\|g^i - g_n^i\|_{\mathcal{W}_T^{\gamma,\delta}} + \sum_{i=1}^\infty \sqrt{\lambda_i}\|g_n^i\|_{\mathcal{W}_T^{\gamma,\delta}} \\
&\le \sum_{i=1}^\infty \sqrt{\lambda_i}\frac{\epsilon}{\sqrt{2^i}} + \sup_{j\ge1}\|g_j\|_{\mathcal{W}_{\delta,T}^{\lambda,\gamma,\infty}} \\
&\le \left(\sum_{i=1}^\infty \lambda_i\right)^{\frac{1}{2}}\sqrt{2}\epsilon + \sup_{n\ge1}\|g_n\|_{\mathcal{W}_{\lambda,T}^{\gamma,\delta,\infty}} < \infty.
\end{aligned}$$

For separability, let $\left[\oplus_{j=1}^\infty \mathcal{W}_T^{\gamma,\delta}\right]_2 = \{f:\mathbb{N}\to\mathcal{W}_T^{\gamma,\delta}; \|f\|_2<\infty\}$ be the l_2-direct sum of the Banach spaces $\mathcal{W}_T^{\gamma,\delta}$ where

$$\|f\|_2 = \left(\sum_{j=1}^\infty \|f^j\|^2_{\mathcal{W}_T^{\gamma,\delta}}\right)^{\frac{1}{2}}.$$

Since trace $Q<\infty$, then

$$\|\cdot\|_{\mathcal{W}_{\lambda,T}^{\gamma,\delta,\infty}} \le (\text{trace } Q)^{\frac{1}{2}}\|\cdot\|_2. \tag{1.12}$$

Of course, $\cup_{n\geq 1}\oplus_{j=1}^{n}\mathcal{W}_T^{\gamma,\delta}\subset\oplus_{j=1}^{\infty}\mathcal{W}_T^{\gamma,\delta}$ and clearly $\cup_{n\geq 1}\oplus_{j=1}^{n}\mathcal{W}_T^{\gamma,\delta}$ is a dense subset of $\big[\oplus_{j=1}^{\infty}\mathcal{W}_T^{\gamma,\delta}\big]_2$. Since $\mathcal{W}_T^{\gamma,\delta}$ is separable, the previous argument shows $\Big[\oplus_{j=1}^{\infty}\mathcal{W}_T^{\gamma,\delta}\Big]_2$ is separable and hence (1.12) implies $\mathcal{W}_{\lambda,T}^{\gamma,\delta,\infty}$ is separable as well. ■

Lemma 1.2.4 *If $\gamma\in\big(\frac{1}{2},H\big)$ and $\gamma+\delta\in(H,1)$, then there exists a Gaussian probability measure $\mu_{\gamma,\delta}^{\infty}$ on $\mathcal{W}_{\lambda,T}^{\gamma,\delta,\infty}$. Therefore, there exists a separable Hilbert space $\mathbf{H}$ continuously imbedded into $\mathcal{W}_{\lambda,T}^{\gamma,\delta,\infty}$ such that $\big(\mathcal{W}_{\lambda,T}^{\gamma,\delta,\infty},\mathbf{H},\mu_{\gamma,\delta}^{\infty}\big)$ is an abstract Wiener space.*

Proof. From Lemma 4.1 in [23], we know there exists a probability measure $\mu_{\gamma,\delta}$ on $\mathcal{W}_T^{\gamma,\delta}$ such that the canonical process is a FBM with Hurst parameter $\frac{1}{2}<H<1$ as long as $\gamma\in\big(\frac{1}{2},H\big)$ and $\gamma+\delta\in(H,1)$. Let $\mathcal{W}_T^{\gamma,\delta,\infty}:=\prod_{j\geq 1}\mathcal{W}_T^{\gamma,\delta}$ be the countable product of the Banach spaces $\mathcal{W}_T^{\gamma,\delta}$ equipped with the product topology which makes $\mathcal{W}_T^{\gamma,\delta,\infty}$ as a topological vector space. Let $\mu_{\gamma,\delta}^{\infty}$ be the product probability measure $\otimes_{j\geq 1}\mu_{\gamma,\delta}$ over $\mathcal{W}_T^{\gamma,\delta,\infty}$ equipped with the usual product sigma-algebra. Then, $\mu_{\gamma,\delta}^{\infty}$ is a Gaussian probability measure (see e.g Example 2.3.8 in [4]). Moreover, we observe

$$\mu_{\gamma,\delta}^{\infty}\big(\mathcal{W}_{\lambda,T}^{\gamma,\delta,\infty}\big)=1.$$

Indeed, by construction, we can take a sequence of $\mu_{\gamma,\delta}$-independent FBMs $\beta^i; i\geq 1$. By using the modulus of continuity of FBM, it is well-known that $\mathbb{E}_{\mu_{\gamma,\delta}}\|\beta^i\|_{\mathcal{W}_T^{\gamma,\delta}}=\mathbb{E}_{\mu_{\gamma,\delta}}\|\beta^1\|_{\mathcal{W}_T^{\gamma,\delta}}<\infty$ for every $i\geq 1$. Therefore,

$$\mathbb{E}_{\mu_{\gamma,\delta}^{\infty}}\sum_{i=1}^{\infty}\sqrt{\lambda_i}\|\beta^i\|_{\mathcal{W}_T^{\gamma,\delta}}=\mathbb{E}_{\mu_{\gamma,\delta}}\|\beta^1\|_{\mathcal{W}_T^{\gamma,\delta}}\sum_{i=1}^{\infty}\sqrt{\lambda_i}<\infty$$

and this proves that $\mu_{\gamma,\delta}^{\infty}$ is a Gaussian probability measure on the Banach space $\mathcal{W}_{\lambda,T}^{\gamma,\delta,\infty}$. This shows that we have an abstract Wiener space structure for $\mu_{\gamma,\delta}^{\infty}$. ■

In the sequel, with a slight abuse of notation, we define $K_H^*:\mathcal{E}\otimes\mathcal{L}_2(U_0,\mathbb{R})\to L^2\big([0,T];\mathcal{L}_2(U_0,\mathbb{R})\big)$ as follows

$$K_H^*(h\otimes\varphi)(s):=\int_s^T h(t)\frac{\partial}{\partial t}K_H(t,s)dt\varphi;\quad h\in\mathcal{E},\varphi\in\mathcal{L}_2(U_0,\mathbb{R}).$$

Clearly,

$$\langle K_H^*(h_1 \otimes \varphi_1), K_H^*(h_2 \otimes \varphi_2)\rangle_{L^2([0,T];\mathcal{L}_2(U_0,\mathbb{R}))} = \langle (h_1 \otimes \varphi_1), (h_2 \otimes \varphi_2)\rangle_{\mathcal{H}\otimes\mathcal{L}_2(U_0,\mathbb{R})},$$

for every $h_1, h_2 \in \mathcal{E}$ and $\varphi_1, \varphi_2 \in \mathcal{L}_2(U_0,\mathbb{R})$ and hence we can extend K_H^* to an isometric isomorphism from $\mathcal{H}\otimes\mathcal{L}_2(U_0,\mathbb{R})$ to $L^2\big([0,T];\mathcal{L}_2(U_0,\mathbb{R})\big)$. Let us also denote $\mathcal{K}_H : L^2\big([0,T];\mathcal{L}_2(U_0,\mathbb{R})\big) \to \mathbf{H}$ by

$$\mathcal{K}_H f(t) := \sqrt{\lambda_i}\int_0^t K_H(t,s) f_s(e_i)ds; 0 \le t \le T, i \ge 1,$$

for $f \in L^2\big([0,T];\mathcal{L}_2(U_0,\mathbb{R})\big)$, where $\mathbf{H} :=$ Range $\mathcal{K}_H$ is the Hilbert space equipped with the norm

$$\|\mathcal{K}_H f\|^2_{\mathbf{H}} := \int_0^T \|f_s\|^2_{\mathcal{L}_2(U_0,\mathbb{R})}ds = \sum_{i=1}^\infty \lambda_i \|f(e_i)\|^2_{L^2([0,T];\mathbb{R})} = \sum_{i=1}^\infty \lambda_i \|\mathcal{K}_{H,1}f(e_i)\|^2_{\mathbb{H}}$$

where $\mathbb{H} :=$ Range $\mathcal{K}_{H,1}$ and

$$\mathcal{K}_{H,1}g(t) := \int_0^t K_H(t,s)g(s)ds; 0 \le t \le T,$$

for $g \in L^2([0,T];\mathbb{R})$. We recall (see Th 3.6 [37]) there exists a constant C such that

$$\|\mathcal{K}_{H,1}g\|_{\mathcal{W}_T^{\gamma,\delta}} \le C\|g\|_{L^2([0,T];\mathbb{R})}$$

for every $g \in L^2([0,T];\mathbb{R})$. Therefore, Cauchy-Schwartz inequality yields

$$\|\mathcal{K}_H f\|_{\mathcal{W}^{\gamma,\delta,\infty}_{\lambda,T}} \le (\text{trace } Q)^{\frac{1}{2}}\|\mathcal{K}_H f\|_{\mathbf{H}}$$

for every $f \in L^2\big([0,T];\mathcal{L}_2(U_0,\mathbb{R})\big)$. Let us set $\mathbb{P} = \mu^\infty_{\gamma,\delta}$ and $\Omega = \mathcal{W}^{\gamma,\delta,\infty}_{\lambda,T}$. Summing up the above computations, we conclude $\mathbf{H}$ is the Cameron-Martin space associated with $\mathbb{P}$ in Lemma 1.2.4, namely

$$\int_\Omega \exp(i\langle \omega, z\rangle_{\Omega,\Omega^*})\mathbb{P}(d\omega) = e^{-\frac{1}{2}\|z\|^2_{\mathbf{H}}}; z \in \Omega^* \tag{1.13}$$

where Ω^* is the topological dual of Ω.

By applying Prop. 4.1.3 in [31] (see also [22]), we arrive at the following result. Let $\mathcal{R}_H := \mathcal{K}_H \circ K_H^*$ be the injection of $\mathcal{L}_2(U_0; \mathcal{H})$ into Ω. We observe $\mathcal{R}_H : \mathcal{L}_2(U_0; \mathcal{H}) \to \Omega$ is a bounded operator with dense range.

Corollary 1.2.5 *If a random variable $Y : \Omega \to \mathbb{R}$ is Frèchet differentiable along directions in the Cameron-Martin space $\mathbf{H}$, then*

$$h \mapsto Y(\omega + \mathcal{R}_H(h))$$

is Fréchet differentiable for each $\omega \in \Omega$, $Y \in \mathbb{D}^{1,2}_{loc}(\mathbb{R})$ and

$$\nabla Y(\cdot)(\mathcal{R}_H h) = \langle \mathbf{D}Y, h \rangle_{\mathcal{L}_2(U_0, \mathcal{H})}$$

locally, for every $h \in \mathcal{L}_2(U_0, \mathcal{H})$.

Chapter 2

Malliavin differentiability of solutions

In this chapter, we discuss differentiability in Malliavin sense (on the probability space defined on Lemma 1.2.4) of SPDE mild $\mathbb{F}$-adapted solutions

$$dX_t = \big(A(X_t) + F(X_t)\big)dt + G(X_t)dB_t \tag{2.1}$$

in a separable Hilbert space E, where $\mathbb{F}$ is the filtration generated by a U-valued FBM B with trace class covariance operator $Q : U \to U$ on a separable Hilbert space U

$$B_t = \sum_{i=1}^{\infty} \sqrt{\lambda_i} e_i \beta_t^i$$

where trace $Q = \sum_{i=1}^{\infty} \lambda_i < \infty$ and additional regularity conditions, namely $\sum_{i=1}^{\infty} \sqrt{\lambda_i} < \infty$ and $\lambda_i > 0$ for all $i \geq 1$. Here, $\big(A, \mathrm{dom}(A)\big)$ is the infinitesimal generator of an analytic semigroup $\{S(t); t \geq 0\}$ on E satisfying

$$\|S(t)\| \leq Me^{-\lambda t} \text{ for some constants } \lambda, M > 0 \text{ and for all } t \geq 0.$$

This allows us to define fractional power $\big((-A)^{\alpha}, \mathrm{Dom}((-A)^{\alpha})\big)$ for any $\alpha \in \mathbb{R}$ (see Sections 2.5 and 2.6 in [35]). The coefficients $F : E \to E$ and $G : E \to \mathcal{L}(U; E)$ will satisfy suitable minimal regularity conditions (see Assumption H1) to ensure well-posedness of (2.1). Let us define $G_i(x) := G(x)(e_i)$ for the orthonormal basis $(e_i)_{i\geq 1}$ of

U. Then, we view the solution as

$$X_t = S(t)x_0 + \int_0^t S(t-s)F(X_s)ds + \int_0^t S(t-s)G(X_s)dB_s \tag{2.2}$$

where the dB differential is understood in Young's sense [39, 18]

$$\int_0^t S(t-s)G(X_s)dB_s = \sum_{i=1}^{\infty} \sqrt{\lambda_i} \int_0^t S(t-s)G_i(X_s)d\beta^i_s; 0 \le t \le T$$

where the convergence of the sum is understood $\mathbb{P}$-a.s in E in the sense of Lemma 2.1.2.

The solution of (2.2) will take values on the domains $\mathrm{Dom}((-A)^\delta)$ of the fractional powers $(-A)^\delta; \delta > 0$. To keep notation simple, we denote $E_\alpha := \mathrm{Dom}((-A)^\alpha)$ for $\alpha > 0$ equipped with the norm $|x|_\alpha := \|(-A)^\alpha x\|_E$ which is equivalent to the graph norm of $(-A)^\alpha$. If $\alpha < 0$, let E_α be the completion of E w.r.t to the norm $|x|_\alpha := \|(-A)^\alpha x\|_E$. If $\alpha = 0$, we set $E_\alpha = E$. Then, $(E_\alpha)_{\alpha\in\mathbb{R}}$ is a family of separable Hilbert spaces such that $E_\delta \hookrightarrow E_\alpha$ whenever $\delta \ge \alpha$. Moreover, $S(t)$ may be extended to E_α as bounded linear operators for $\alpha < 0$ and $t \ge 0$. Moreover, $S(t)$ maps E_α to E_δ for every $\alpha \in \mathbb{R}$ and $\delta \ge 0$. To keep notation simple, we denote $\|\cdot\|_{\beta\to\alpha}$ as the norm operator of the space of bounded linear operator $\mathcal{L}(E_\beta, E_\alpha)$ from E_β to E_α and we set $\|\cdot\| = \|\cdot\|_{0\to 0}$. The space of bounded multilinear operators from the n-fold space E^n_α to E_α is equipped with the usual norm $\|\cdot\|_{(n),\alpha\to\alpha}$ for $\alpha \ge 0$.

In order to prove differentiability in Frechét sense, it is crucial to play with linear SPDE solutions living in Banach spaces which are "sensible" to the Hölder -type norm of the noise space $\mathcal{W}^{\gamma,\delta}_{\lambda,T}$. For this purpose, we make use of the algebraic/analytic formalism developed by [17] in the framework of rough paths. Even though we are working with a regular noise $\frac{1}{2} < H < 1$, the techniques developed by [17, 18] allow us to derive better estimates than usual Riemman's sum approach or fractional calculus given by [29].

2.1 Algebraic integration

For completeness of presentation, let us summarize the basic objects of [17, 18] which will be important for us. At first, we fix some notation. We denote by $\mathcal{C}_k(V)$

the set of continuous functions $g : [0,T]^k \to V$ such that $g_{t_1 \dots t_k} = 0$ whenever $t_i = t_{i+1}$ for some $i \le k-1$. We define $\delta : \mathcal{C}_n(V) \to \mathcal{C}_{n+1}(V)$ by

$$(\delta F)_{t_1,\dots t_{n+1}} := \sum_{j=1}^{n+1} (-1)^j F_{t_1,\dots \hat{t}_j \dots t_{n+1}}; F \in \mathcal{C}_n(V)$$

where $\hat{t}_j$ means that this particular argument is omitted. We are mostly going to use the two special cases:

If $F \in \mathcal{C}_1(V)$, then

$$(\delta F)_{ts} = F_t - F_s; (t,s) \in [0,T]^2.$$

If $F \in \mathcal{C}_2(V)$, then

$$(\delta F)_{tsu} = -F_{su} + F_{tu} - F_{ts}; (t,s,u) \in [0,T]^3.$$

We measure the size of the increments by Hölder norms defined as follows: For $f \in \mathcal{C}_2(V)$ and $\mu \ge 0$, let

$$\|f\|_\mu := \sup_{s,t \in [0,T]} \frac{|f_{st}|}{|t-s|^\mu}$$

and we denote $\mathcal{C}_2^\mu(V) := \{f \in \mathcal{C}_2(V); \|f\|_\mu < \infty\}$ and $\mathcal{C}_1^\mu(V) := \{f \in \mathcal{C}_1(V); \|\delta f\|_\mu < \infty\}$. In the same way, for $h \in \mathcal{C}_3(V)$, we set

$$\|h\|_{\gamma,\rho} := \sup_{s,u,t \in [0,T]} \frac{|h_{tus}|}{|t-u|^\rho |s-u|^\gamma}$$

and

$$\|h\|_\mu := \inf \Big\{ \sum_i \|h_i\|_{\rho_i, \mu-\rho_i}; h = \sum_i h_i, 0 < \rho_i < \mu \Big\},$$

where the last infimum is taken over all sequences $\{h_i \in \mathcal{C}_3(V)\}$ such that $h = \sum_i h_i$ and for all choices of numbers $\rho_i \in (0,\mu)$. Then, $\|\cdot\|_\mu$ is a norm on the space $\mathcal{C}_3(V)$, and we set

$$\mathcal{C}_3^\mu(V) := \{h \in \mathcal{C}_3(V); \|h\|_\mu < \infty\}.$$

Let us denote $\mathcal{ZC}_k(V) := \mathcal{C}_k(V) \cap \mathrm{Ker}\delta|_{\mathcal{C}_k(V)}$ and $\mathcal{BC}_k(V) := \mathcal{C}_k(V) \cap \mathrm{Range}\ \delta|_{\mathcal{C}_{k-1}(V)}$.

We have $\mathcal{ZC}_{k+1}(V) = \mathcal{BC}_{k+1}(V)$ for $k \geq 1$.

The convolutional increments will be defined as follows. Let $\mathcal{S}_n = \{(t_1, \ldots, t_n); T \geq t_1 \geq t_2 \geq \ldots t_n \geq 0\}$. For a Banach space V, $\hat{\mathcal{C}}_n(V)$ denotes the space of continuous functions from $\mathcal{S}_n$ to V. We also need a modified version of basic increments distorted by the semigroup as follows: Let $\hat{\delta} : \hat{\mathcal{C}}_n(E) \to \hat{\mathcal{C}}_{n+1}(E)$ given by

$$(\hat{\delta} F)_{t_1,\ldots t_{n+1}} := (\delta F)_{t_1,\ldots t_{n+1}} - a_{t_1 t_2} F_{t_2 \ldots t_n}$$

where $a_{t_1 t_2} := S(t_1 - t_2) - \mathrm{Id}$ for $(t_1, t_2) \in \mathcal{S}_2$.

Hölder-type space of increments. We need to define Hölder-type subspaces of $\hat{\mathcal{C}}_k$ for $1 \leq k \leq 3$ associated with $E_\alpha; \alpha \in \mathbb{R}$. For $0 < \mu < 1$ and $g \in \hat{\mathcal{C}}_2(E_\alpha)$, we define the norm

$$\|g\|_{\mu,\alpha} := \sup_{t,s \in \mathcal{S}_2} \frac{|g_{ts}|_\alpha}{|t-s|^\mu}$$

and the spaces

$$\hat{\mathcal{C}}_2^{\mu,\alpha} := \{g \in \hat{\mathcal{C}}_2(E_\alpha); \|g\|_{\mu,\alpha} < \infty\}$$

and

$$\hat{\mathcal{C}}_1^{\mu,\alpha} := \{f \in \hat{\mathcal{C}}_1(E_\alpha); \|\hat{\delta} f\|_{\mu,\alpha} < \infty\},$$

$$\mathcal{C}_1^{\mu,\alpha} := \{f \in \hat{\mathcal{C}}_1(E_\alpha); \|\delta f\|_{\mu,\alpha} < \infty\}.$$

We denote $\hat{\mathcal{C}}_1^{0,\alpha} := \hat{\mathcal{C}}_1(E_\alpha)$ equipped with the norm

$$\|f\|_{0,\alpha} := \sup_{0 \leq t \leq T} |f_t|_\alpha.$$

We also equip $\mathcal{C}_1^{\mu,\alpha}$ and $\hat{\mathcal{C}}_1^{\mu,\alpha}$ with the norms given, respectively, by

$$\|f\|_{\mathcal{C}_1^{\mu,\alpha}} := \|f\|_{0,\alpha} + \|\delta f\|_{\mu,\alpha}$$

and

$$\|f\|_{\hat{\mathcal{C}}_1^{\mu,\alpha}} := \|f\|_{0,\alpha} + \|\hat{\delta} f\|_{\mu,\alpha}.$$

We observe that

$$\hat{\mathcal{C}}_1^{\mu,\mu} \hookrightarrow \mathcal{C}_1^{\mu,0} \tag{2.3}$$

for every $\mu \in (0,1)$ due to the following estimate: For $\lambda \geq \mu$,

$$\|\delta f\|_{\mu,0} \leq \|\hat{\delta} f\|_{\mu,\lambda} + C|T|^{\lambda-\mu}\|f\|_{0,\lambda} \tag{2.4}$$

whenever $f \in \hat{\mathcal{C}}_1^{\mu,\lambda}$ (see Lemma 2.4 in [12]).

Let us now consider the 3-increment spaces. If $h \in \hat{\mathcal{C}}_3(E_\alpha)$, we define

$$\|h\|_{\eta,\rho,\alpha} := \sup_{t,u,s\in\mathcal{S}_3} \frac{|h_{tus}|_\alpha}{|t-u|^\eta |u-s|^\rho}$$

and

$$\|h\|_{\mu,\alpha} := \inf\Big\{\sum_i \|h_i\|_{\rho_i,\mu-\rho_i,\alpha}; h = \sum_i h_i, 0 < \rho_i < \mu\Big\}$$

where the last infimum is taken over all sequences h_i such that $h = \sum_i h_i$ and for all choices of the numbers $\rho_i \in (0,\mu)$. One can check $\|\cdot\|_{\mu,\alpha}$ it is a norm and we define

$$\hat{\mathcal{C}}_3^{\mu,\alpha} := \{h \in \hat{\mathcal{C}}_3(E_\alpha); \|h\|_{\mu,\alpha} < \infty\}.$$

We also need Hölder-type spaces for operator-valued increments. For $0 \leq \mu \leq 1$ and $\alpha, \beta \in \mathbb{R}$, we set

$$\hat{\mathcal{C}}_2^\mu \mathcal{L}^{\beta,\alpha} := \hat{\mathcal{C}}_2^\mu\big(\mathcal{L}(E_\beta; E_\alpha)\big) = \{f : \mathcal{S}_2 \to \mathcal{L}(E_\beta; E_\alpha); \|f\|_{\mu,\beta\to\alpha} < \infty\}$$

where

$$\|f\|_{\mu,\beta\to\alpha} := \sup_{t,s\in\mathcal{S}_2} \frac{\|f_{ts}\|_{\beta\to\alpha}}{|t-s|^\mu}.$$

In order to work with the convolution sewing map, we define

$$\mathcal{Z}\hat{\mathcal{C}}_j^{\mu,\alpha} := \hat{\mathcal{C}}_j^{\mu,\alpha} \cap \ker \hat{\delta}|_{\hat{\mathcal{C}}_j}; j = 2,3.$$

We recall Range $\hat{\delta}|_{\hat{\mathcal{C}}_j} = \text{Ker } \hat{\delta}|_{\hat{\mathcal{C}}_{j+1}}; j \geq 1$. Let $\mathcal{E}_2^{\mu,\alpha} := \cap_{\epsilon \leq \mu\wedge 1^-} \hat{\mathcal{C}}_2^{\mu-\epsilon,\alpha+\epsilon}$ where $\epsilon \leq \mu \wedge 1^-$ means $\epsilon \in [0,\mu] \cap [0,1)$.

Infinite-dimensional regularized noise: We define

$$X^{x,i}_{ts} := S(t-s)(\delta x^i)_{ts}\sqrt{\lambda_i};\,(t,s)\in\mathcal{S}_2, \tag{2.5}$$

for $x=(x^i)_{i\geq 1}\in\mathcal{W}^{\gamma,\delta,\infty}_{\lambda,T}$ and $\frac{1}{2}<\gamma<H<1, \gamma+\delta\in(H,1)$. Let us now collect some important properties of the regularized noise.

Lemma 2.1.1 *The following properties hold true: $X^{x,i}\in\hat{\mathcal{C}}^{\gamma}_2\mathcal{L}^{\beta,\alpha}$ for $i\geq 1$ and for every $(\alpha,\beta)\in\mathbb{R}^2$ such that $\beta\geq\alpha$. Moreover, there exists a constant C which depends on (α,β) such that*

$$\sup_{(t,s)\in\mathcal{S}_2}\frac{\|X^{x,i}_{ts}\|_{\beta\to\alpha}}{|t-s|^{\gamma}}\leq C\sqrt{\lambda_i}\|x^i\|_{\mathcal{W}^{\gamma,\delta}_T} \tag{2.6}$$

for every $i\geq 1$. Moreover, the following algebraic condition holds

$$(\hat{\delta}X^{x,i})_{tsu} = (X^{x,i}a)_{tsu};\,(t,s,u)\in\mathcal{S}_3 \tag{2.7}$$

where $a_{su}=S(s-u)-Id;\,(s,u)\in\mathcal{S}_2$.

Proof. We observe if $\beta\geq\alpha$, then there exists $C_{\alpha,\beta}$ such that $\sup_{0\leq r\leq T}\|S(r)\|_{\beta\to\alpha}\leq C_{\alpha,\beta}<\infty$. This is obviously true for $\alpha=\beta$. In case, $\beta>\alpha$, we observe if $x\in E_\beta$, then

$$\begin{aligned}|S(r)x|_\alpha = \|(-A)^\alpha S(r)x\|_E = \|S(r)(-A)^\alpha x\|_E &= \|S(r)(-A)^{\alpha-\beta}(-A)^\beta x\|_E\\ &\leq \|S(r)(-A)^{\alpha-\beta}\|_{0\to 0}|x|_\beta\end{aligned}$$

because $(-A)^{\alpha-\beta}$ is a bounded operator on E (see Section 2.6 in [35]) whenever $\beta>\alpha$. Therefore, $\|S(r)\|_{\beta\to\alpha}\leq\|S(r)(-A)^{\alpha-\beta}\|_{0\to 0}\leq\|S(r)\|_{0\to 0}\|(-A)^{\alpha-\beta}\|_{0\to 0}$ for every $r\in[0,T]$. This proves our first claim. Therefore,

$$\|X^{x,i}_{ts}\|_{\beta\to\alpha}\leq\|S(t-s)\|_{\beta\to\alpha}|x^i_t-x^i_s|\sqrt{\lambda_i}$$

which implies (2.6). By definition,

$$\begin{aligned}(\hat{\delta}X^{x,i})_{tsu} &= X^{x,i}_{tu}-X^{x,i}_{ts}-S(t-s)X^{x,i}_{su}\\ &= S(t-u)(x^i_t-x^i_s)\sqrt{\lambda_i}-S(t-s)(x^i_t-x^i_s)\sqrt{\lambda_i}\\ &= S(t-s)[S(s-u)-\mathrm{Id}](x^i_t-x^i_s)\sqrt{\lambda_i}\end{aligned}$$

$$= \quad X^{x,i}_{ts} a_{su} = (X^{x,i} a)_{tsu}; (t,s,u) \in \mathcal{S}_3.$$

This shows (2.7). ■

Lemma 2.1.2 *Let us fix* $x = (x^i)_{i\geq 1} \in \mathcal{W}^{\gamma,\delta,\infty}_{\lambda,T}$ *and* $\frac{1}{2} < \gamma < H < 1, \gamma + \delta \in (H,1)$. *Assume* $z = (z^i)_{i\geq 1}$ *satisfies* $\sup_{i\geq 1} \|z^i\|_{\hat{\mathcal{C}}^{\eta,\beta}_1} < \infty$ *for* $\eta + \gamma > 1$. *Then*

$$\mathcal{J}_{t_1 t_2}(\hat{d}xz) := \sum_{i=1}^{\infty} \sqrt{\lambda_i} X^{x,i}_{t_1 t_2} z^i_{t_2} + \sum_{i=1}^{\infty} \sqrt{\lambda_i} \hat{\Lambda}\big(X^{x,i} \hat{\delta} z^i\big)_{t_1 t_2}$$

satisfies:

(i) There exists a constant C such that

$$\|\hat{\delta}\mathcal{J}(\hat{d}xz)\|_{\gamma,\alpha} \leq C \|x\|_{\mathcal{W}^{\gamma,\delta,\infty}_{\lambda,T}} \sup_{i\geq 1}\{\|z^i\|_{0,\beta} + \|\hat{\delta} z^i\|_{\eta,\beta}\}$$

for $\alpha \leq \beta$.

(ii)

$$\mathcal{J}_{t_1 t_2}(\hat{d}xz) = \sum_{i=1}^{\infty} \sqrt{\lambda_i} \int_{t_2}^{t_1} S(t_1 - u) z^i_u dx^i_u \text{ in } E_\alpha,$$

for each $(t_1, t_2) \in \mathcal{S}_2$.

Proof. In the sequel, C is a constant which may defer from line to line and we fix $\alpha \leq \beta$. At first, the algebraic property (2.7) yields

$$\hat{\delta} X^{x,i} z^i = -X^{x,i} \hat{\delta} z^i; i \geq 1.$$

Indeed,

$$\begin{aligned} -X^{x,i}\hat{\delta} z^i &= -X^{x,i}\delta z^i + X^{x,i} a z^i \\ &= -X^{x,i}\delta z^i + (\hat{\delta} X^{x,i}) z^i = -X^{x,i}\delta z^i + (\delta X^{x,i}) z^i - aX^{x,i} z^i \ . \end{aligned}$$

On the other hand, Lemma 3.2 in [18] yields

$$\hat{\delta} X^{x,i} z^i \quad = \quad \delta X^{x,i} z^i - a X^{x,i} z^i$$

$$= \quad (\delta X^{x,i})z^i - X^{x,i}\delta z^i - aX^{x,i}z^i$$

which implies $\hat{\delta} X^{x,i}z^i = -X^{x,i}\hat{\delta}z^i$. We also observe $-X^{x,i}\hat{\delta}z^i \in \mathcal{Z}\hat{\mathcal{C}}_3^{\mu,\alpha}$ for $\mu > 1$. Indeed, let us take $\eta + \gamma > 1$. Then,

$$\begin{aligned}|X^{x,i}_{t_1t_2}(\hat{\delta}z^i)_{t_2t_3}|_\alpha &\leq \|X^{x,i}_{t_1t_2}\|_{\beta\to\alpha}|\hat{\delta}z^i_{t_2t_3}|_\beta \\ &\leq C\sqrt{\lambda_i}\|x^i\|_{\mathcal{W}_T^{\gamma,\delta}}|t_1-t_2|^\gamma\|\hat{\delta}z^i\|_{\eta,\beta}|t_2-t_3|^\eta.\end{aligned}$$

Therefore,

$$\|X^{x,i}\hat{\delta}z^i\|_{\gamma,\eta,\alpha} = \sup_{(t_1,t_2,t_3)\in\mathcal{S}_3}\frac{|X^{x,i}_{t_1t_2}(\hat{\delta}z^i)_{t_2t_3}|_\alpha}{|t_1-t_2|^\gamma|t_2-t_3|^\eta} \leq C\sqrt{\lambda_i}\|x^i\|_{\mathcal{W}_T^{\gamma,\delta}}\|\hat{\delta}z^i\|_{\eta,\beta}. \tag{2.8}$$

By taking $\mu = \gamma + \eta > 1$, we conclude $X^{x,i}\hat{\delta}z^i \in \mathcal{Z}\hat{\mathcal{C}}_3^{\mu,\alpha}$ for each $i \geq 1$. The Sewing property yields $\hat{\delta}\hat{\Lambda}\big(\hat{\delta}X^{x,i}z^i\big) = \hat{\delta}X^{x,i}z^i$ and hence

$$\hat{\delta}\Big(X^{x,i}z^i - \hat{\Lambda}(-X^{x,i}\hat{\delta}z^i)\Big) = 0$$

so that $X^{x,i}z^i + \hat{\Lambda}(X^{x,i}\hat{\delta}z^i) \in \mathrm{Ker}|_{\hat{\mathcal{C}}_2} = \mathrm{Range}\ \hat{\delta}|_{\hat{\mathcal{C}}_1}$. Therefore, there exists $f^i \in \hat{\mathcal{C}}_1$ such that

$$\begin{aligned}\hat{\delta}f^i &= X^{x,i}z^i + \hat{\Lambda}(X^{x,i}\hat{\delta}z^i) \\ &= \big(\mathrm{Id} - \hat{\Lambda}\hat{\delta}\big)X^{x,i}z^i; i \geq 1.\end{aligned} \tag{2.9}$$

The Sewing map yields $\hat{\Lambda}(X^{x,i}\hat{\delta}z^i) \in \mathcal{E}_2^{\gamma+\eta,\alpha}$ and we observe

$$|X^{x,i}_{ts}z^i_s|_\alpha \leq C\sqrt{\lambda_i}\|x^i\|_{\mathcal{W}_T^{\gamma,\delta}}|t-s|^\gamma\|z^i\|_{0,\beta} \tag{2.10}$$

so the best we can get is $X^{x,i}z^i\ + \hat{\Lambda}\big(X^{x,i}\hat{\delta}z^i\big) \in \hat{\mathcal{C}}_2^{\gamma,\alpha}$. The Sewing property (Th. 3.5 in [18]) and (2.8) yield

$$|\hat{\Lambda}\big(X^{x,i}\hat{\delta}z^i\big)_{ts}|_\alpha \leq C\|X^{x,i}\hat{\delta}z^i\|_{\gamma,\eta,\alpha}|t-s|^{\gamma+\eta} \leq C\sqrt{\lambda_i}\|x^i\|_{\mathcal{W}_T^{\gamma,\delta}}\|\hat{\delta}z^i\|_{\eta,\beta}|t-s|^{\gamma+\eta} \tag{2.11}$$

Therefore, (2.10) and (2.11) imply

$$\|\hat{\delta} f^i\|_{\gamma,\alpha} \le C\|z^i\|_{0,\beta}\sqrt{\lambda_i}\|x^i\|_{\mathcal{W}_T^{\gamma,\delta}} + C\sqrt{\lambda_i}\|x^i\|_{\mathcal{W}_T^{\gamma,\delta}}\|\hat{\delta} z^i\|_{\eta,\beta}.$$

Therefore, we conclude (i). In order to check (ii), we just need to apply Corollary 3.6 in [18] and observe (2.9). Then, for each $(t,s) \in \mathcal{S}_2$, we have

$$\begin{aligned}(\hat{\delta} f^i)_{ts} &= \lim_{\|\Pi_{st}\|\to 0} \sum_{r_j \in \Pi_{st}} S(t - r_{j+1})(X^{x,i})_{r_{j+1} r_j} z^i_{r_j} \\ &= \lim_{\|\Pi_{st}\|\to 0} \sqrt{\lambda_i} \sum_{r_j \in \Pi_{st}} S(t - r_j) z^i_{r_j} (\delta x^i)_{r_{j+1} r_j}\end{aligned}$$

where convergence holds (for each $i \ge 1$) in E_α as the mesh $|\Pi|_{st}$ of the partition of $[s,t]$ vanishes. Therefore,

$$(\hat{\delta} f^i)_{ts} = \int_s^t S(t-u) z^i_u dx^i_u; (t,s) \in \mathcal{S}_2, i \ge 1. \tag{2.12}$$

Representation (ii) is a consequence of (i) and (2.12). This concludes the proof. ■

2.2 The Itô map

For a given $y_0 = \psi \in E$, the Itô map $x \mapsto y$ is defined as the solution of the equation

$$y_t = S(t-s)y_s + \int_s^t S(t-u)F(y_u)du + \mathcal{J}_{ts}(\hat{d}xG(y)); (t,s) \in \mathcal{S}_2$$

which can be rewritten in terms of the increment operator $\hat{\delta}$,

$$(\hat{\delta} y)_{ts} = \int_s^t S(t-u)F(y_u)du + \mathcal{J}_{ts}(\hat{d}xG(y)); y_0 = \psi. \tag{2.13}$$

Next, we list the basic assumptions needed for the existence and uniqueness of the SPDE solution. Before that, let us check that we may choose the correct set of parameters.

Lemma 2.2.1 *For given $\frac{1}{2} < H < 1$ and $\frac{1}{2} > \kappa > \frac{1}{4}$, there exist $\tilde{\gamma}, \kappa_0$ satisfying*

$\tilde{\gamma} > \kappa_0 > \kappa > \frac{1}{4}$ *with* $\tilde{\gamma} + \kappa > 1$, $\tilde{\gamma} - \kappa \geq \kappa_0$ *such that*

$$X^{x,i} \in \hat{\mathcal{C}}_2^{\tilde{\gamma}} \mathcal{L}^{0,-\kappa} \cap \hat{\mathcal{C}}_2^{\kappa_0} \mathcal{L}^{\kappa,\kappa} \tag{2.14}$$

for every $i \geq 1$.

Proof. From Lemma 2.1.1 and the definition of the spaces $\mathcal{W}_T^{\gamma,\delta}$, there exists a constant C (which does not depend on $i \geq 1$) such that

$$\|X^{x,i}\|_{H-\epsilon,0\to-\kappa} \leq C\sqrt{\lambda_i}\|x^i\|_{\mathcal{W}_T^{H-\epsilon,\delta}}$$

and

$$\|X^{x,i}\|_{H-\eta,\kappa\to\kappa} \leq C\sqrt{\lambda_i}\|x^i\|_{\mathcal{W}_T^{H-\eta,\delta}}$$

for every $\kappa > 0$, $\epsilon \in (0, H), \eta \in (0, H)$ and $\delta > 0$ such that $H - \epsilon + \delta \in (H, 1)$ and $H - \eta + \delta \in (H, 1)$. For a given $\frac{1}{2} < H < 1$ and $\frac{1}{2} > \kappa > \frac{1}{4}$. Choose $\epsilon = \epsilon(\kappa, H) \in (0, H)$ such that

$$H - \epsilon + \kappa > 1. \tag{2.15}$$

Choose $\eta = \eta(\epsilon, H)$ such that

$$\eta > \frac{1}{2} + \epsilon \quad \text{and } H - \kappa > \eta. \tag{2.16}$$

Of course, (2.16) implies $\frac{1}{2} + \epsilon < \eta < H - \kappa$. Choose δ accordingly to these conditions. We then set $\tilde{\gamma} = H - \epsilon$, $\kappa_0 = H - \eta$ where ϵ and η satisfy (2.15) and (2.16). Then, by construction $\tilde{\gamma} + \kappa = H - \epsilon + \kappa > 1$ due to (2.15) and $\tilde{\gamma} > \kappa_0 > \kappa > \frac{1}{4}$ due to (2.16). Moreover, $\eta - \epsilon > \frac{1}{2} > \kappa > \frac{1}{4}$ so that

$$\tilde{\gamma} - \kappa_0 > \frac{1}{2} > \kappa > \frac{1}{4}.$$

Finally, we stress the choice of ϵ and η does not depend on the index $i \geq 1$. This concludes the proof. ■

Let us assume the following regularity assumptions on F, G:

Assumption H1: For $\frac{1}{2} > \kappa > \frac{1}{4}$, we assume that $F, G_i : E_\kappa \to E_\kappa$ is Lipschitz (uniformly in $i \geq 1$) and they have linear growth

$$|G_i(x)|_\kappa \leq C(1 + |x|_\kappa), \quad |F(x)|_\kappa \leq C(1 + |x|_\kappa); x \in E_\kappa,$$

for every $i \geq 1$. Furthermore, we suppose that F, G_i can also be seen as maps from E to E, and when considered as such, it holds that F, G_i are Lipschitz (uniformly in $i \geq 1$).

In the sequel, recall $\hat{\mathcal{C}}_1^{\kappa,\kappa}$ is the subspace of $\hat{\mathcal{C}}_1(E_\kappa)$ such that

$$\|z\|_{\hat{\mathcal{C}}_1^{\kappa,\kappa}} = \|z\|_{0,\kappa} + \|\hat{\delta} z\|_{\kappa,\kappa} < \infty.$$

In what follows, $x \in \mathcal{W}_{\lambda,T}^{\tilde{\gamma},\delta,\infty}$ where $\tilde{\gamma} + \delta \in (H, 1), \frac{1}{2} < \tilde{\gamma} < H$,

$$\tilde{\gamma} > \kappa_0 > \kappa > \frac{1}{4} \tag{2.17}$$

and $\tilde{\gamma} + \kappa > 1, \tilde{\gamma} - \kappa \geq \kappa_0$. By Lemma 2.2.1, X^x satisfies (2.14). By using Assumption H1, the following result is a straightforward application of Theorem 4.3 in [18].

Proposition 2.2.2 *Under Assumption H1 and the choice of indexes (2.17), for each $\psi \in E_\kappa$ there exists a unique global solution to (2.13) in $\hat{\mathcal{C}}_1^{\kappa,\kappa}$.*

By noticing (see Lemma 1.2.4) that $(\beta^i)_{i \geq 1} \in \mathcal{W}_{\lambda,T}^{\tilde{\gamma},\delta,\infty}$ a.s, Proposition 2.2.2 yields the following result.

Proposition 2.2.3 *Under Assumption H1 and the choice of indexes (2.17), for each initial condition $x_0 \in E_\kappa$, there exists a unique adapted process X which is solution to (2.1).*

2.3 Fréchet differentiability

Let us now devote our attention to the Fréchet differentiability of the Itô map

$$\Phi : \mathcal{W}_{\lambda,T}^{\tilde{\gamma},\delta,\infty} \to \hat{\mathcal{C}}_1^{\kappa,\kappa} \quad x \mapsto y$$

where y is the mild solution of (2.13) driven by $x \in \mathcal{W}^{\tilde{\gamma},\delta,\infty}_{\lambda,T}$ where the indices $\tilde{\gamma}, \delta, \kappa_0, \kappa$ satisfy (2.17). Then, the Fréchet derivative is a mapping

$$\nabla\Phi : \mathcal{W}^{\tilde{\gamma},\delta,\infty}_{\lambda,T} \to \mathcal{L}\big(\mathcal{W}^{\tilde{\gamma},\delta,\infty}_{\lambda,T}; \hat{\mathcal{C}}^{\kappa,\kappa}_{1}\big).$$

The importance of Fréchet differentiability lies on the following argument: Once we have Fréchet differentiability of the Itô map $x \mapsto y$, we shall use the Fréchet derivative chain rule to infer that $\langle X_t, h\rangle_E$ is Fréchet differentiable along the direction of the Cameron-Martin space $\mathbf{H}$ for a given $h \in E$ and $t \in [0,T]$. Hence, Corollary 1.2.5 implies

$$\langle X_t, h\rangle_E \in \mathbb{D}^{1,2}_{loc}(\mathbb{R}).$$

Then, we must use Lemma 1.2.2 and try to conclude a representation. We follow the idea contained in the work of Nualart and Saussereau [33]. At first, we list a set of assumptions on the vector fields which will be important in this section.

Assumption A1: The vector fields, $G_i, F : E_\kappa \to E_\kappa$ are Fréchet differentiable and also differentiable when considering from E to E. Moreover,

$$\sup_{i\geq 1}\sup_{x\in E_\kappa} \|\nabla G_i(x)\|_{\kappa\to\kappa} + \sup_{x\in E_\kappa} \|\nabla F(x)\|_{\kappa\to\kappa} < \infty$$

and $\sup_{i\geq 1}\sup_{x\in E}\|\nabla G_i(x)\| + \sup_{x\in E}\|\nabla F(x)\| < \infty$.

Assumption A2:

$$\sup_{i\geq 1}\sup_{g\in E}\|\nabla^{(2)}G_i(g)\|_{(2),q\to q} + \sup_{f\in E}\|\nabla^{(2)}F(f)\|_{(2),\kappa\to\kappa} < \infty,$$

for $q = 0, \kappa$ and there exists a constant C such that

$$\sup_{i\geq 1}\|\nabla G_i(f) - \nabla G_i(g)\| + \sup_{i\geq 1}\big\|\nabla^{(2)}G_i(f) - \nabla^{(2)}G_i(g)\big\|_{(2),0\to 0} \leq C\|f-g\|_E$$

for every $f, g \in E$.

At first, it is necessary to investigate flow properties of linear equations. We start with the following corollary whose proof is entirely analogous to Proposition 2.2.2, so we omit the details.

Corollary 2.3.1 *Suppose F, G satisfy Assumptions A1 and H1 and let us fix $(x, y) \in \mathcal{W}^{\tilde{\gamma},\delta,\infty}_{\lambda,T} \times \hat{\mathcal{C}}^{\kappa,\kappa}_1$ and $t_0 \in [0, T]$. Then, for every $\eta \in \hat{\mathcal{C}}^{\kappa,\kappa}_1$,*

$$v_t = \eta_t + \int_{t_0}^{t} S(t-s)\nabla F(y_s)v_s ds + \mathcal{J}_{tt_0}(\hat{d}x \nabla G(y)v)$$

admits a unique solution in $v \in \hat{\mathcal{C}}^{\kappa,\kappa}_1$ on the interval $[t_0, T]$.

The following lemma plays a key role on the Fréchet differentiability of the Itô map.

Lemma 2.3.2 *Let $[s_0, t_0]$ be a subset of $[0, T]$ and let*

$$Z_t = \sum_{i\geq 1} \sqrt{\lambda_i} \int_{s_0}^{t} S(t-s) z^i_s dx^i_s; s_0 \leq t \leq t_0$$

where $x \in \mathcal{W}^{\tilde{\gamma},\delta,\infty}_{\lambda,T}$ and assume $\sup_{i\geq 1} \|z^i\|_{0,\eta} + \sup_{i\geq 1} \|\hat{\delta} z^i\|_{\zeta,\eta-\alpha} < \infty$ on the interval $[s_0, t_0]$ for some $\eta \geq 0$ where $0 \leq \alpha \leq min(\zeta, \eta)$, $0 \leq \zeta \leq \tilde{\gamma}$ and $\tilde{\gamma} + \zeta > 1$. Then, there exists a constant C which depends on η and $\tilde{\gamma}$ such that

$$\|\hat{\delta} Z\|_{\tilde{\gamma},\eta} \leq C \|x\|_{\mathcal{W}^{\tilde{\gamma},\delta,\infty}_{\lambda,T}} \Big\{ \sup_{i\geq 1} \|z^i\|_{0,\eta} + |t_0 - s_0|^{\zeta-\alpha} \sup_{i\geq 1} \|\hat{\delta} z^i\|_{\zeta,\eta-\alpha} \Big\} \tag{2.18}$$

and

$$\|\hat{\delta} Z\|_{\zeta,\eta} \leq C \|x\|_{\mathcal{W}^{\tilde{\gamma},\delta,\infty}_{\lambda,T}} \Big\{ |t_0 - s_0|^{\tilde{\gamma}-\zeta} \sup_{i\geq 1} \|z^i\|_{0,\eta} + |t_0 - s_0|^{\tilde{\gamma}-\alpha} \sup_{i\geq 1} \|\hat{\delta} z^i\|_{\zeta,\eta-\alpha} \Big\} \tag{2.19}$$

on the interval $[s_0, t_0]$.

Proof. In the sequel, C is a constant which may defer from line to line. To keep notation simple, without loss of generality, we set $s_0 = 0, t_0 = T$. We observe $(\hat{\delta} Z)_{ts} = \sum_{i\geq 1} \sqrt{\lambda_i} \int_s^t S(t-u) z^i_u dx^i_u$. From the proof of Lemma 2.1.2, we know that

$$\int_s^t S(t-u) z^i_u dx^i_u = X^{x,i}_{ts} z^i_s + \hat{\Lambda}\big(X^{x,i} \hat{\delta} z^i\big)_{ts}; (t,s) \in \mathcal{S}_2,$$

where $X^{x,i} \in \hat{\mathcal{C}}_2^{\tilde{\gamma}}\mathcal{L}^{\eta,\eta}$ due to Lemma 2.1.1. Then, checking the proof of Lemma 2.1.2, we have $X^{x,i}\hat{\delta}z^i \in \mathcal{Z}\hat{\mathcal{C}}_3^{\zeta+\tilde{\gamma},\theta}$ for $\theta \leq \eta - \alpha$. Now,

$$\Big|\sum_{i\geq 1}\sqrt{\lambda_i}\int_s^t S(t-u)z_u^i dx_u^i\Big|_\eta \leq \sum_{i\geq 1}\sqrt{\lambda_i}\Big|\int_s^t S(t-u)z_u^i dx_u^i\Big|_\eta$$
$$\leq C\sum_{i\geq 1}\sqrt{\lambda_i}\|X_{ts}^{x,i}\|_{\eta\to\eta}|z_s^i|_\eta + \sum_{i\geq 1}\sqrt{\lambda_i}\Big|\hat{\Lambda}\big(X^{x,i}\hat{\delta}z^i\big)_{ts}\Big|_\eta.$$

That is,

$$\Big|\sum_{i\geq 1}\sqrt{\lambda_i}\int_s^t S(t-u)z_u^i dx_u^i\Big|_\eta \leq C\|x\|_{\mathcal{W}_{\lambda,T}^{\tilde{\gamma},\delta,\infty}}|t-s|^{\tilde{\gamma}}\sup_{i\geq 1}\|z^i\|_{0,\eta}$$
$$+\sum_{i\geq 1}\sqrt{\lambda_i}\Big|\hat{\Lambda}\big(X^{x,i}\hat{\delta}z^i\big)_{ts}\Big|_\eta; (t,s)\in\mathcal{S}_2. \tag{2.20}$$

By applying the "convolution" Sewing lemma (Th 3.5 in [18]), there exists a constant $C_{\zeta+\tilde{\gamma}}$ such that

$$\|\hat{\Lambda}X^{x,i}\hat{\delta}z^i\|_{\zeta+\tilde{\gamma}-\epsilon,\theta+\epsilon} \leq C_{\zeta+\tilde{\gamma},\epsilon}\|X^{x,i}\hat{\delta}z^i\|_{\zeta+\tilde{\gamma},\theta}$$

for every $\epsilon \in [0,\zeta+\tilde{\gamma}]\cap[0,1)$. Take $\theta = \eta-\alpha$ and $\epsilon = \alpha$. Then,

$$\Big|\hat{\Lambda}\big(X^{x,i}\hat{\delta}z^i\big)_{ts}\Big|_{\eta=\theta+\epsilon} \leq C_{\zeta+\tilde{\gamma},\epsilon}\|X^{x,i}\hat{\delta}z^i\|_{\zeta+\tilde{\gamma},\theta}|t-s|^{\zeta+\tilde{\gamma}-\epsilon}. \tag{2.21}$$

On the other hand, $(X^{x,i}\hat{\delta}z^i)$ is a 3-increment where

$$\|X^{x,i}\hat{\delta}z^i\|_{\zeta+\tilde{\gamma},\eta-\alpha} = \inf\Big\{\sum_j\|h_j\|_{\rho_j,\zeta+\tilde{\gamma}-\rho_j,\eta-\alpha}; X^{x,i}\hat{\delta}z^i = \sum_j h_j, 0<\rho_j<\zeta+\tilde{\gamma}\Big\}$$

and the last infimum is taken over all sequences h_j such that $X^{x,i}\hat{\delta}z^i = \sum_j h_j$ and for all choices of the numbers $\rho_j \in (0,\zeta+\tilde{\gamma})$ and we recall for any 3-increment f, we have

$$\|f\|_{\rho_j,\zeta+\tilde{\gamma}-\rho_j,\eta-\alpha} = \sup_{t,u,s\in\mathcal{S}_3}\frac{|f_{tus}|_{\eta-\alpha}}{|t-u|^{\rho_j}|u-s|^{\zeta+\tilde{\gamma}-\rho_j}}.$$

Take $h_j = X^{x,i}\hat{\delta}z^i$ and $\rho_j = \tilde{\gamma}$. By definition, $(X^{x,i}\hat{\delta}z^i)_{tus} = X_{tu}^{x,i}\hat{\delta}z_{us}^i$, then

$$\|X^{x,i}\hat{\delta}z^i\|_{\zeta+\tilde{\gamma},\eta-\alpha} \leq \sup_{t,u,s\in\mathcal{S}_3}\frac{|X_{tu}^{x,i}\hat{\delta}z_{us}^i|_{\eta-\alpha}}{|t-u|^{\tilde{\gamma}}|u-s|^{\zeta}} \leq \sup_{t,u,s\in\mathcal{S}_3}\frac{|X_{tu}^{x,i}|_{\eta-\alpha\to\eta-\alpha}|\hat{\delta}z_{us}^i|_{\eta-\alpha}}{|t-u|^{\tilde{\gamma}}|u-s|^{\zeta}}$$

$$\leq C\|x^i\|_{\mathcal{W}_T^{\tilde{\gamma},\delta}}\|\hat{\delta}z^i\|_{\zeta,\eta-\alpha}.$$

Then, (2.21) yields

$$\sum_{i\geq 1}\sqrt{\lambda_i}\Big|\hat{\Lambda}\big(X^{x,i}\hat{\delta}z^i\big)_{ts}\Big|_\eta \leq C_{\zeta+\tilde{\gamma},\alpha}|t-s|^{\zeta+\tilde{\gamma}-\alpha}\|x\|_{\mathcal{W}_{\lambda,T}^{\tilde{\gamma},\delta,\infty}}\sup_{i\geq 1}\|\hat{\delta}z^i\|_{\zeta,\eta-\alpha}. \tag{2.22}$$

Finally, we shall plug (2.22) into (2.20) and we conclude the proof of (2.18). By observing (2.22) and (2.20), we conclude (2.19). ■

Lemma 2.3.3 *Let y be the solution of (2.13) driven by $x\in\mathcal{W}_{\lambda,T}^{\tilde{\gamma},\delta,\infty}$ and assume Assumption (A1-A2) hold true. Then, the mapping*

$$L:\mathcal{W}_{\lambda,T}^{\tilde{\gamma},\delta,\infty}\times\hat{\mathcal{C}}_1^{\kappa,\kappa}\to\hat{\mathcal{C}}_1^{\kappa,\kappa}$$

defined by

$$(x,y)\mapsto L(x,y)_t := y_t - S_t\psi - \int_0^t S(t-s)F(y_s)ds - \mathcal{J}_{t0}\big(\hat{d}(x)G(y)\big)$$

is Fréchet differentiable. In particular, for each $(x,y)\in\mathcal{W}_{\lambda,T}^{\tilde{\gamma},\delta,\infty}\times\hat{\mathcal{C}}_1^{\kappa,\kappa}$ and $(q,v)\in\mathcal{W}_{\lambda,T}^{\tilde{\gamma},\delta,\infty}\times\hat{\mathcal{C}}_1^{\kappa,\kappa}$, we have

$$\nabla_1 L(x,y)(q)_t = -\mathcal{J}_{t0}\big(\hat{d}qG(y)\big) \tag{2.23}$$

and

$$\nabla_2 L(x,y)(v)_t = v_t - \int_0^t S(t-s)\nabla F(y_s)v_s ds - \mathcal{J}_{t0}\big(\hat{d}x\nabla G(y)v\big); 0\leq t\leq T. \tag{2.24}$$

Moreover, for each $x\in\mathcal{W}_{\lambda,T}^{\tilde{\gamma},\delta,\infty}$, the mapping $\nabla_2 L(x,\Phi(x)):\hat{\mathcal{C}}_1^{\kappa,\kappa}\to\hat{\mathcal{C}}_1^{\kappa,\kappa}$ is a homeomorphism.

Proof. In the sequel, C is a constant which may defer form line to line. By the very definition,

$$L(x+h,y+v)_t - L(x,y)_t = (y_t+v_t) - S(t)\psi - \int_0^t S(t-u)F(y_u+v_u)du$$

$$
\begin{aligned}
&-\sum_{i\geq 1}\sqrt{\lambda_i}\int_0^t S(t-s)G_i(y_u+v_u)d(x_u^i+h_u^i)-y_t+S(t)\psi\\
&+\int_0^t S(t-u)F(y_u)du+\sum_{i\geq 1}\sqrt{\lambda_i}\int_0^t S(t-u)G_i(y_u)dx_u^i\\
=\,& v_t-\int_0^t S(t-u)\big[F(y_u+v_u)-F(y_u)\big]du-\sum_{i\geq 1}\sqrt{\lambda_i}\int_0^t S(t-u)(G_i(y_u+v_u))dh_u^i\\
&-\sum_{i\geq 1}\sqrt{\lambda_i}\int_0^t S(t-u)(G_i(y_u+v_u)-G_i(y_u))dx_u^i.
\end{aligned}
$$

Let us write the increments in terms of the Taylor formula (see e.g [10]),

$$
\begin{aligned}
F(y_u+v_u)-F(y_u)&=\nabla F(y_u)v_u+z_u(y,v)\\
G_i(y_u+v_u)-G_i(y_u)&=\nabla G_i(y_u)v_u+c_u^i(y,v)\\
G_i(y_u+v_u)&=G_i(y_u)+e_u^i(y,v)
\end{aligned}
$$

where

$$
\begin{aligned}
z_u(y,v)&:=\left(\int_0^1(1-r)\nabla^{(2)}F(y_u+rv_u)dr\right)(v_u,v_u)\\
c_u^i(y,v)&:=\left(\int_0^1(1-r)\nabla^{(2)}G_i(y_u+rv_u)dr\right)(v_u,v_u)\\
e_u^i(y,v)&:=\left(\int_0^1\nabla G_i(y_u+rv_u)dr\right)v_u
\end{aligned}
$$

for $i\geq 1$ and $0\leq u\leq t$. Therefore,

$$
L(x+h,y+v)_t-L(x,y)_t-\nabla_1 L(x,y)(h)_t-\nabla_2 L(x,y)(v)_t=R_1(y,v)_t+R_2(y,v)_t+R_3(y,v)_t
$$

where

$$
\begin{aligned}
R_1(y,v)_t&:=-\int_0^t S(t-u)z_u(y,v)du\\
R_2(y,v)_t&:=-\sum_{i\geq 1}\sqrt{\lambda_i}\int_0^t S(t-u)c_u^i(y,v)dx_u^i\\
R_3(y,v)_t&:=-\sum_{i\geq 1}\sqrt{\lambda_i}\int_0^t S(t-u)e_u^i(y,v)dh_u^i.
\end{aligned}
$$

We need to check

$$\|R_1(y,v)+R_2(y,v)+R_3(y,v)\|_{\hat{\mathcal{C}}_1^{\kappa,\kappa}} = o\big(\|h\|^2_{\mathcal{W}^{\tilde{\gamma},\delta,\infty}_{\lambda,T}} + \|v\|^2_{\hat{\mathcal{C}}_1^{\kappa,\kappa}}\big)^{\frac{1}{2}}. \tag{2.25}$$

The first term is easy. Indeed, if the second order derivative of F is bounded, then the norm of the bilinear form $z_u(y,v)$ can be estimated as follows $\|z_u(y,v)\|_{(2),\kappa\to\kappa} \le C|v_u|^2_\kappa \le C\|v\|^2_{\hat{\mathcal{C}}_1^{\kappa,\kappa}}$. Therefore,

$$\|R_1(u,v)\|_{\hat{\mathcal{C}}_1^{\kappa,\kappa}} \le C\|v\|^2_{\hat{\mathcal{C}}_1^{\kappa,\kappa}}.$$

Then,

$$\frac{\|R_1(u,v)\|_{\hat{\mathcal{C}}_1^{\kappa,\kappa}}}{\big(\|h\|^2_{\mathcal{W}^{\tilde{\gamma},\delta,\infty}_{\lambda,T}} + \|v\|^2_{\hat{\mathcal{C}}_1^{\kappa,\kappa}}\big)^{\frac{1}{2}}} \le \frac{\|R_1(u,v)\|_{\hat{\mathcal{C}}_1^{\kappa,\kappa}}}{\big(\|v\|^2_{\hat{\mathcal{C}}_1^{\kappa,\kappa}}\big)^{\frac{1}{2}}} \le C\|v\|_{\hat{\mathcal{C}}_1^{\kappa,\kappa}}. \tag{2.26}$$

Let us now estimate $R_2(y,v)$. At first, since $R_2(y,v)_0 = 0$, then

$$\|R_2(y,v)\|_{\hat{\mathcal{C}}_1^{\kappa,\kappa}} \le (2+T^\kappa)\|\hat{\delta}R_2(y,v)\|_{\kappa,\kappa} \tag{2.27}$$

where

$$-(\hat{\delta}R_2(y,v))_{ts} = \sum_{i\ge 1}\sqrt{\lambda_i}\int_s^t S(t-u)c^i_u(y,v)dx^i_u = \mathcal{J}_{ts}\big(\hat{d}xc(y,v)\big)$$

so that $\|\hat{\delta}R_2(y,v)\|_{\kappa,\kappa} = \|\mathcal{J}\big(\hat{d}xc(y,v)\big)\|_{\kappa,\kappa}$. By Lemma 2.3.2, there exists a constant C such that

$$\|\mathcal{J}\big(\hat{d}xc(y,v)\big)\|_{\kappa,\kappa} \le C\|x\|_{\mathcal{W}^{\gamma,\delta,\infty}_{\lambda,T}}\Big\{\sup_{i\ge 1}\|c^i(y,v)\|_{0,\kappa} + \sup_{i\ge 1}\|\hat{\delta}c^i(y,v)\|_{\kappa,0}\Big\}. \tag{2.28}$$

By definition,

$$(\hat{\delta}c^i(y,v))_{ts} = c^i_t(y,v) - c^i_s(y,v) + c^i_s(y,v) - S(t-s)c^i_s(y,v); (t,s)\in\mathcal{S}_2.$$

By viewing $\nabla^{(2)}G_i : E_\kappa\times E_\kappa \to E_\kappa$ as a bounded bilinear form where $\kappa > 0$, we observe $c^i_s(y,v)\in E_\kappa$ and this little gain of spatial regularity allows us to estimate

$$\|(\hat{\delta}c^i(y,v))_{ts}\| \le \|(\delta c^i(y,v))_{ts}\| + \|\big(S(t-s)-\mathrm{Id}\big)c^i_s(y,v)\| \tag{2.29}$$

where (see e.g Th 6.13 in [35])

$$\begin{aligned}\|\big(S(t-s)-\mathrm{Id}\big)c^i_s(y,v)\| &\le C|t-s|^\kappa|c^i_s(y,v)|_\kappa \\ &\le C|t-s|^\kappa|v_s|^2_\kappa \le C|t-s|^\kappa\|v\|^2_{0,\kappa} \qquad (2.30)\end{aligned}$$

and the estimate (2.30) is due to the boundedness $\sup_{i\ge1}\sup_{a\in E_\kappa}\|\nabla^{(2)}G_i(a)\|_{(2),\kappa\to\kappa}<\infty$.

We now observe for each $i\ge1$ and $u\in[0,t]$, $\int_0^1(1-r)\nabla^{(2)}G_i(y_u+rv_u)dr: E\times E\to E$ is a bounded bilinear form so that we shall estimate

$$\begin{aligned}&\|c^i_u(y,v)-c^i_{u'}(y,v)\|\le\Big\|c^i_u(y,v)-\Big(\int_0^1(1-r)\nabla^{(2)}G_i(y_u+rv_u)dr\Big)(v_{u'},v_{u'})\Big\| \\ &\quad+\Big\|\Big(\int_0^1(1-r)\nabla^{(2)}G_i(y_u+rv_u)dr\Big)(v_{u'},v_{u'})-c^i_{u'}(y,v)\Big\| \\ &=\Big\|\Big(\int_0^1(1-r)\nabla^{(2)}G_i(y_u+rv_u)dr\Big)(v_{u'},v_{u'}) \\ &\quad-\Big(\int_0^1(1-r)\nabla^{(2)}G_i(y_u+rv_u)dr\Big)(v_u,v_u)\Big\|+\Big\|\Big(\int_0^1(1-r)\big[\nabla^{(2)}G_i(y_u+rv_u) \\ &\quad-\nabla^{(2)}G_i(y_{u'}+rv_{u'})\big]dr\Big)(v_{u'},v_{u'})\Big\| \\ &=\Big\|\Big(\int_0^1(1-r)\nabla^{(2)}G_i(y_u+rv_u)dr\Big)(v_{u'}-v_u,v_{u'}) \\ &\quad+\Big(\int_0^1(1-r)\nabla^{(2)}G_i(y_u+rv_u)dr\Big)(v_u,v_{u'}-v_u)\Big\| \\ &\quad+\Big\|\Big(\int_0^1(1-r)\big[\nabla^{(2)}G_i(y_u+rv_u)-\nabla^{(2)}G_i(y_{u'}+rv_{u'})\big]dr\Big)(v_{u'},v_{u'})\Big\| \\ &\le C\|v_{u'}-v_u\|\|v_{u'}\|+C\|v_{u'}-v_u\|\|v_u\| \\ &\quad+\int_0^1(1-r)\big\|\nabla^{(2)}G_i(y_u+rv_u)-\nabla^{(2)}G_i(y_{u'}+rv_{u'})\big\|_{(2),0\to0}dr\|v_{u'}\|^2.\end{aligned}$$

By using the Lipschitz property on the bilinear form $\nabla^{(2)}G_i$, we have

$$\begin{aligned}&\int_0^1(1-r)\big\|\nabla^{(2)}G_i(y_u+rv_u)-\nabla^{(2)}G_i(y_{u'}+rv_{u'})\big\|_{(2),0\to0}dr \\ &\qquad\le C\int_0^1(1-r)\|y_u-y_{u'}\|dr+\int_0^1(1-r)r\|v_u-v_{u'}\|dr \\ &\qquad\le C\|y_u-y_{u'}\|+C\|v_u-v_{u'}\|.\end{aligned}$$

Now, we observe $\hat{\mathcal{C}}_1^{\kappa,\kappa} \hookrightarrow \mathcal{C}_1^{\kappa,0}$ (see (2.4)) and $E_\kappa \hookrightarrow E$. Therefore,

$$\begin{aligned}\frac{\|c_u^i(y,v)-c_{u'}^i(y,v)\|}{|u-u'|^\kappa} &\le C\frac{\|v_u-v_{u'}\|}{|u-u'|^\kappa}\|v_{u'}\| + \frac{\|v_u-v_{u'}\|}{|u-u'|^\kappa}\|v_u\| \\ &\quad + C\left(\frac{\|y_u-y_{u'}\|}{|u-u'|^\kappa} + \frac{\|v_u-v_{u'}\|}{|u-u'|^\kappa}\right)\|v_{u'}\|^2 \\ &\le C2\|v\|^2_{\hat{\mathcal{C}}_1^{\kappa,\kappa}} + C\|v\|^3_{\hat{\mathcal{C}}_1^{\kappa,\kappa}}. \end{aligned} \tag{2.31}$$

By assumption, $\sup_{i\ge 1}\sup_{p\in E_\kappa}\|\nabla^2 G_i(p)\|_{(2),\kappa\to\kappa} < \infty$ so that

$$\sup_{i\ge 1}\|c^i(y,v)\|_{0,\kappa} \le C\|v\|^2_{\hat{\mathcal{C}}_1^{\kappa,\kappa}}. \tag{2.32}$$

Plugging (2.32), (2.31), (2.30) and (2.29) into (2.28), we conclude from (2.27) that $\|R_2(y,v)\|_{\hat{\mathcal{C}}_1^{\kappa,\kappa}} \le C\|v\|^2_{\hat{\mathcal{C}}_1^{\kappa,\kappa}}$.

Let us now estimate $R_3(y,v)$. Similar to (2.27), from Lemma 2.3.2, we know there exists a constant C such that

$$\|\mathcal{J}\big(\hat{d}xe(y,v)\big)\|_{\kappa,\kappa} \le C\|h\|_{\mathcal{W}^{\gamma,\delta,\infty}_{\lambda,T}}\Big\{\sup_{i\ge 1}\|e^i(y,v)\|_{0,\kappa} + \sup_{i\ge 1}\|\hat{\delta}e^i(y,v)\|_{\kappa,0}\Big\}. \tag{2.33}$$

Clearly, Assumption A1 yields

$$\sup_{i\ge 1}\|e^i(y,v)\|_{0,\kappa} \le C\|v\|_{0,\kappa} \le C\|v\|_{\hat{\mathcal{C}}_1^{\kappa,\kappa}}. \tag{2.34}$$

Similar to (2.29) and (2.30), we observe

$$\|(\hat{\delta}e^i(y,v))_{ts}\| \le \|(\delta e^i(y,v))_{ts}\| + \|\big(S(t-s)-\mathrm{Id}\big)e^i_s(y,v)\| \tag{2.35}$$

where

$$\begin{aligned}\|\big(S(t-s)-\mathrm{Id}\big)e^i_s(y,v)\| &\le C|t-s|^\kappa|e^i_s(y,v)|_\kappa \\ &\le C|t-s|^\kappa|v_s|_\kappa \le C|t-s|^\kappa\|v\|_{0,\kappa};\,(t,s)\in\mathcal{S}_2. \end{aligned}\tag{2.36}$$

The boundedness and the Lipschitz property on ∇G_i (Assumption A2) allow us to estimate

$$\begin{aligned}\|e^i_u(y,v)-e^i_{u'}(y,v)\| &\le \left\|\left(\int_0^1 \nabla G_i(y_u+rv_u)dr\right)(v_u-v_{u'})\right\| \\ &\quad+\left\|\left(\int_0^1 [\nabla G_i(y_u+rv_u)-\nabla G_i(y_{u'}+rv_{u'})]dr\right)v_{u'}\right\| \\ &\le C\|v_u-v_{u'}\|+C\|v_{u'}\|\Big\{\|y_u-y_{u'}\|+\|v_u-v_{u'}\|\Big\}.\end{aligned}$$

Then, (2.4) yields

$$\begin{aligned}\frac{\|e^i_u(y,v)-e^i_{u'}(y,v)\|}{|u-u'|^\kappa} &\le C\|\delta v\|_{\kappa,0}+\|v\|_{\hat{\mathcal{C}}^{\kappa,\kappa}_1}\{\|\delta y\|_{\kappa,0}+\|\delta v\|_{\kappa,0}\} \\ &\le C\|v\|_{\hat{\mathcal{C}}^{\kappa,\kappa}_1}+\|v\|_{\hat{\mathcal{C}}^{\kappa,\kappa}_1}\{\|y\|_{\hat{\mathcal{C}}^{\kappa,\kappa}_1}+\|v\|_{\hat{\mathcal{C}}^{\kappa,\kappa}_1}\}.\end{aligned} \tag{2.37}$$

By using (2.33), (2.34), (2.35), (2.36) and (2.37), we infer

$$\|\mathcal{J}\big(\hat{d}xe(y,v)\big)\|_{\kappa,\kappa}=O\big(\|h\|_{\mathcal{W}^{\tilde{\gamma},\delta,\infty}_{\lambda,T}}\times\|v\|_{\hat{\mathcal{C}}^{\kappa,\kappa}_1}\big).$$

One can check $(x,y)\mapsto\nabla_1L(x,y)\in\mathcal{L}(\mathcal{W}^{\tilde{\gamma},\delta,\infty}_{\lambda,T};\hat{\mathcal{C}}^{\kappa,\kappa}_1)$ and $(x,y)\mapsto\nabla_2L(x,y)\in\mathcal{L}(\hat{\mathcal{C}}^{\kappa,\kappa}_1;\hat{\mathcal{C}}^{\kappa,\kappa}_1)$ are both continuous. Summing up all the above steps, we conclude L is Fréchet differentiable and formulas (2.23) and (2.24) hold true. It remains to check $\nabla_2L(x,\Phi(x))$ is a $\hat{\mathcal{C}}^{\kappa,\kappa}_1$ - homeomorphism. By open mapping theorem, this is an immediate consequence of Corollary 2.3.1 (which proves it is an isomorphism). The continuity can be easily checked so we left the details of this point to the reader. ■

By applying implicit function theorem, $x\mapsto\Phi(x)$ is continuously Fréchet differentiable and the following formula holds true

$$\nabla\Phi(x)=-\nabla_2L(x,\Phi(x))^{-1}\circ\nabla_1L(x,\Phi(x));x\in\mathcal{W}^{\tilde{\gamma},\delta,\infty}_{\lambda,T} \tag{2.38}$$

The inverse operator yields $\nabla_2L(x,\Phi(x))\big(\nabla_2L(x,\Phi(x))^{-1}(v)\big)=v$ so that

$$\begin{aligned}\nabla_2L(x,\Phi(x))^{-1}(v)_t &= v_t+\int_0^t S(t-u)\nabla F(\Phi(x)_u)\nabla_2L(x,\Phi(x))^{-1}(v)_udu \\ &\quad+\sum_{i\ge1}\sqrt{\lambda_i}\int_0^t S(t-u)\nabla G_i\big(\Phi(x)_u\big)\nabla_2L(x,\Phi(x))^{-1}(v)_udx^i_u;0\le t\le T\end{aligned}$$

for each $v \in \hat{\mathcal{C}}_1^{\kappa,\kappa}$. Therefore, for each $x, h \in \mathcal{W}_{\lambda,T}^{\tilde{\gamma},\delta,\infty}$, $\nabla\Phi(x)(h)$ is the unique solution of

$$\begin{aligned}\nabla\Phi(x)(h)_t = &\sum_{i\geq 1}\sqrt{\lambda_i}\int_0^t S(t-u)G_i\big(\Phi(x)_u\big)dh_u^i \\ &+\sum_{i\geq 1}\sqrt{\lambda_i}\int_0^t S(t-u)\nabla G_i\big(\Phi(x)_u\big)\nabla\Phi(x)(h)_u dx_u^i \\ &+\int_0^t S(t-u)\nabla F\big(\Phi(x)_u\big)\nabla\Phi(x)(h)_u du; 0\leq t\leq T.\end{aligned} \tag{2.39}$$

Now, by Corollary 2.3.1, for each $u \in (0,T), x \in \mathcal{W}_{\lambda,T}^{\tilde{\gamma},\delta,\infty}$ and $i \geq 1$, the mapping $t \mapsto \Psi_{t,u}^i(x)$ given by

$$\begin{aligned}\Psi_{t,u}^i(x) := &S(t-u)G_i\big(\Phi(x)_u\big) + \sum_{j\geq 1}\sqrt{\lambda_j}\int_u^t S(t-\ell)\nabla G_j\big(\Phi(x)_\ell\big)\Psi_{\ell,u}^i(x)dx_\ell^j \\ &+\int_u^t S(t-\ell)\nabla F\big(\Phi(x)_\ell\big)\Psi_{\ell,u}^i(x)d\ell\end{aligned} \tag{2.40}$$

where $\Psi_{t,u}^i(x) = 0$ for $u > t$, it is a well-defined element of $\hat{\mathcal{C}}_1^{\kappa,\kappa}$ over $[u,T]$. Let us denote $\Gamma_{x,u,u'}^i(t) := \Psi_{t,u}^i(x) - \Psi_{t,u'}^i(x)$ for $0 \leq u' \leq u \leq t \leq T$. It is simple to check that

$$\begin{aligned}\Gamma_{x,u,u'}^i(t) = &S(t-u)\Big[\Psi_{u,u}^i(x) - \Psi_{u,u'}^i(x)\Big] \\ &+\sum_{j\geq 1}\sqrt{\lambda_j}\int_u^t S(t-\ell)\nabla G_j(\Phi(x)_\ell)\Gamma_{x,u,u'}^i(\ell)dx_\ell^j \\ &+\int_u^t S(t-\ell)\nabla F(\Phi(x)_\ell)\Gamma_{x,u,u'}^i(\ell)d\ell.\end{aligned}$$

The following technical lemma is important to derive an alternative representation for $\Phi'(x)(h)$.

Lemma 2.3.4 *For each $x \in \mathcal{W}_{\lambda,T}^{\tilde{\gamma},\delta,\infty}$ there exists a positive constant C which only depends on $\|x\|_{\mathcal{W}_{\lambda,T}^{\tilde{\gamma},\delta,\infty}}$ and $\|\delta\Phi(x)\|_{\kappa,\kappa}$ such that*

$$|\Gamma_{x,u,u'}^i(t)|_\kappa \leq C|\Psi_{u,u}^i(x) - \Psi_{u,u'}^i(x)|_\kappa$$

for every $0 \leq u' < u \leq t \leq T$ and $i \geq 1$.

Proof. Fix $0 \leq u' < u \leq T$, $i \geq 1$, $0 \leq \alpha \leq \min\{\kappa,\eta\}$ for $\eta \geq 0$. Let us denote

$\varphi^i_{x,u,u'} = \big[\Psi^i_{u,u}(x) - \Psi^i_{u,u'}(x)\big]$. In the sequel, C is a constant which may defer form line to line. Of course,

$$\begin{aligned}
|\hat{\delta}\Gamma^i_{x,u,u'}\|_{\kappa,\eta} \le \|\hat{\delta}S(\cdot - u)\varphi^i_{x,u,u'}\|_{\kappa,\eta} \\
+ \sum_{j\ge 1}\sqrt{\lambda_j}\left\|\hat{\delta}\int_u^{\cdot} S(\cdot - \ell)\nabla G_j(\Phi(x)_\ell)\Gamma^i_{x,u,u'}(\ell)dx^j_\ell\right\|_{\kappa,\eta} \\
+ \left\|\hat{\delta}\int_u^{\cdot} S(\cdot - \ell)\nabla F(\Phi(x)_\ell)\Gamma^i_{x,u,u'}(\ell)d\ell\right\|_{\kappa,\eta} =: I_1 + I_2 + I_3.
\end{aligned}$$

At first, we observe $S(t-u)\varphi^i_{x,u,u'} - S(t-s)S(s-u)\varphi^i_{x,u,u'} = 0$ so that $I_1 = 0$.

By Lemma 2.3.2 (see (2.19)), we observe there exists a constant C such that

$$\begin{aligned}
I_2 &\le C\sum_{j\ge 1}\sqrt{\lambda_j}\left\|\hat{\delta}\int_u^{\cdot} S(\cdot - \ell)\nabla G_j(\Phi(x)_\ell)\Gamma^i_{x,u,u'}(\ell)dx^j_\ell\right\|_{\kappa,\eta} \\
&\le C\|x\|_{\mathcal{W}^{\tilde{\gamma},\delta,\infty}_{\lambda,T}}\Big\{\sup_{j\ge 1}\|z^{ij}_{x,u,u'}\|_{0,\eta}|T-u|^{\tilde{\gamma}-\kappa} + |T-u|^{\tilde{\gamma}-\alpha}\sup_{j\ge 1}\|\hat{\delta}z^{ij}_{x,u,u'}\|_{\kappa,\eta-\alpha}\Big\}
\end{aligned}$$

where $z^{ij}_{x,u,u'}(\ell) = \nabla G_j\big(\Phi(x)_\ell\big)\Gamma^i_{x,u,u'}(\ell)$. Let us take $\eta = \kappa = \alpha$. We observe

$$|z^{ij}_{x,u,u'}(\ell)|_\kappa \le \|\nabla G_j\big(\Phi(x)_\ell\big)\|_{\kappa\to\kappa}|\Gamma^i_{x,u,u'}(\ell)|_\kappa$$

so that the boundedness assumption on the gradient ∇G_j yields

$$\|z^{ij}_{x,u,u'}\|_{0,\kappa} \le C\|\Gamma^i_{x,u,u'}\|_{0,\kappa} \le C\|\Gamma^i_{x,u,u'}\|_{\hat{\mathcal{C}}^{\kappa,\kappa}_1}. \tag{2.41}$$

Triangle inequality yields

$$\begin{aligned}
\big\|(\hat{\delta}z^{ij}_{x,u,u'})_{ts}\big\|_E &\le \big|[\nabla G_j\big(\Phi(x)_t\big) - \nabla G_j\big(\Phi(x)_s\big)]\Gamma^i_{x,u,u'}(t)\big\|_E \\
&\quad + \big\|\nabla G_j\big(\Phi(x)_s\big)\big(\Gamma^i_{x,u,u'}(t) - \Gamma^i_{x,u,u'}(s)\big)\big\|_E \\
&\quad + \big\|[\mathrm{Id} - S(t-s)]\nabla G_j\big(\Phi(x)_s\Gamma^i_{x,u,u'}(s)\big\|_E \\
&\le \big\|[\nabla G_j\big(\Phi(x)_t\big) - \nabla G_j\big(\Phi(x)_s\big)]\big\|_{0\to 0}\big\|\Gamma^i_{x,u,u'}(t)\big\|_E \\
&\quad + \|\nabla G_j\big(\Phi(x)_s\big)\|_{0\to 0}\|(\delta\Gamma^i_{x,u,u'})_{ts}\|_E \\
&\quad + \big\|[\mathrm{Id} - S(t-s)]\nabla G_j\big(\Phi(x)_s\big)\Gamma^i_{x,u,u'}(s)\big\|_E =: I_4 + I_5 + I_6,
\end{aligned}$$

where $\nabla G_j\big(\Phi(x)_s\big)\Gamma^i_{x,u,u'}(s) \in E_\kappa$. We observe

$$\begin{aligned} I_6 &\le C|t-s|^\kappa |\nabla G_j\big(\Phi(x)_s\big)\Gamma^i_{x,u,u'}(s)|_\kappa \\ &\le C|t-s|^\kappa \|\Gamma^i_{x,u,u'}\|_{0,\kappa}. \end{aligned} \tag{2.42}$$

The imbedding (2.4) yields

$$\begin{aligned} I_5 &\le C\|(\delta\Gamma^i_{x,u,u'})_{ts}\|_E \\ &\le C|t-s|^\kappa\{\|\hat{\delta}\Gamma^i_{x,u,u'}\|_{\kappa,\kappa} + \|\Gamma^i_{x,u,u'}\|_{0,\kappa}\} = C|t-s|^\kappa\|\Gamma^i_{x,u,u'}\|_{\hat{\mathcal{C}}_1^{\kappa,\kappa}}. \end{aligned} \tag{2.43}$$

We observe

$$I_4 \le C\|\delta\Phi(x)\|_{\kappa,\kappa}|t-s|^\kappa\|\Gamma^i_{x,u,u'}\|_{0,\kappa}. \tag{2.44}$$

Summing up (2.44), (2.43) and (2.42), we have

$$\|\hat{\delta} z^{ij}_{x,u,u'}\|_{\kappa,0} \le C\Big(1 + \|\delta\Phi(x)\|_{\kappa,\kappa}\Big)\|\Gamma^i_{x,u,u'}\|_{\hat{\mathcal{C}}_1^{\kappa,\kappa}}. \tag{2.45}$$

This shows that

$$I_2 \le C\|x\|_{\mathcal{W}^{\tilde{\gamma},\delta,\infty}_{\lambda,T}}\Big\{\|\Gamma^i_{x,u,u'}\|_{\hat{\mathcal{C}}_1^{\kappa,\kappa}}|T-u|^{\tilde{\gamma}-\kappa} + |T-u|^{\tilde{\gamma}-\alpha}\Big(1+\|\delta\Phi(x)\|_{\kappa,\kappa}\Big)\|\Gamma^i_{x,u,u'}\|_{\hat{\mathcal{C}}_1^{\kappa,\kappa}}\Big\}.$$

We notice that

$$I_3 \le C\sup_{u\le s<t\le T}\frac{\|\int_s^t S(t-\ell)\nabla F(\Phi(x)_\ell)\Gamma^i_{x,u,u'}(\ell)d\ell\|_\kappa}{|t-s|^\kappa} = C\|\Gamma^i_{x,u,u'}\|_{0,\kappa}|T-u|^{1-\kappa}.$$

Summing up the above inequalities, we have

$$\begin{aligned} \|\hat{\delta}\Gamma^i_{x,u,u'}\|_{\kappa,\kappa} \le C\|x\|_{\mathcal{W}^{\tilde{\gamma},\delta,\infty}_{\lambda,T}}&\Big\{C|T-u|^{\tilde{\gamma}-\kappa}\Big(1+\|\delta\Phi(x)\|_{\kappa,\kappa}\Big)\|\Gamma^i_{x,u,u'}\|_{\hat{\mathcal{C}}_1^{\kappa,\kappa}}\Big\} \\ &+ C\|\Gamma^i_{x,u,u'}\|_{0,\kappa}|T-u|^{1-\kappa}. \end{aligned} \tag{2.46}$$

Therefore,

$$\begin{aligned} \|\Gamma^i_{x,u,u'}\|_{\hat{\mathcal{C}}_1^{\kappa,\kappa}} &\le \|S(\cdot-u)\varphi^i_{x,u,u'}\|_{0,\kappa} \\ &\qquad + C\|x\|_{\mathcal{W}^{\tilde{\gamma},\delta,\infty}_{\lambda,T}}\Big\{C|T-u|^{\tilde{\gamma}-\kappa}\Big(1+\|\delta\Phi(x)\|_{\kappa,\kappa}\Big)\|\Gamma^i_{x,u,u'}\|_{\hat{\mathcal{C}}_1^{\kappa,\kappa}}\Big\} \end{aligned}$$

$$+ C\|\Gamma^i_{x,u,u'}\|_{0,\kappa}|T-u|^{1-\kappa} \tag{2.47}$$

where $\|S(\cdot - u)\varphi^i_{x,u,u'}\|_{0,\kappa} \le C|\varphi^i_{x,u,u'}|_\kappa$. Finally, by working on a small interval and performing a standard patching argument, the estimate (2.47) allows us to conclude

$$\|\Gamma^i_{x,u,u'}\|_{\hat{\mathcal{C}}_1^{\kappa,\kappa}} \le C_{x,y,T}|\varphi^i_{x,u,u'}|_\kappa$$

where $C_{x,y,T} = g\big(\|x\|_{\mathcal{W}^{\tilde{\gamma},\delta,\infty}_{\lambda,T}}, \|\delta\Phi(x)\|_{\kappa,\kappa}, T\big)$ for a function $g : \mathbb{R}^3_+ \to \mathbb{R}_+$ growing with its arguments. This implies

$$|\Gamma^i_{x,u,u'}(t)|_\kappa = |\Psi^i_{t,u}(x) - \Psi^i_{t,u'}(x)|_\kappa \le C_{x,y,T}\big|\Psi^i_{u,u}(x) - \Psi^i_{u,u'}(x)\big|_\kappa.$$

■

We are now in position to state the main result of this section. Let $\mathcal{C}^\infty_{0,\lambda}$ be the subset of $\mathcal{W}^{\tilde{\gamma},\delta,\infty}_{\lambda,T}$ composed by functions $g : \mathbb{N} \to \mathcal{C}^\infty_0$.

Theorem 2.3.5 *Under Assumptions (H1-A1-A2), the Itô map $x \mapsto \Phi(x)$ is continuously Fréchet differentiable and for each $x, h \in \mathcal{W}^{\tilde{\gamma},\delta,\infty}_{\lambda,T}$, $\nabla\Phi(x)(h)$ is the unique solution of the equation (2.39). In addition, the following representation formula holds true*

$$\nabla\Phi(x)(h)_t = \sum_{i\ge1}\sqrt{\lambda_i}\int_0^t \Psi^i_{t,u}(x)dh^i_u \in E_\kappa; 0 \le t \le T \tag{2.48}$$

for each $(x,h) \in \mathcal{W}^{\tilde{\gamma},\delta,\infty}_{\lambda,T} \times \mathcal{C}^\infty_{0,\lambda}$.

Proof. The fact that $x \mapsto \Phi(x)$ is continuously Fréchet differentiable and it satisfies (2.39) are consequences of (2.38). Obviously,

$$\begin{aligned}\sum_{i\ge1}\sqrt{\lambda_i}\int_0^t \Psi^i_{t,u}(x)dh^i_u &= \sum_{i\ge1}\sqrt{\lambda_i}\int_0^t S(t-u)G_i\big(\Phi(x)_u\big)dh^i_u \\ &+ \sum_{i\ge1}\sqrt{\lambda_i}\int_0^t\int_u^t S(t-\ell)\nabla F\big(\Phi(x)_\ell\big)\Psi^i_{\ell,u}(x)d\ell dh^i_u \\ &+ \sum_{i\ge1}\sqrt{\lambda_i}\sum_{j\ge1}\sqrt{\lambda_j}\int_0^t\int_u^t S(t-\ell)\nabla G_j\big(\Phi(x)_\ell\big)\Psi^i_{\ell,u}(x)dx^j_\ell dh^i_u; \\ &\qquad 0 \le t \le T.\end{aligned}$$

Let us fix $i \geq 1$ and $x \in \mathcal{W}^{\tilde{\gamma},\delta,\infty}_{\lambda,T}$. By Lemma 2.3.4 and noticing

$$\begin{aligned}\Psi^i_{u,u}(x) - \Psi^i_{u,u'}(x) &= G_i\big(\Phi(x)_u\big) - S(u-u')G_i\big(\Phi(x)_{u'}\big)\\ &- \sum_{j\geq 1}\sqrt{\lambda_j}\int_{u'}^{u} S(u-\ell)\nabla G_j\big(\Phi(x)_\ell\big)\Psi^i_{\ell,u'}(x)dx^j_\ell\\ &- \int_{u'}^{u} S(u-\ell)\nabla F\big(\Phi(x)_\ell\big)\Psi^i_{\ell,u'}(x)d\ell; 0 \leq u' < u \leq T; i \geq 1, \quad (2.49)\end{aligned}$$

we clearly have $u \mapsto \Psi^i_{t,u}(x)$ is continuous, so that we shall apply Fubini's theorem to get

$$\int_0^t\int_u^t S(t-\ell)\nabla F\big(\Phi(x)_\ell\big)\Psi^i_{\ell,u}(x)d\ell dh^i_u = \int_0^t\int_0^\ell S(t-\ell)\nabla F\big(\Phi(x)_\ell\big)\Psi^i_{\ell,u}(x)dh^i_u d\ell$$

and

$$\int_0^t\int_u^t S(t-\ell)\nabla G_j\big(\Phi(x)_\ell\big)\Psi^i_{\ell,u}(x)dx^j_\ell dh^i_u = \int_0^t\int_0^\ell S(t-\ell)\nabla G_j\big(\Phi(x)_\ell\big)\Psi^i_{\ell,u}(x)dh^i_u dx^j_\ell;$$

$$0 \leq t \leq T, i \geq 1.$$

Therefore,

$$\begin{aligned}\sqrt{\lambda_i}\int_0^t \Psi^i_{t,u}(x)dh^i_u &= \sqrt{\lambda_i}\int_0^t S(t-u)G_i\big(\Phi(x)_u\big)dh^i_u\\ &+ \int_0^t S(t-\ell)\nabla F\big(\Phi(x)_\ell\big)\sqrt{\lambda_i}\int_0^\ell \Psi^i_{\ell,u}(x)dh^i_u d\ell\\ &+ \sum_{j\geq 1}\sqrt{\lambda_j}\int_0^t S(t-\ell)\nabla G_j\big(\Phi(x)_\ell\big)\sqrt{\lambda_i}\int_0^\ell \Psi^i_{\ell,u}(x)dh^i_u dx^j_\ell;\\ &\qquad 0 \leq t \leq T.\end{aligned}$$

At this point, in order to complete the proof of representation (2.48), we only need to check

$$\sup_{i\geq 1}\sup_{0\leq t\leq T}\|\Psi^i_{t,\cdot}\|_{0,\kappa} < \infty. \quad (2.50)$$

Since Ψ^i is the solution of the linear equation (2.40), a completely similar argument as detailed in the proof of Lemma 2.3.4 yields

$$|\Psi^i_{t,u}(x)|_\kappa \le C_{x,y,T} \sup_{0\le r\le T} |S(r-u)G_i(\Phi(x)_u)|_\kappa$$

for each $0 \le u \le t \le T$, where $C_{x,y,T} = g\big(\|x\|_{\mathcal{W}^{\tilde{\gamma},\delta,\infty}_{\lambda,T}}, \|\delta\Phi(x)\|_{\kappa,\kappa}, T\big)$ for a function $g : \mathbb{R}^3_+ \to \mathbb{R}_+$ growing with its arguments. This completes the proof. ■

Let us now check Malliavin differentiability. Let us fix $t \in [0,T]$, $g \in E$ and we now look the mapping $\mathcal{W}^{\tilde{\gamma},\delta,\infty}_{\lambda,T} \ni x \mapsto \langle \Phi(x)_t, g\rangle_E \in \mathbb{R}$. We can represent $\Phi(x)_t = \tau_t(\Phi(x))$ where $\tau_t : \hat{\mathcal{C}}^{\kappa,\kappa}_1 \to E$ is the evaluation map which is a bounded linear operator for every $t \in [0,T]$. Then the Fréchet derivative of $x \mapsto \Phi(x)_t$ equals to the linear operator

$$\mathcal{W}^{\tilde{\gamma},\delta,\infty}_{\lambda,T} \ni f \mapsto \nabla\Phi(x)(f)_t \in E_\kappa \subset E.$$

Similarly, the Fréchet derivative of $x \mapsto \langle \Phi(x)_t, g\rangle_E$ equals to

$$f \mapsto \langle \nabla\Phi(x)(f)_t, g\rangle_E.$$

We must find an $\mathcal{L}_2(U_0,\mathcal{H})$-valued random element $\omega \mapsto a(\omega)$ such that

$$\langle \nabla\Phi(\cdot)(\mathcal{R}_H h)_t, g\rangle_E = \langle a(\cdot), h\rangle_{\mathcal{L}_2(U_0,\mathcal{H})}\ a.s$$

for each $h \in \mathcal{L}_2(U_0;\mathcal{H})$. If this is the case, then $a = \mathbf{D}_\cdot\langle X_t, g\rangle_E$ a.s.

Lemma 2.3.6 *If $h \in \mathcal{C}^\infty_0$ and $\varphi \in \mathcal{L}_2(U_0;\mathbb{R})$, then*

$$\mathcal{R}_H(h \otimes \varphi) \in \mathcal{C}^\infty_{0,\lambda}.$$

Proof. By definition, if $(h\otimes\varphi) \in \mathcal{C}^\infty_0 \otimes \mathcal{L}_2(U_0.\mathbb{R})$, then

$$K^*_H(h\otimes\varphi)_s(\sqrt{\lambda_i}e_i) = \sqrt{\lambda_i}\int_s^T h(t)\frac{\partial K_H}{\partial t}(t,s)dt.\varphi(e_i); i \ge 1$$

where

$$\frac{\partial K_H}{\partial t}(t,s) = c_H\Big(\frac{t}{s}\Big)^{H-\frac{1}{2}}(t-s)^{H-\frac{3}{2}}; s<t,$$

$$\mathcal{R}_H(h\otimes\varphi)_t = \sqrt{\lambda_i}\int_0^t K_H(t,s)K_H^*(h\otimes\varphi)_s(e_i)ds$$
$$= \sqrt{\lambda_i}\int_0^t \Big(\int_0^u \frac{\partial K_H}{\partial u}(u,s)K_H^*(h\otimes\varphi)_s(e_i)ds\Big)du; i\geq 1. \quad (2.51)$$

We observe
$$u\mapsto \int_0^u \frac{\partial K_H}{\partial u}(u,s)K_H^*(h\otimes\varphi)_s(e_i)ds$$
is continuous (hence bounded) and this implies that

$$\sum_{i\geq 1}\lambda_i \sup_{0\leq u\leq T}\Big|\int_0^u \frac{\partial K_H}{\partial u}(u,s)K_H^*(h\otimes\varphi)_s(e_i)ds\Big| =$$
$$= \sum_{i\geq 1}\lambda_i \sup_{0\leq u\leq T}\Big|\int_0^u \frac{\partial K_H}{\partial u}(u,s)\int_s^T h(t)\frac{\partial K_H}{\partial t}(t,s)dt.\varphi(e_i)ds\Big|$$
$$= \sum_{i\geq 1}\lambda_i|\varphi(e_i)| \sup_{0\leq u\leq T}\Big|\int_0^u \frac{\partial K_H}{\partial u}(u,s)\int_s^T h(t)\frac{\partial K_H}{\partial t}(t,s)dtds\Big|$$
$$\leq \big(\text{trace } Q\big)^{\frac{1}{2}}\|\varphi\|_{\mathcal{L}_2(U_0,\mathbb{R})} \sup_{0\leq u\leq T}\Big|\int_0^u \frac{\partial K_H}{\partial u}(u,s)\int_s^T h(t)\frac{\partial K_H}{\partial t}(t,s)dtds\Big|.$$

■

Corollary 2.3.7 *Under the probability space given in Lemma 1.2.4, the random variable $\langle X_t, g\rangle_E\in\mathbb{D}^{1,2}_{loc}(\mathbb{R})$ and $\mathbf{D}\langle X_t,g\rangle_E\in\mathcal{L}_2(U_0;\mathcal{H})$ is the Hilbert-Schmidt linear operator defined by*
$$\mathbf{D}\langle X_t,g\rangle_E(\sqrt{\lambda_i}e_i) := \langle\sqrt{\lambda_i}\Psi^i_{t,\cdot}, g\rangle_E \ a.s$$
for every $t\in[0,T]$ and $g\in E$.

Proof. Let us fix $t\in[0,T]$ and $g\in E$. By Lemma 1.2.4, we shall represent $X_t(\omega) = \Phi(\omega)_t; (\omega,t)\in\mathcal{W}^{\tilde{\gamma},\delta,\infty}_{\lambda,T}\times[0,T]$. Since $\mathbf{H}\subset\mathcal{W}^{\gamma,\delta,\infty}_{\lambda,T}$, then

$$f\mapsto\langle X_t(f),g\rangle_E = \langle\Phi(f)_t,g\rangle_E$$

is Fréchet differentiable at all vectors $f\in\mathbf{H}$. In this case, Corollary 1.2.5 yields $\langle X_t,g\rangle_E\in\mathbb{D}^{1,2}_{loc}(\mathbb{R})$ and

$$\langle\nabla\Phi(\cdot)(\mathcal{R}_H v)_t, g\rangle_E = \langle\mathbf{D}\langle X_t,g\rangle_E, v\rangle_{\mathcal{L}_2(U_0;\mathcal{H})} \text{ locally in } \Omega$$

for each $v \in \mathcal{L}_2(U_0;\mathcal{H})$. Let us take $v = (h \otimes \varphi) \in \mathcal{C}_0^\infty \otimes \mathcal{L}_2(U_0.\mathbb{R})$. By using (2.48)

$$\Big\langle \nabla\Phi(\mathcal{R}_H v)_t, g\Big\rangle_E = \sum_{i\geq 1}\sqrt{\lambda_i}\Bigg\langle \int_0^t \Psi^i_{t,u}(x)d(\mathcal{R}_H v^i)_u, g\Bigg\rangle_E. \tag{2.52}$$

From (2.51), we have

$$(\mathcal{R}_H v^i)'_u = \int_0^u \frac{\partial K_H}{\partial u}(u,s)K_H^*(h\otimes\varphi)_s(e_i)ds$$

Therefore,

$$\begin{aligned}
\int_0^t \Psi^i_{t,u}(x)d(\mathcal{R}_H v^i)_u &= \sqrt{\lambda_i}\int_0^t \Psi^i_{t,u}(x)\left(\int_0^u \frac{\partial K_H}{\partial u}(u,s)K_H^*(h\otimes\varphi)_s(e_i)ds\right)du\\
&= \sqrt{\lambda_i}\int_0^T \Psi^i_{t,u}(x)\left(\int_0^u \frac{\partial K_H}{\partial u}(u,s)K_H^*(h\otimes\varphi)_s(e_i)ds\right)du\\
&= \sqrt{\lambda_i}\int_0^T K_H^*(h\otimes\varphi)_s(e_i)\left(\int_s^T \frac{\partial K_H}{\partial u}(u,s)\Psi^i_{t,u}(x)du\right)ds.
\end{aligned}$$

Then,

$$\begin{aligned}
\langle \nabla\Phi(\mathcal{R}_H v)_t, g\rangle_E &= \sum_{i=1}^\infty \lambda_i \int_0^T K_H^*\big(h\otimes\varphi\big)_s(e_i)K_H^*\Big(\langle \Psi^i_{t,\cdot}, g\rangle_E\Big)_s ds\\
&= \sum_{i=1}^\infty \lambda_i \Big\langle (h\otimes\varphi)(e_i), \langle \Psi^i_{t,\cdot}, g\rangle_E\Big\rangle_{\mathcal{H}}\\
&= \sum_{i=1}^\infty \Big\langle (h\otimes\varphi)(\sqrt{\lambda_i}e_i), \langle \sqrt{\lambda_i}\Psi^i_{t,\cdot}, g\rangle_E\Big\rangle_{\mathcal{H}}
\end{aligned}$$

where we observe (recall that this function is continuous (except at one point) for every $x \in \Omega$) $\langle \sqrt{\lambda_i}\Psi^i_{t,\cdot}(x), g\rangle_E \in L^{\frac{1}{H}}([0,T];\mathbb{R}) \subset |\mathcal{H}|$. The candidate is then the linear operator defined by

$$\mathbf{D}\langle X_t, g\rangle_E(\sqrt{\lambda_i}e_i) := \langle \sqrt{\lambda_i}\Psi^i_{t,\cdot}, g\rangle_E \ a.s. \tag{2.53}$$

We observe (2.53) provides a well-defined Hilbert-Schmidt operator from U_0 to $\mathcal{H}$ because

$$\sum_{i=1}^{\infty}\lambda_i\int_0^T\big|K_H^*\big(\langle\Psi^i_{t,\cdot}(\omega),g\rangle_E\big)_s\big|^2ds \;=\; \sum_{i=1}^{\infty}\lambda_i\int_0^T\Big|\int_s^T\frac{\partial K_H}{\partial u}(u,s)\langle\Psi^i_{t,u}(\omega),g\rangle_E du\Big|^2 ds$$
$$\le\int_0^T\Big(\int_s^T\big|\frac{\partial K_H}{\partial u}(u,s)\big|du\Big)^2ds\|g\|^2_E\sup_{i\ge1}\|\Psi^i_{t,\cdot}(\omega)\|^2_{0,\kappa}\text{Trace }Q<\infty$$

for each $\omega\in\Omega$. This concludes the proof. ■

We are now able to state the main result of this section.

Theorem 2.3.8 *If assumptions H1-A1-A2 hold true, then $X_t\in\mathbb{D}^{1,2}_{loc}(E)$ for each $t\in[0,T]$ and the following formula holds*

$$\mathbf{D}_sX_t=S(t-s)G(X_s)+\int_s^t S(t-r)\nabla F(X_r)\mathbf{D}_sX_rdr$$
$$+\sum_{i=1}^{\infty}\sqrt{\lambda_i}\int_s^t S(t-r)\nabla G_i(X_r)\mathbf{D}_sX_rd\beta^i_r \tag{2.54}$$

where $\mathbf{D}_sX_t=0$ for $s>t$.

Proof. At first, we observe the postulated object $\mathbf{D}X_t$ takes values on $\mathcal{H}\otimes\mathcal{L}_2(U_0;\mathbb{R})\otimes E\equiv\mathcal{L}_2(U_0;\mathcal{H}\otimes E)$. Let us compute

$$\Big\langle\mathbf{D}\langle X_t,g\rangle_E,v\Big\rangle_{\mathcal{L}_2(U_0;\mathcal{H})}$$

for a given $g\in E$ and $v=(\varphi\otimes h)\in\mathcal{L}_2(U_0;\mathcal{H})$. By definition,

$$\Big\langle\mathbf{D}\langle X_t,g\rangle_E,v\Big\rangle_{\mathcal{L}_2(U_0;\mathcal{H})}=\sum_{i=1}^{\infty}\Big\langle\langle\sqrt{\lambda_i}\Psi^i_{t,\cdot},g\rangle_E,(\varphi\otimes h)(\sqrt{\lambda_i}e_i)\Big\rangle_{\mathcal{H}}$$
$$=\sum_{i=1}^{\infty}\varphi(e_i)\lambda_i\Big\langle\langle\Psi^i_{t,\cdot},g\rangle_E,h\Big\rangle_{\mathcal{H}}.$$

Let us define a Hilbert-Schmidt operator $\Psi_{t,\cdot}(\omega):U_0\to L^{\frac{1}{H}}([0,T];E)\hookrightarrow\mathcal{H}\otimes E$ as follows

$$\Psi_{t,\cdot}(\omega)(\sqrt{\lambda_i}e_i):=\sqrt{\lambda_i}\Psi^i_{t,\cdot}(\omega);\omega\in\Omega.$$

By (1.6), we observe

$$\sum_{i=1}^{\infty}\|\Psi_{t,\cdot}(\sqrt{\lambda_i}e_i)\|^2_{\mathcal{H}\otimes E} \le C\sum_{i=1}^{\infty}\|\Psi_{t,\cdot}(\sqrt{\lambda_i}e_i)\|^2_{L^{\frac{1}{H}}([0,T];E)}$$
$$\le C\sum_{i=1}^{\infty}\lambda_i\|\Psi^i_{t,\cdot}\|^2_{0,\kappa} \le C\sup_{i\ge1}\|\Psi^i_{t,\cdot}\|_{0,\kappa}.\text{trace } Q < \infty \ a.s.$$

We claim that $X_t \in \mathbb{D}^{1,2}_{loc}(E)$ and

$$\mathbf{D}_{\cdot}X_t = \Psi_{t,\cdot} \ a.s. \tag{2.55}$$

Indeed, we observe Ψ_t satisfies

$$\Psi_{t,s} = S(t-s)G(X_s) + \int_s^t S(t-r)\nabla F(X_r)\Psi_{r,s}dr$$
$$+\sum_{i=1}^{\infty}\sqrt{\lambda_i}\int_s^t S(t-r)\nabla G_i(X_r)\Psi_{r,s}d\beta^i_r \ a.s$$

where $\Psi_{t,s} = 0$ for $t < s$. Moreover,

$$\Big\langle \mathbf{D}\langle X_t, g\rangle_E, v\Big\rangle_{\mathcal{L}_2(U_0;\mathcal{H})} = \sum_{i=1}^{\infty}\varphi(e_i)\lambda_i\Big\langle\langle\Psi^i_{t,\cdot}, g\rangle_E, h\Big\rangle_{\mathcal{H}}$$
$$= \sum_{i=1}^{\infty}\Big\langle\langle\Psi_{t,\cdot}(\sqrt{\lambda_i}e_i), g\rangle_E, \varphi(e_i)\sqrt{\lambda_i}h\Big\rangle_{\mathcal{H}}$$
$$= \langle\mathbf{D}X_t g, v\rangle_{\mathcal{L}_2(U_0;\mathcal{H})} \ a.s.$$

By applying Lemma 1.2.2 and Corollary 2.3.7, we conclude the proof. ■

2.4 The right inverse of the Jacobian of the SPDE solution

From now on, it will be useful to make clear the dependence on the initial conditions of (2.1). Let us write X^y as the solution of (2.1) for an initial condition $y \in E_\kappa$. In previous section, we made use of the $\hat{\mathcal{C}}^{\kappa,\kappa}_1$-topology to get differentiability of $X^{x_0}_t$ (in Malliavin's sense) for each initial condition at $x_0 \in E_\kappa$. Even though we are interested in establishing the existence of densities $\mathcal{T}(X^{x_0}_t)$ for initial conditions on dom $(A^\infty) \subset E_\kappa$, it is important to work with the solution map $E \to \mathcal{C}_1^{\alpha,0}$ given by

$$y \mapsto X^y \in \mathcal{C}_1^{\alpha,0} \tag{2.56}$$

for some $\alpha > 1 - H$. One drawback to keep the flow from E_κ to $\hat{\mathcal{C}}_1^{\kappa,\kappa}$ is that X^{x_0} does not belong to $\mathcal{C}_1^{\kappa,\kappa}$ and the best we can get is $X^{x_0} \in \mathcal{C}_1^{\kappa,0}$ a.s. For this purpose, we need to impose further regularity assumptions as described in Th 3.2 in [29], which we list here for the sake of preciseness:

Assumption A3: There exists $\gamma_1, \gamma_2 \in (0, 2H-1)$ and c_1 such that

$$\|S(r)G(x)\| \leq \frac{c_1}{r^{\gamma_1}}(1 + \|x\|_E)$$

and

$$\|S(r)\big(G(x) - G(y)\big)\| \leq \frac{c_1}{r^{\gamma_2}}\|x - y\|_E$$

for every $x, y \in E$. Furthermore, for $\alpha > 1-H$, $\alpha < \min\left(\frac{1}{2}(1-\gamma_1), \frac{1}{2}(1-\gamma_2)\right)$, assume there exist constants $c_2, 0 \leq \eta < 1-\alpha, \tilde{\beta} \in (\alpha, \frac{1}{2})$ such that

$$\begin{gathered}
\|\nabla S(r)F(x)\| + \|\|\nabla S(r)G_i(x)\| \leq c_2 \\
\|\nabla S(r)F(x) - \nabla S(r)F(y)\| + \|\nabla S(r)G_i(x) - \nabla S(r)G_i(y)\| \leq \frac{c_2}{r^\eta}\|x - y\|_E, \\
\|\nabla(S(r) - S(s))F(x)\| + \|\nabla\big(S(r) - S(s)\big)G_i(x)\| \leq c_2(r-s)^{\tilde{\beta}}s^{-\tilde{\beta}}
\end{gathered}$$

for every $r \in (0, T], 0 < s < r, x, y \in E$ and $i \geq 1$.

Under these conditions, the map (2.56) is well-defined (see Th 3.2 in [29]). Moreover, it is not difficult to check the map $E \ni y \mapsto X^y \in \mathcal{C}_1^{\alpha,0}$ is Fréchet differentiable. In other words,

$$\mathbf{J}_{0\to t}(y; v) = \nabla_y X_t^y(v)$$

for each $t \in [0, T]$ and $y, v \in E$. The proof of this fact is quite standard and the main arguments do not defer too much from the classical Brownian motion driving case (see e.g Th. 3.9 in [19]), so we left the details to the reader. Moreover, (see [32]) there exists $\alpha \in \left(1 - H, \frac{1}{2}\right)$ and $\kappa \in \left(\frac{1}{2}, \frac{1}{4}\right)$ satisfying

$$\mathcal{C}_1^{\kappa,0} \subset W^{\alpha,\infty}(0,T;E)$$

where $W^{\alpha,\infty}(0,T;E)$ is the space of all measurable functions $f:[0,T] \to E$ such that

$$\|f\|_{\alpha,\infty} := \left(|f|_{0,0} + \sup_{0\le t\le T} \int_0^t \frac{\|f(t) - f(s)\|}{|t-s|^{1+\alpha}}\right) < \infty.$$

Therefore, under Assumptions H1 and A3, the uniqueness of the flow described in Th 3.2 in [29] and (2.3) imply that all solutions X^y generated by Proposition 2.2.2 coincides with the ones given by [29] for every $y \in E_\kappa$. In addition, by applying Th 3.2 in [29], $\mathbf{J}_{0\to t}(y;v)$ satisfies the following linear equation

$$\begin{aligned}\mathbf{J}_{0\to t}(y;v) = S(t)v &+ \int_0^t S(t-s)\nabla F(X_s^y)\mathbf{J}_{0\to s}(y;v)ds \\ &+ \sum_{i=1}^{\infty} \sqrt{\lambda_i} \int_0^t S(t-s)\nabla G_i(X_s^y)\mathbf{J}_{0\to s}(y;v)d\beta_s^i. \end{aligned} \tag{2.57}$$

Of course, $v \mapsto \mathbf{J}_{0\to t}(y;v) \in \mathcal{L}(E;E)$ for each $t \in [0,T]$ and $y \in E$. Then, we shall see $t \mapsto \mathbf{J}_{0\to t}(y)$ as an operator-valued process as follows

$$\begin{aligned}\mathbf{J}_{0\to t}(y) = S(t) &+ \int_0^t S(t-s)\nabla F(X_s^y)\mathbf{J}_{0\to s}(y)ds \\ &+ \sum_{i=1}^{\infty} \sqrt{\lambda_i} \int_0^t S(t-s)\nabla G_i(X_s^y)\mathbf{J}_{0\to s}(y)d\beta_s^i; 0 \le t \le T.\end{aligned}$$

Remark 2.4.1 *Recall that infinitesimal generators of analytic semigroups are sectorial (see e.g Prop 3.16 [25]). Then, it is known that (see e.g Corollary 2.1.7 in [26]) that $S(t)$ is one-to-one for every $t \ge 0$. We also observe the left-inverse linear operator $S(-t)$ of $S(t)$ defined on the subspace $S(t)E$ is, in general, unbounded.*

Example 2.4.2 *Let $E = L^2(0,1)$ with Dirichlet boundary conditions. Take the orthonormal basis*

$$e_n(x) = \sqrt{2} sin(\pi n x); 0 < x < 1,$$

with eigenvalues $\lambda_n = \pi^2 n$. Then the heat semigroup generated by the Laplacian $A = \Delta$

is given by

$$S(t)f = \sum_{n=1}^{\infty} e^{-\lambda_n t}\langle f, e_n\rangle_E e_n$$

for $f \in E$. This is an analytic semigroup where

$$S(-t)g = \sum_{n=1}^{\infty} e^{\lambda_n t}\langle g, e_n\rangle_E e_n$$

for $g \in S(t)E$.

In order to obtain a right-inverse operator-valued process for the Jacobian, we need to assume the following regularity conditions. In the sequel, we denote $S^{-}(t) := S(-t); t \geq 0$ where $S(-t)$ stands the left-inverse linear operator on $S(t)E$.

Assumption B1: Let $\alpha > 1 - H$ be a constant as defined in Assumption A3. For each path $f \in \mathcal{C}_1^{\alpha,0}$,

$$\sup_{i\geq 1} \Big\{ \|S^{-}\nabla G_i(f)S\|_{0,0\to 0} + \|\delta S^{-}\nabla G_i(f)S\|_{\mu,0\to 0} \Big\} < \infty$$

for $\mu + \tilde{\gamma} > 1$ where $\frac{1}{2} < \tilde{\gamma} < H$ satisfies (2.17).

Assumption B2: For each path $f \in \mathcal{C}_1^{\alpha,0}$, $\|S^{-}\nabla F(f)S\|_{0,0\to 0} < \infty$.

In Assumptions B1-B2, we assume

$$\nabla F(w)z \in S(T)E \text{ and } \nabla G_i(w)z \in S(T)E \tag{2.58}$$

for every $w, z \in E$ and $i \geq 1$.

Remark 2.4.3 *Since $S(T)z = S(t)S(T-t)z$ for every $0 \leq t \leq T$ and $z \in E$, then $S(T)E \subset S(t)E_\beta$ for every $0 \leq t \leq T$ and $\beta \geq 0$.*

Remark 2.4.4 *We implicitly assume in Assumptions B1-B2 that $\nabla F(f_t)S(t)x \in S(t)E$ and $\nabla G_i(f_t)S(t)x \in S(t)E$ for every $t \geq 0$, $x \in E$ and $i \geq 1$. This property holds true under (2.58) due to Remark 2.4.3. In this case, taking into account that S is a differ-*

entiable semigroup, then (see e.g Prop 3.12 in [25]) we have $\nabla F(w)z \in \cap_{n=1}^{\infty} dom(A^n)$ *and* $\nabla G_i(w)z \in \cap_{n=1}^{\infty} dom(A^n)$ *for every* $w, z \in E$ *and* $i \geq 1$.

In the sequel, we freeze an initial condition $y \in E_\kappa$. Let us now investigate the existence of an operator-valued process $\mathbf{J}^{+}_{0\to t}(y)$ such that

$$\mathbf{J}_{0\to t}(y)\mathbf{J}^{+}_{0\to t}(y) = \mathrm{Id} \ a.s; 0 \leq t \leq T$$

where Id is the identity operator on $S(t)E$. We start the analysis with the following equation

$$\begin{aligned} U_t(y) = &-\int_0^t \big[\mathrm{Id} + U_r(y)\big] S(-r)\nabla F(X^y_r)S(r)dr \\ &-\sum_{i=1}^{\infty}\sqrt{\lambda_i}\int_0^t \big[\mathrm{Id} + U_r(y)\big] S(-r)\nabla G_i(X^y_r)S(r)d\beta^i_r. \end{aligned} \tag{2.59}$$

Let $\mathcal{C}_1^{\mu,0\to 0}$ be the linear space of $\mathcal{L}(E;E)$-valued functions $r \mapsto f_r$ such that

$$\|f\|_{\mathcal{C}_1^{\mu,0\to 0}} := \|f\|_{0,0\to 0} + \|\delta f\|_{\mu,0\to 0} < \infty.$$

Lemma 2.4.5 *Under Assumptions B1-B2, there exists a unique adapted solution* $U(y)$ *of (2.59) such that* $U(y) \in \mathcal{C}_1^{\mu,0\to 0}$ *a.s for* $\mu + \tilde{\gamma} > 1$ *and* $0 < \mu < \tilde{\gamma}$.

Proof. For a given $g \in \mathcal{W}^{\tilde{\gamma},\delta,\infty}_{\lambda,T}$ and $w \in \mathcal{C}_1^{\alpha,0}$, let us define $\Gamma : \mathcal{C}_1^{\mu,0\to 0} \to \mathcal{C}_1^{\mu,0\to 0}$ by

$$\begin{aligned} \Gamma(U)_t := &-\int_0^t \big[\mathrm{Id} + U_r\big] S(-r)\nabla F(w_r)S(r)dr \\ &-\sum_{i=1}^{\infty}\sqrt{\lambda_i}\int_0^t \big[\mathrm{Id} + U_r\big] S(-r)\nabla G_i(w_r)S(r)dg^i_r. \end{aligned}$$

We claim that Γ is a contraction map on a small interval $[0,T]$. Indeed, for $U, V \in \mathcal{C}_1^{\mu,0\to 0}$, if $q_t = \Gamma(U)_t - \Gamma(V)_t$, then

$$\begin{aligned} q_t = &\int_0^t \big[V_r - U_r\big] S(-r)\nabla F(w_r)S(r)dr \\ &+\sum_{i=1}^{\infty}\sqrt{\lambda_i}\int_0^t \big[V_r - U_r\big] S(-r)\nabla G_i(w_r)S(r)dg^i_r = q^1_t + \sum_{i=1}^{\infty} q^{2,i}_t. \end{aligned}$$

Assumption B2 implies the existence of a constant C_F such that

$$\|q^1\|_{0,0\to0} = \sup_{0\le t\le T}\|q^1_t\| \le \int_0^T \Big\|\big[V_r - U_r\big]S(-r)\nabla F(w_r)S(r)\Big\|_{0\to0}dr$$
$$\le C_F T\|U-V\|_{0,0\to0} \tag{2.60}$$

and

$$\frac{\|q^1_t - q^1_s\|}{|t-s|^\mu} \le C_F\|U-V\|_{0,0\to0}|t-s|^{1-\mu} \le C_F T^{1-\mu}\|U-V\|_{0,0\to0}.$$

Then,

$$\|\delta q^1\|_{\mu,0\to0} \le C_F T^{1-\mu}\|U-V\|_{0,0\to0}. \tag{2.61}$$

Young-Loeve's inequality yields

$$\begin{aligned}\Big\|\sum_{i=1}^\infty (q^{2,i}_t - q^{2,i}_s)\Big\| &\le \frac{1}{2^{\mu+\tilde\gamma}-2}\sum_{i=1}^\infty \big\|\delta[V-U]S^-\nabla G_i(w)S\big\|_{\mu,0\to0}\sqrt{\lambda_i}\big\|g^i\big\|_{\mathcal{W}^{\tilde\gamma,\delta,\infty}_{\lambda,T}}|t-s|^{\mu+\tilde\gamma}\\ &\quad+\sum_{i=1}^\infty \big\|[V_s-U_s]S(-s)\nabla G_i(w_s)S(s)\big\||(\delta g^i_{ts})|\sqrt{\lambda_i}\\ &\le \frac{1}{2^{\mu+\tilde\gamma}-2}\sum_{i=1}^\infty \big\|\delta[V-U]S^-\nabla G_i(w)S\big\|_{\mu,0\to0}\sqrt{\lambda_i}\big\|g^i\big\|_{\mathcal{W}^{\tilde\gamma,\delta,\infty}_{\lambda,T}}|t-s|^{\mu+\tilde\gamma}\\ &\quad+\sum_{i=1}^\infty \big\|[V_s-U_s]S(-s)\nabla G_i(w_s)S(s)\big\|\|g^i\|_{\mathcal{W}^{\tilde\gamma,\delta,\infty}_{\lambda,T}}|t-s|^{\tilde\gamma}\sqrt{\lambda_i}\end{aligned} \tag{2.62}$$

where by linearity,

$$\begin{aligned}\big\|\delta[V-U]S^-\nabla G_i(w)S\big\|_{\mu,0\to0} &\le \|S^-\nabla G_i(w)S\|_{0,0\to0}\|\delta(V-U)\|_{\mu,0\to0}\\ &\quad+\|V-U\|_{0,0\to0}\|\delta S^-\nabla G_i(w.)S\|_{\mu,0\to0}\\ &\le C_G\|V-U\|_{\mathcal{C}^{\mu,0\to0}_1},\end{aligned} \tag{2.63}$$

for a constant C_G coming from Assumption B1. Summing up (2.62) and (2.63), we have

$$\Big\|\sum_{i=1}^\infty \delta q^{2,i}\Big\|_{\mu,0\to0} \le C_G\|V-U\|_{\mathcal{C}^{\mu,0\to0}_1}T^{\tilde\gamma}\|g\|_{\mathcal{W}^{\tilde\gamma,\delta,\infty}_{\lambda,T}} + C_G\|V-U\|_{\mathcal{C}^{\mu,0\to0}_1}\|g\|_{\mathcal{W}^{\tilde\gamma,\delta,\infty}_{\lambda,T}}T^{\tilde\gamma-\mu} \tag{2.64}$$

where we recall $\tilde{\gamma} > \mu$. In addition, (2.62) yields

$$\Big\| \sum_{i=1}^{\infty} q^{2,i} \Big\|_{0,0\to 0} \le C_G \|V-U\|_{\mathcal{C}_1^{\mu,0\to 0}} T^{\mu+\tilde{\gamma}} \|g\|_{\mathcal{W}_{\lambda,T}^{\tilde{\gamma},\delta,\infty}} + C_G \|V-U\|_{\mathcal{C}_1^{\mu,\kappa\to\kappa}} \|g\|_{\mathcal{W}_{\lambda,T}^{\tilde{\gamma},\delta,\infty}} T^{\tilde{\gamma}}. \tag{2.65}$$

Summing up (2.60), (2.61), (2.64) and (2.65), we conclude

$$\|q\|_{\mathcal{C}_1^{\mu,0\to 0}} \le \Big[C_F(T^{1-\mu}+T) + (C_G\|g\|_{\mathcal{W}_{\lambda,T}^{\tilde{\gamma},\delta,\infty}})(2T^{\tilde{\gamma}} + T^{\tilde{\gamma}-\mu} + T^{\mu+\tilde{\gamma}}) \Big] \|U-V\|_{\mathcal{C}_1^{\mu,0\to 0}}. \tag{2.66}$$

where $q = \Gamma(U) - \Gamma(V)$. By making T small in (2.66), we conclude there exists a unique fixed point for Γ on small interval $[0, \bar{T}]$ whose size does not depend on the initial condition. The construction of a global unique solution from the solution in $[0, \bar{T}]$ is standard and it is left to the reader for sake of conciseness. This pathwise argument clearly provides a unique adapted process U realizing (2.59). ■

Now, we set $R_t(y) = U_t(y) + \mathrm{Id}$ and we observe that

$$\begin{aligned} R_t(y) = \mathrm{Id} &- \int_0^t R_s(y) S(-s) \nabla F(X_s^y) S(s) ds \\ &- \sum_{i=1}^{\infty} \sqrt{\lambda_i} \int_0^t R_s(y) S(-s) \nabla G_i(X_s^y) S(s) d\beta_s^i; 0 \le t \le T. \end{aligned} \tag{2.67}$$

We the arrive at the following result which will play a key role in representing the Malliavin matrix.

Proposition 2.4.6 *If Assumptions H1-A1-A2-A3-B1-B2 hold then, for each initial condition $y \in E_k$, the Jacobian $\mathbf{J}_{0\to t}(y)$ admits a right-inverse adapted process $\mathbf{J}^+_{0\to t}(y)$ which satisfies*

$$\begin{aligned} \mathbf{J}^+_{0\to t}(y) = S(-t) &- \int_0^t \mathbf{J}^+_{0\to s}(y) \nabla F(X_s^y) S(s-t) ds \\ &- \sum_{i=1}^{\infty} \sqrt{\lambda_i} \int_0^t \mathbf{J}^+_{0\to s}(y) \nabla G_i(X_s^y) S(s-t) d\beta_s^i; 0 \le t \le T. \end{aligned} \tag{2.68}$$

Proof. The candidate is $\mathbf{J}^+_{0\to t}(y) := R_t(y) S^-(t)$ defined on $S(t)E$. At first, we observe

$$S(s)S(-t) = S(s-t) \text{ on } S(t)E \subset S(t-s)E$$

for every $s < t$. Then, (2.68) is well-defined in view of Assumptions B1-B2. Let us check it is the right-inverse. Let

$$V_t(y) = \int_0^t S(-s)\nabla F(X_s^y)S(s)\big[\mathrm{Id} + V_s(y)\big]ds$$
$$+\sum_{i=1}^{\infty}\sqrt{\lambda_i}\int_0^t S(-s)\nabla G_i(X_s^y)S(s)\big[\mathrm{Id} + V_s(y)\big]d\beta_s^i; 0 \le t \le T. \tag{2.69}$$

By following a similar proof of Lemma 2.4.5, we can safely state there exists a unique adapted solution $V(y)$ of (2.69) such that $V(y) \in \mathcal{C}_1^{\mu,0\to 0}$ a.s for $\mu < \tilde{\gamma}$ and $\mu + \tilde{\gamma} > 1$. Let us define $P_t(y) = V_t(y) + \mathrm{Id}$ and notice that $S(t)S(-s) = S(t-s)$ on $S(s)E$ for every $t > s \ge 0$. Then,

$$P_t(y) = \mathrm{Id} + \int_0^t S(-s)\nabla F(X_s^y)S(s)P_s(y)ds$$
$$+\sum_{i=1}^{\infty}\sqrt{\lambda_i}\int_0^t S(-s)\nabla G_i(X_s^y)S(s)P_s(y)d\beta_s^i\ , \tag{2.70}$$

and therefore $\mathbf{J}_{0\to t}(y) = S(t)P_t(y)$. Equations (2.67), (2.70) and integration by parts in Hilbert spaces yield

$$\big\langle P_t(y)R_t(y)w, w'\big\rangle_E = \big\langle R_t(y)w, P_t^*(y)w'\big\rangle_E = \big\langle w, w'\big\rangle_E + \int_0^t \big\langle dR_s(y)w, P_s^*(y)w'\big\rangle_E$$
$$+\int_0^t \big\langle R_s(y)w, dP_s^*(y)w'\big\rangle_E$$

for each $w, w' \in E$ where P^* is the adjoint. To keep notation simple, we set $I_1 = \int_0^t \big\langle dR_s(y)w, P_s^*(y)w'\big\rangle_E$ and $I_2 = \int_0^t \big\langle R_s(y)w, dP_s^*(y)w'\big\rangle_E$. We observe

$$I_1 = -\int_0^t \big\langle R_s(y)S(-s)\nabla F(X_s^y)S(s)w, P_s^*(y)w'\big\rangle_E ds$$
$$-\sum_{i=1}^{\infty}\sqrt{\lambda_i}\int_0^t \big\langle R_s(y)S(-s)\nabla G_i(X_s^y)S(s)w, P_s^*(y)w'\big\rangle_E d\beta_s^i$$
$$= -\int_0^t \big\langle P_s(y)R_s(y)S(-s)\nabla F(X_s^y)S(s)w, w'\big\rangle_E ds$$
$$-\sum_{i=1}^{\infty}\sqrt{\lambda_i}\int_0^t \big\langle P_s(y)R_s(y)S(-s)\nabla G_i(X_s^y)S(s)w, w'\big\rangle_E d\beta_s^i.$$

In addition, Assumption B1 allows us to represent

$$
\begin{aligned}
I_2 &= \int_0^t \left\langle R_s(y)w, \left(S(-s)\nabla F(X_s^y)S(s)P_s(y)\right)^* w' \right\rangle_E ds \\
&\quad + \sum_{i=1}^{\infty} \sqrt{\lambda_i} \int_0^t \left\langle R_s(y)w, \left(S(-s)\nabla G_i(X_s^y)S(s)P_s(y)\right)^* w' \right\rangle_E d\beta_s^i \\
&= \int_0^t \left\langle S(-s)\nabla F(X_s^y)S(s)P_s(y)R_s(y)w, w' \right\rangle_E ds \\
&\quad + \sum_{i=1}^{\infty} \sqrt{\lambda_i} \int_0^t \left\langle S(-s)\nabla F(X_s^y)S(s)P_s(y)R_s(y)w, w' \right\rangle_E d\beta_s^i.
\end{aligned}
$$

This shows that

$$
\begin{aligned}
P_t(y)R_t(y) = \mathrm{Id} &+ \int_0^t S(-s)\nabla F(X_s^y)S(s)(P_s(y)R_s(y)ds \\
&+ \sum_{i=1}^{\infty} \sqrt{\lambda_i} \int_0^t S(-s)\nabla G_i(X_s^y)S(s)P_s(y)R_s(y)d\beta_s^i \\
&- \int_0^t P_s(y)R_s(y)S(-s)\nabla F(X_s^y)S(s)ds \\
&- \sum_{i=1}^{\infty} \sqrt{\lambda_i} \int_0^t P_s(y)R_s(y)S(-s)\nabla G_i(X_s^y)S(s)d\beta_s^i. \qquad (2.71)
\end{aligned}
$$

We now observe there exists a unique solution of (2.71). To see this, let $Q_t(y) = P_t(y)R_t(y) - \mathrm{Id}$ and from (2.71), we have

$$
\begin{aligned}
Q_t(y) &= \int_0^t S(-s)\nabla F(X_s^y)S(s)Q_s(y)ds + \sum_{i=1}^{\infty} \sqrt{\lambda_i} \int_0^t S(-s)\nabla G_i(X_s^y)S(s)Q_s(y)d\beta_s^i \\
&\quad - \int_0^t Q_s(y)S(-s)\nabla F(X_s^y)S(s)ds - \sum_{i=1}^{\infty} \sqrt{\lambda_i} \int_0^t Q_s(y)S(-s)\nabla G_i(X_s^y)S(s)d\beta_s^i.
\end{aligned}
\qquad (2.72)
$$

The same argument of the proof of Lemma 2.4.5 yields the existence of a unique solution of equation (2.72). This obviously implies that (2.71) admits only one solution. Since Id solves (2.71), we do have $P_t(y)R_t(y) = \mathrm{Id}$ for every $t \in [0,T]$ and we conclude $\mathbf{J}_{0\to t}(y)\mathbf{J}_{0\to t}^{+}(y) = S(t)P_t(y)R_t(y)S^{-}(t) = \mathrm{Id}$ *a.s.* ■

Chapter 3

Existence of densities under Hörmander's bracket condition

Here, we examine the existence of the densities for random variables of the form $\mathcal{T}(X_t^{x_0})$ for a bounded linear operator $\mathcal{T}: E \rightarrow \mathbb{R}^d$ for a given $t \in (0,T]$.

3.1 The Hörmander's bracket condition

Throughout this section, we fix a set of parameters $\kappa, \kappa_0, \tilde{\gamma}, \delta, \lambda$ as described in (2.17). In order to state a Hörmander's bracket condition, we need to work with smooth vector fields $F, G_i; i \geq 1$. Let

$$\text{dom}(A^n) := \{h \in E; h \in \text{dom}(A^{n-1}) \text{ and } A^{n-1}h \in \text{dom}(A)\},$$

$$\|h\|^2_{\text{dom}(A^n)} := \sum_{i=0}^{n} \|A^i h\|^2_E,$$

$$\text{dom}(A^\infty) := \cap_{n=1}^{\infty} \text{dom}(A^n).$$

We observe $\text{dom}(A^\infty)$ is a Frechét space equipped with the family of seminorms $\|\cdot\|_{\text{dom}(A^n)}; n \geq 0$. In the sequel, for each $t \in [0,T]$, we equip $S(t)E$ with the following inner product

$$\langle S(t)x, S(t)y\rangle_{S(t)E} := \langle x, y\rangle_E;\ x, y \in E. \tag{3.1}$$

Notice that this is a well-defined inner product due to the injectivity of the semigroup. One can easily check $S(t)E$ is a separable Hilbert space equipped with the norm associated with (3.1). Moreover, for each $x_0 \in E_\kappa$ and $t \in [0,T]$, $\mathbf{J}^{+}_{0\to t}(x_0) : S(t)E \to E$ admits an adjoint as a bounded linear operator from E to $S(t)E$. Indeed, let $\mathbf{J}^{+,*}_{0\to t}(x_0) : E \to S(t)E$ be the linear operator defined by

$$y \mapsto \mathbf{J}^{+,*}_{0\to t}(x_0)y := S(t)R^*_t(x_0)y.$$

Then,

$$\begin{aligned}\left\langle \mathbf{J}^{+}_{0\to t}(x_0)S(t)x, y\right\rangle_E &= \left\langle R_t(x_0)S(-t)S(t)x, y\right\rangle_E \\ &= \left\langle x, R^*_t(x_0)y\right\rangle_E = \left\langle S(t)x, \mathbf{J}^{+,*}_{0\to t}(x_0)y\right\rangle_{S(t)E}\end{aligned}$$

where $\|\mathbf{J}^{+,*}_{0\to t}(x_0)y\|_{S(t)E} = \|R^*_t(x_0)y\|_E \le \|R^*_t(x_0)\|\|y\|_E$. This proves our claim. We observe $R^*_t(x_0) = \mathrm{Id} + U^*_t(x_0)$ where

$$\begin{aligned}U^*_t(x_0) = &- \int_0^t \left(S(-r)\nabla F(X_r^{x_0})S(r)\right)^*\left(\mathrm{Id} + U^*_r(x_0)\right)dr \\ &- \sum_{i=1}^{\infty}\sqrt{\lambda_i}\int_0^t \left(S(-r)\nabla G_i(X_r^{x_0})S(r)\right)^*\left(\mathrm{Id} + U^*_r(x_0)\right)d\beta^i_r\end{aligned}$$

so that

$$\begin{aligned}R^*_t(x_0) = \mathrm{Id} &- \int_0^t \left(S(-r)\nabla F(X_r^{x_0})S(r)\right)^* R^*_r(x_0)dr \\ &- \sum_{i=1}^{\infty}\sqrt{\lambda_i}\int_0^t \left(S(-r)\nabla G_i(X_r^{x_0})S(r)\right)^* R^*_r(x_0)d\beta^i_r. \end{aligned} \tag{3.2}$$

In other words,

$$\begin{aligned}\mathbf{J}^{+,*}_{0\to t}(x_0) = S(t) &- \int_0^t S(t)\left(S(-r)\nabla F(X_r^{x_0})S(r)\right)^* R^*_r(x_0)dr \\ &- \sum_{i=1}^{\infty}\sqrt{\lambda_i}\int_0^t S(t)\left(S(-r)\nabla G_i(X_r^{x_0})S(r)\right)^* R^*_r(x_0)d\beta^i_r.\end{aligned}$$

Definition 3.1.1 *A vector field V on an open subset $U \subset M$ of a Fréchet space M is a smooth map $V : U \to M$.*

Let us recall the concept of Lie brackets (see e.g [14]) between two vector fields $V_1, V_2 : \mathrm{dom}(A^\infty) \to \mathrm{dom}(A^\infty)$

$$[V_1, V_2](r) := \nabla V_2(r) V_1(r) - \nabla V_1(r) V_2(r)$$

for each $r \in \mathrm{dom}(A^\infty)$. We observe $[V_1, V_2] : \mathrm{dom}(A^\infty) \to \mathrm{dom}(A^\infty)$ is a well-defined vector field whenever V_1, V_2 are vector fields on $\mathrm{dom}(A^\infty)$. Moreover, $\frac{1}{4} < \kappa < 1$ implies $\mathrm{dom}(A) \subset \mathrm{dom}(-A)^\kappa$ so that $\mathrm{dom}(A^\infty) \subset E_\kappa$.

Assumption C1: $G : E \to \mathcal{L}_2\big(U_0; S(T)E\big)$ satisfies

(i) $x \mapsto G_i(x)$ is an $S(T)\mathrm{dom}(A)$-valued continuous mapping for each $i \geq 1$. Moreover,

(ii)
$$\mathbb{E} \int_0^T \|G(X_r^{x_0})\|^2_{\mathcal{L}_2(U_0, S(T)E)} dr < \infty.$$

Assumption C2: $F, G_i : E \to \mathrm{dom}(A^\infty)$ are smooth mappings with bounded derivatives for every $i \geq 1$ with the property that

$$\sup_{\ell \geq 1} \sup_{y \in E} \|\nabla^n G_\ell(y)\|_{(n), E \to \mathrm{dom}(A^m)} < \infty,$$

for every $n, m \geq 1$. There exists a constant C such that

$$\|G_\ell(y)\|_{\mathrm{dom}(A)} \leq C(1 + \|y\|_{\mathrm{dom}(A)}),\ y \in \mathrm{dom}(A)$$

for every $\ell \geq 1$. Moreover, if $V = F,\ G_i; i \geq 1 : \mathrm{dom}(A^k) \to \mathrm{dom}(A^k)$ are C^∞-bounded for every $k \geq 1$, i.e., for a given $k \geq 1$, each derivative $\nabla^\ell V : \mathrm{dom}(A^k) \to \mathcal{L}_{(\ell)}\big(\mathrm{dom}^\ell(A^k); \mathrm{dom}(A^k)\big)$ is a bounded function for every $\ell \geq 1$, where $\mathrm{dom}^\ell(A^k) := \mathrm{dom}(A^k) \times \cdots \times \mathrm{dom}(A^k)$ (ℓ-fold).

Assumption C3: For every $n \geq 1$, $\nabla^n G_p(x) v \in S(T)\mathrm{dom}(A)$ and $\nabla^n F(x) v \in S(T)\mathrm{dom}(A)$ for every $x \in \mathrm{dom}(A)$ and $v \in \mathrm{dom}^n(A)$.

Under Assumption C2, if we assume that $x_0 \in \mathrm{dom}(A^\infty)$, then we can construct a solution process with α-Hölder continuous trajectories in $\mathrm{dom}(A^\infty)$. This is true because the Picard approximation procedure converges in every Hilbert space $\mathrm{dom}(A^m)$, and the topology of $\mathrm{dom}(A^\infty)$ is the projective limit of the ones on $\mathrm{dom}(A^m)$. We summarize this fact into the following remark.

Remark 3.1.2 *Under Assumption C2, for each initial condition $x_0 \in dom(A)$, (2.1) has a unique strong solution. If $x \in dom(A^\infty)$ then we can construct a solution of (2.1) taking values on $dom(A^\infty)$ and such that*

$$\|\delta X^{x_0}\|_{\alpha, dom(A^m)} < \infty$$

for every $m \geq 1$.

Remark 3.1.3 *Assumption C3 plays a rule in constructing the argument towards the existence of densities which requires*

$$[G_0, V](X_t^{x_0}) \in S(t)E$$

in order to belong to the domain of $\mathbf{J}^{+}_{0\to t}(x_0)$ for every $V \in \mathcal{V}_k;\ k \geq 0$ (see (3.11)), where G_0 is the vector field given by (3.10).

The following elementary result is useful.

Lemma 3.1.4 *If $V : E \to dom(A^\infty)$ is a smooth mapping with bounded derivatives, then*

$$\sup_{y\in E} \|\nabla^n V(y)\|_{(n),0\to 0} < \infty.$$

Proof. The n-th Fréchet derivative of V viewed as a map from E to $\mathrm{dom}(A)$ is given by $\nabla^n V : E \to \mathcal{L}_n\big(E^n; \mathrm{dom}(A)\big)$, where

$$\|\nabla^n V(x)(h_1, \ldots, h_n)\|_{\mathrm{dom}(A)} \leq \|\nabla^n V(x)\|_{(n),E\to\mathrm{dom}(A)} \|h_1\| \times \ldots \times \|h_n\|_E$$

Then,

$$\|\nabla^n V(x)(h_1, \ldots, h_n)\|_E \leq \|\nabla^n V(x)(h_1, \ldots, h_n)\|_{\mathrm{dom}(A)}$$

$$\leq \|\nabla^n V(x)\|_{(n),E\to \mathrm{dom}(A)} \|h_1\|_E \times \ldots \times \|h_n\|_E$$
$$\leq \sup_{y\in E} \|\nabla^n V(y)\|_{(n),E\to \mathrm{dom}(A)} \|h_1\|_E \times \ldots \times \|h_n\|_E$$

and hence $\|\nabla^n V(x)\|_{(n),0\to 0} \leq \sup_{y\in E} \|\nabla^n V(y)\|_{(n),E\to \mathrm{dom}(A)}$ for every $x \in E$. $\blacksquare$

Let us now investigate the existence of densities for the SPDE (2.1). We start with some preliminary results.

Lemma 3.1.5 *Under Assumptions H1-A1-A2-A3-B1-B2-C1-C2, for each $x_0 \in dom(A)$, we have*

$$\mathbf{D}_r X_t^{x_0} = \mathbf{J}_{0\to t}(x_0)\mathbf{J}_{0\to r}^{+}(x_0) G(X_r^{x_0}) \ a.s \tag{3.3}$$

for every $r < t$. Therefore,

$$\mathbf{D}_r \mathcal{T}(X_t^{x_0}) = \mathcal{T}\big(\mathbf{J}_{0\to t}(x_0)\mathbf{J}_{0\to r}^{+}(x_0) G(X_r^{x_0})\big) \ a.s \tag{3.4}$$

for every $r < t$.

Proof. On one hand, Remark 3.1.2 and (2.54) yields

$$\mathbf{D}_r X_t^{x_0} = G(X_r^{x_0}) + \int_r^t \nabla F(X_\ell^{x_0}) \mathbf{D}_r X_\ell^{x_0} d\ell + \sum_{i=1}^{\infty} \int_r^t \nabla G_i(X_\ell^{x_0}) \mathbf{D}_r X_\ell^{x_0} d\beta_\ell^i \tag{3.5}$$

for $0 \leq r < t$. On the other hand, Assumption C2 implies that (2.57) has a strong solution for $y = x_0 \in \mathrm{dom}(A)$ and for each $v = G_j(X_r^{x_0})$. Having said that, let us fix $0 \leq r < t$ and a positive integer $j \geq 1$. The fact that $G_j(E) \subset S(T)E$ and Remark 2.4.3 yield

$$\begin{aligned}
&G_j(X_r^{x_0}) + \int_r^t \nabla F(X_\ell^{x_0}) \mathbf{J}_{0\to \ell}(x_0) \mathbf{J}_{0\to r}^{+}(x_0) G_j(X_r^{x_0}) d\ell \\
&\qquad + \sum_{i=1}^{\infty} \int_r^t \nabla G_i(X_\ell^{x_0}) \mathbf{J}_{0\to \ell}(x_0) \mathbf{J}_{0\to r}^{+}(x_0) G_j(X_r^{x_0}) d\beta_\ell^i \\
&= G_j(X_r^{x_0}) + \Bigg(\int_r^t \nabla F(X_\ell^{x_0}) \mathbf{J}_{0\to \ell}(x_0) d\ell \\
&\qquad + \sum_{i=1}^{\infty} \int_r^t \nabla G_i(X_\ell^{x_0}) \mathbf{J}_{0\to \ell}(x_0) d\beta_\ell^i \Bigg) \mathbf{J}_{0\to r}^{+}(x_0) G_j(X_r^{x_0}) \\
&= G_j(X_r^{x_0}) + \Big(\mathbf{J}_{0\to t}(x_0) - \mathbf{J}_{0\to r}(x_0) \Big) \mathbf{J}_{0\to r}^{+}(x_0) G_j(X_r^{x_0})
\end{aligned}$$

$$= \mathbf{J}_{0\to t}(x_0)\mathbf{J}^{+}_{0\to r}(x_0)G_j(X^{x_0}_r)\ a.s.$$

By invoking (2.55), (2.49), Lemma 2.3.4, (3.5) and Assumption C1(i), we know that both $(r,t)\mapsto \mathbf{D}_r X^{x_0}_t$ and $(r,t)\mapsto \mathbf{J}_{0\to t}(x_0)\mathbf{J}^{+}_{0\to r}(x_0)G_j(X^{x_0}_r)$ are jointly continuous a.s on the simplex $\{(r,t); 0\le r\le t\le T\}$. This fact combined with the uniqueness of the SPDE solution of (3.5) (for each fixed r) implies that they are indistinguishable

$$\big(\mathbf{D}_{\cdot}X^{x_0}_{\cdot}\big)(\sqrt{\lambda_j}e_j) = \mathbf{J}_{0\to\cdot}(x_0)\mathbf{J}^{+}_{0\to\cdot}(x_0)G_j(X^{x_0}_{\cdot})\ a.s$$

for each $j\ge 1$. Assumption C1 (ii) implies

$$r\mapsto \mathbf{J}_{0\to t}(x_0)\mathbf{J}^{+}_{0\to r}(x_0)G(X^{x_0}_r)\in \mathcal{L}_2\big(U_0; L^{\frac{1}{H}}([0,T];E)\big)\subset \mathcal{L}_2(U_0;\mathcal{H}\otimes E)\ a.s$$

for every $t\in(0,T]$. Summing up the above arguments, we shall conclude (3.3) holds true. The chain rule yields representation (3.4). ■

In what follows, let us denote

$$\gamma_t := \Big(\big\langle \mathbf{D}\mathcal{T}_i(X^{x_0}_t), \mathbf{D}\mathcal{T}_j(X^{x_0}_t)\big\rangle_{\mathcal{L}_2(U_0;\mathcal{H})}\Big)_{1\le i,j\le d} \tag{3.6}$$

where $\mathcal{T} = (\mathcal{T}_1,\dots\mathcal{T}_d) : E\to\mathbb{R}^d$. In order to investigate non-degeneracy of the Malliavin derivative, it is convenient to work with a *reduced* Malliavin operator. Let us define the self-adjoint linear operator $\mathcal{C}_t : E\to E$ by the following quadratic form

$$\begin{aligned}\langle \mathcal{C}_t y, y\rangle_E &:= \alpha_H \sum_{\ell=1}^{\infty}\int_0^t\int_0^t \big\langle \mathbf{J}^{+}_{0\to u}(x_0)G_\ell(X^{x_0}_u), y\big\rangle_E \big\langle \mathbf{J}^{+}_{0\to v}(x_0)G_\ell(X^{x_0}_v), y\big\rangle_E \\ &\qquad |u-v|^{2H-2}dudv \\ &= \sum_{\ell=1}^{\infty}\Big\|\big\langle \mathbf{J}^{+}_{0\to\cdot}(x_0)G_\ell(X^{x_0}_{\cdot}), y\big\rangle_E\Big\|^2_{\mathcal{H}} = \sum_{\ell=1}^{\infty}\Big\|\big\langle G_\ell(X^{x_0}_{\cdot}), \mathbf{J}^{+,*}_{0\to\cdot}(x_0)y\big\rangle_{S(\cdot)E}\Big\|^2_{\mathcal{H}}\end{aligned} \tag{3.7}$$

for $y\in E$ and $0<t\le T$. In (3.7), the norm in $\mathcal{H}$ is computed over $[0,t]$. We observe $\mathcal{C}_t$ is a well-defined bounded linear operator due to Assumption C1 (ii) and $\frac{1}{H}<2$.

Lemma 3.1.6 *Under Assumptions H1-A2-A2-A3-B1-B2-C1-C2, we have for each $x_0\in dom(A)$,*

$$\gamma_t = \big(\mathcal{T}\circ \mathbf{J}_{0\to t}(x_0)\big)\ \mathcal{C}_t\ \big(\mathcal{T}\circ\mathbf{J}_{0\to t}(x_0)\big)^{*}.$$

Proof. In one hand, Lemma 3.1.5 and (1.7) yield

$$
\begin{aligned}
\gamma_t^{ij} &= \langle \mathbf{D}\mathcal{T}_i(X_t^{x_0}), \mathbf{D}\mathcal{T}_j(X_t^{x_0})\rangle_{\mathcal{L}_2(U_0;\mathcal{H})} \\
&= \sum_{\ell=1}^{\infty} \Big\langle \mathbf{D}\mathcal{T}_i(X_t^{x_0})(\sqrt{\lambda_\ell}e_\ell), \mathbf{D}\mathcal{T}_j(X_t^{x_0})(\sqrt{\lambda_\ell}e_\ell)\Big\rangle_{\mathcal{H}} \\
&= \sum_{\ell=1}^{\infty} \alpha_H \int_0^t \int_0^t \mathbf{D}_u\mathcal{T}_i(X_t^{x_0})(\sqrt{\lambda_\ell}e_\ell)\mathbf{D}_v\mathcal{T}_j(X_t^{x_0})(\sqrt{\lambda_\ell}e_\ell)|u-v|^{2H-2}dudv \\
&= \sum_{\ell=1}^{\infty} \alpha_H \int_{[0,t]^2} \mathcal{T}_i\big(\mathbf{J}_{0\to t}(x_0)\mathbf{J}_{0\to u}^{+}(x_0)G_\ell(X_u^{x_0})\big)\mathcal{T}_j\big(\mathbf{J}_{0\to t}(x_0)\mathbf{J}_{0\to r}^{+}(x_0)G_\ell(X_r^{x_0})\big) \\
&\qquad\qquad |u-v|^{2H-2}dudv.
\end{aligned} \tag{3.8}
$$

On the other hand, if $(b_i)_{i=1}^d$ is the canonical orthonormal basis of $\mathbb{R}^d$, we observe

$$
\Big\langle \big(\mathcal{T}\circ\mathbf{J}_{0\to t}(x_0)\big)\,\mathcal{C}_t\,\big(\mathcal{T}\circ\mathbf{J}_{0\to t}(x_0)\big)^* b_i, b_j\Big\rangle_{\mathbb{R}^d} = \Big\langle \mathcal{C}_t\,\big(\mathcal{T}\circ\mathbf{J}_{0\to t}(x_0)\big)^* b_i, \big(\mathcal{T}\circ\mathbf{J}_{0\to t}(x_0)\big)^* b_j\Big\rangle_E. \tag{3.9}
$$

Now, use the definition (3.7) and the polarization formula

$$
\langle \mathcal{C}_t x, y\rangle_E = \frac{1}{2}\Big\{\langle \mathcal{C}_t(x+y),(x+y)\rangle_E - \langle \mathcal{C}_t x, x\rangle_E - \langle \mathcal{C}_t y, y\rangle_E\Big\}
$$

to conclude (3.9) equals to (3.8). ■

Let us now investigate the existence of densities. At first, we recall the following result which is an immediate consequence of the classical result in Malliavin calculus. See e.g Th 2.1.1 in [31].

Lemma 3.1.7 *Assume* $\mathcal{T}_i(X_t^{x_0}) \in \mathbb{D}_{loc}^{1,2}(\mathbb{R})$ *for* $i = 1, \ldots, d$ *and* γ_t *is invertible a.s for an initial condition* $x_0 \in dom(A)$ *and* $t \in (0,T]$. *Then, the law of* $\mathcal{T}(X_t^{x_0})$ *has a density w.r.t Lebesgue measure in* $\mathbb{R}^d$.

We can now turn to our first main result of this paper. Let us define

$$
G_0(x) := Ax + F(x); x \in \mathrm{dom}(A^\infty). \tag{3.10}
$$

Given the SPDE (2.1), define a collection of vector fields $\mathcal{V}_k$ by

$$
\mathcal{V}_0 = \{G_i; i \geq 1\}, \quad \mathcal{V}_{k+1} := \mathcal{V}_k \cup \big\{[G_j, V]; V \in \mathcal{V}_k \text{ and } j \geq 0\big\}. \tag{3.11}
$$

We also define the vector spaces $\mathcal{V}_k(x_0) := \text{span}\{V(x_0); V \in \mathcal{V}_k\}$ and we set

$$\mathcal{D}(x_0) := \cup_{k\geq 1}\mathcal{V}_k(x)$$

for each $x_0 \in \text{dom}(A^\infty)$.

Note that under Assumption C2, all the Lie brackets in (3.11) are well-defined as vector fields $\text{dom}(A^\infty) \to \text{dom}(A^\infty)$.

Proposition 3.1.8 *If Assumptions H1-A1-A2-A3-B1-B2-C1-C2-C3 hold true, then for each $x_0 \in dom(A^\infty)$, we have*

$$\begin{aligned}\mathbf{J}^{+}_{0\to t}(x_0)V(X^{x_0}_t) = V(x_0) + \int_0^t \mathbf{J}^{+}_{0\to s}(x_0)[G_0, V](X^{x_0}_s)ds \\ + \sum_{\ell=1}^{\infty}\sqrt{\lambda_\ell}\int_0^t \mathbf{J}^{+}_{0\to s}(x_0)[G_\ell, V](X^{x_0}_s)d\beta^\ell_s;\ 0 \leq t \leq T,\end{aligned} \tag{3.12}$$

where $V \in \mathcal{V}_n$ for $n = 0, 1, 2, \ldots$.

Proof. At first, we take $V \in \mathcal{V}_0$. Assumptions C2-C3 yield $V(X^{x_0}_\cdot) \in S(T)E_\kappa$, $[G_0, V](X^{x_0}_\cdot) \in S(T)E$, and $[G_\ell, V](X^{x_0}_\cdot) \in S(T)E$ a.s. Moreover, change of variables for Young integrals yields

$$V(X^{x_0}_t) = V(x_0) + \int_0^t \nabla V(X^{x_0}_s)G_0(X^{x_0}_s)ds + \sum_{\ell=1}^{\infty}\sqrt{\lambda_\ell}\int_0^t \nabla V(X^{x_0}_s)G_\ell(X^{x_0}_s)d\beta^\ell_s \tag{3.13}$$

where $G_0(X^{x_0}_s) = A(X^{x_0}_s) + F(X^{x_0}_s); 0 \leq s \leq T$. We observe Young-Loeve's inequality and A1-A2-A3 allow us to state the Young integral in (3.13) is well-defined. Recall the Lie bracket $[G_0, V](X^{x_0}_s) = \nabla V(X^{x_0}_s)G_0(X^{x_0}_s) - \nabla G_0(X^{x_0}_s)V(X^{x_0}_s)$, so that we can actually rewrite

$$\begin{aligned}V(X^{x_0}_t) = V(x_0) + \int_0^t \Big(\nabla G_0(X^{x_0}_s)V(X^{x_0}_s) + [G_0, V](X^{x_0}_s)\Big)ds \\ + \sum_{\ell=1}^{\infty}\sqrt{\lambda_\ell}\int_0^t \nabla V(X^{x_0}_s)G_\ell(X^{x_0}_s)d\beta^\ell_s\end{aligned}$$

where $\nabla G_0(X^{x_0}_s)V(X^{x_0}_s) = A(V(X^{x_0}_s)) + \nabla F(X^{x_0}_s)V(X^{x_0}_s); 0 \leq s \leq T$. This implies that $V(X^{x_0})$ can be written as the mild solution of

$$V(X_t^{x_0}) = S(t)V(x_0) + \int_0^t S(t-s)\big(\nabla F(X_s^{x_0})V(X_s^{x_0}) + [G_0,V](X_s^{x_0})\big)ds + \sum_{\ell=1}^{\infty}\sqrt{\lambda_\ell}\int_0^t S(t-s)\nabla V(X_s^{x_0})G_\ell(X_s^{x_0})d\beta_s^\ell$$

so that

$$S(-t)V(X_t^{x_0}) = V(x_0) + \int_0^t S(-s)\big(\nabla F(X_s^{x_0})V(X_s^{x_0}) + [G_0,V](X_s^{x_0})\big)ds + \sum_{\ell=1}^{\infty}\sqrt{\lambda_\ell}\int_0^t S(-s)\nabla V(X_s^{x_0})G_\ell(X_s^{x_0})d\beta_s^\ell; 0\le t\le T. \tag{3.14}$$

The adjoint operator $\mathbf{J}^{+,*}_{0\to t}(x_0)$ yields

$$\langle \mathbf{J}^{+}_{0\to t}(x_0)V(X_t^{x_0}), y\rangle_E = \langle V(X_t^{x_0}), \mathbf{J}^{+,*}_{0\to t}(x_0)y\rangle_{S(t)E} = \langle S(-t)V(X_t^{x_0}), R_t^*(x_0)y\rangle_E$$

for a given $y\in E$. Hence, integration by parts yields

$$\langle \mathbf{J}^{+}_{0\to t}(x_0)V(X_t^{x_0}), y\rangle_E = \langle V(x_0), y\rangle_E + \int_0^t \langle dS(-s)V(X_s^{x_0}), R_s^*(x_0)y\rangle_E + \int_0^t \langle S(-s)V(X_s^{x_0}), dR_s^*(x_0)y\rangle_E; 0\le t\le T.$$

By combining (3.14) and (3.2), we conclude that (3.12) holds true for $V\in\mathcal{V}_0$. Now, we take $V=[G_i,G_p]$ or $V=[G_0,G_p]$ for $p,i=1,2,\ldots$. In this case, C2-C3 yield $V(X_\cdot^{x_0})\in S(T)E$, $[G_0,V](X_\cdot^{x_0})\in S(T)E$, and $[G_\ell,V](X_\cdot^{x_0})\in S(T)E$. From the above argument for vector fields in $\mathcal{V}_0$, we learn that in order to prove (3.12), it is sufficient to ensure that the Young integral in the right-hand side of (3.13) is well-defined, i.e.,

$$\sup_{\ell\ge 1}\|\delta\nabla V(X_\cdot^{x_0})G_\ell(X_\cdot^{x_0})\|_{\alpha,0} < \infty \ a.s. \tag{3.15}$$

At first, we observe if $W:\mathrm{dom}(A^\infty)\to\mathrm{dom}(A^\infty)$ is smooth, then

$$\nabla[G_0,W](x)(h) = \nabla^2W(x)(h,Ax) + \nabla W(x)A(h) + \nabla^2W(x)(h,F(x)) + \nabla W(x)\nabla F(x)h - A\nabla W(x)h$$

$$- \nabla^2 F(x)(h, W(x)) - \nabla F(x)\nabla W(x)h; h \in \mathrm{dom}(A^\infty), \tag{3.16}$$

and

$$\begin{aligned}\nabla[G_p, W](x)(h) &= \nabla^2 W(x)(h, G_p(x)) + \nabla W(x)\nabla G_p(x)(h) \\ &\quad - \nabla^2 G_p(x)(h, W(x)) - \nabla G_p(x)\nabla W(x)(h)\end{aligned} \tag{3.17}$$

for $h \in \mathrm{dom}(A^\infty)$ and $p \geq 1$. If $V = [G_0, G_p]$, we observe

$$\begin{aligned}\nabla V(X_t^{x_0})G_\ell(X_t^{x_0}) &= -A\nabla G_p(X_t^{x_0})G_\ell(X_t^{x_0}) - \nabla^2 F(X_t^{x_0})(G_p(X_t^{x_0}), G_\ell(X_t^{x_0})) \\ &\quad - \nabla F(X_t^{x_0})\nabla G_p(X_t^{x_0})G_\ell(X_t^{x_0}) + \nabla^2 G_p(X_t^{x_0})(AX_t^{x_0}, G_\ell(X_t^{x_0})) \\ &\quad + \nabla G_p(X_t^{x_0})AG_\ell(X_t^{x_0}) + \nabla^2 G_p(X_t^{x_0})(F(X_t^{x_0}), G_\ell(X_t^{x_0})) \\ &\quad + \nabla G_p(X_t^{x_0})\nabla F(X_t^{x_0})G_\ell(X_t^{x_0}) \\ &=: \sum_{i=1}^{7} I_{i,p,\ell}(t).\end{aligned}$$

Since $F, G_i : E \to \mathrm{dom}(A^\infty)$ has bounded derivatives of all orders (by C2), we shall use Lemma 3.1.4 to get

$$\begin{aligned}\|I_{1,p,\ell}(t) - I_{1,p,\ell}(s)\|_E &\leq \|\nabla G_p(X_t^{x_0})G_\ell(X_t^{x_0}) - \nabla G_p(X_t^{x_0})G_\ell(X_s^{x_0})\|_{\mathrm{dom}(A)} \\ &\quad + \|A\nabla G_p(X_t^{x_0})G_\ell(X_s^{x_0}) - A\nabla G_p(X_s^{x_0})G_\ell(X_s^{x_0})\|_E \\ &\leq \sup_{y\in E} \|\nabla G_p(y)\|_{E\to\mathrm{dom}(A)}\|G_\ell(X_t^{x_0}) - G_\ell(X_s^{x_0})\|_E \\ &\quad + \|G_p(X_t^{x_0}) - G_p(X_s^{x_0})\|_{E\to\mathrm{dom}(A)}\|G_\ell(X_s^{x_0})\|_E \\ &\leq C\sup_{y\in E} \|\nabla G_p(y)\|_{E\to\mathrm{dom}(A)}\|\delta X_{ts}^{x_0}\|_E \\ &\quad + C\sup_{y\in E} \|\nabla G_p(y)\|_{E\to\mathrm{dom}(A)}\|\delta X_{ts}^{x_0}\|_E(1 + \|X^{x_0}\|_{0,0}),\end{aligned}$$

$$\begin{aligned}\|I_{2,p,\ell}(t) - I_{2,p,\ell}(s)\|_E &\leq \|\nabla^2 F(X_t^{x_0})(G_p(X_t^{x_0}), G_\ell(X_t^{x_0})) - \nabla^2 F(X_s^{x_0})(G_p(X_t^{x_0}), G_\ell(X_t^{x_0}))\|_E \\ &\quad + \|\nabla^2 F(X_s^{x_0})(G_p(X_t^{x_0}), G_\ell(X_t^{x_0})) - \nabla^2 F(X_s^{x_0})(G_p(X_s^{x_0}), G_\ell(X_t^{x_0}))\|_E \\ &\quad + \|\nabla^2 F(X_s^{x_0})(G_p(X_s^{x_0}), G_\ell(X_t^{x_0})) - \nabla^2 F(X_s^{x_0})(G_p(X_s^{x_0}), G_\ell(X_s^{x_0}))\|_E \\ &\leq \|\nabla^2 F(X_t^{x_0}) - \nabla^2 F(X_s^{x_0})\|_{(2),0\to 0}\|G_p(X_t^{x_0})\|_E\|G_\ell(X_t^{x_0})\|_E\end{aligned}$$

$$
\begin{aligned}
&+ \|\nabla^2 F(X_s^{x_0})\|_{(2),0\to 0}\|G_p(X_t^{x_0}) - G_p(X_s^{x_0})\|_E\|G_\ell(X_t^{x_0})\|_E \\
&+ \|\nabla^2 F(X_s^{x_0})\|_{(2),0\to 0}\|G_\ell(X_t^{x_0}) - G_\ell(X_s^{x_0})\|_E\|G_p(X_s^{x_0})\|_E \\
&\qquad \le C\|\delta X_{ts}^{x_0}\|_E(1+\|X^{x_0}\|_{0,0})^2 \\
&+ 2C\sup_{y\in E}\|\nabla^2 F(y)\|_{(2),0\to 0}(1+\|X^{x_0}\|_{0,0})\|\delta X_{ts}^{x_0}\|_E,
\end{aligned}
$$

$$
\begin{aligned}
\|I_{3,p,\ell}(t) - I_{3,p,\ell}(s)\|_E &\le \|\nabla F(X_t^{x_0})\nabla G_p(X_t^{x_0})G_\ell(X_t^{x_0}) - \nabla F(X_s^{x_0})\nabla G_p(X_t^{x_0})G_\ell(X_t^{x_0})\|_E \\
&+ \|\nabla F(X_s^{x_0})\nabla G_p(X_t^{x_0})G_\ell(X_t^{x_0}) - \nabla F(X_s^{x_0})\nabla G_p(X_s^{x_0})G_\ell(X_t^{x_0})\|_E \\
&+ \|\nabla F(X_s^{x_0})\nabla G_p(X_s^{x_0})G_\ell(X_t^{x_0}) - \nabla F(X_s^{x_0})\nabla G_p(X_s^{x_0})G_\ell(X_s^{x_0})\|_E \\
&\qquad \le C\sup_{y\in E}|\nabla^2 F(y)|_{(2),0\to 0}\|\delta X_{ts}^{x_0}\|_E(1+\|X^{x_0}\|_{0,0}) \\
&+ C\sup_{y\in E}\|\nabla F(y)\|\sup_{y\in E}\|\nabla^2 G_p(y)\|_{(2),0\to 0}\|\delta X_{ts}^{x_0}\|_E(1+\|X^{x_0}\|_{0,0}),
\end{aligned}
$$

$$
\begin{aligned}
\|I_{4,p,\ell}(t) - I_{4,p,\ell}(s)\|_E &\le \|\nabla^2 G_p(X_t^{x_0})(AX_t^{x_0}, G_\ell(X_t^{x_0})) - \nabla^2 G_p(X_s^{x_0})(AX_t^{x_0}, G_\ell(X_t^{x_0}))\|_E \\
&+ \|\nabla^2 G_p(X_s^{x_0})(AX_t^{x_0}, G_\ell(X_t^{x_0})) - \nabla^2 G_p(X_s^{x_0})(AX_s^{x_0}, G_\ell(X_t^{x_0}))\|_E \\
&+ \|\nabla^2 G_p(X_s^{x_0})(AX_s^{x_0}, G_\ell(X_t^{x_0})) - \nabla^2 G_p(X_s^{x_0})(AX_s^{x_0}, G_\ell(X_s^{x_0}))\|_E \\
&\qquad \le \sup_{y\in E}\|\nabla^3 G_p(y)\|_{(3),0\to 0}\|\delta X_{ts}^{x_0}\|_E\|X^{x_0}\|_{0,\mathrm{dom}(A)}(1+\|X^{x_0}\|_{0,0}) \\
&+ \sup_{y\in E}\|\nabla^2 G_p(y)\|_{(2),0\to 0}\|\delta X_{ts}^{x_0}\|_{\mathrm{dom}(A)}(1+\|X^{x_0}\|_{0,0}) \\
&+ \sup_{y\in E}\|\nabla^2 G_p(y)\|_{(2),0\to 0}\|\delta X_{ts}^{x_0}\|_E(1+\|X^{x_0}\|_{0,\mathrm{dom}(A)}),
\end{aligned}
$$

$$
\begin{aligned}
\|I_{5,p,\ell}(t) - I_{5,p,\ell}(t)\|_E &\le \|\nabla G_p(X_t^{x_0})AG_\ell(X_t^{x_0}) - \nabla G_p(X_s^{x_0})AG_\ell(X_t^{x_0})\|_E \\
&+ \|\nabla G_p(X_s^{x_0})AG_\ell(X_t^{x_0}) - \nabla G_p(X_s^{x_0})AG_\ell(X_s^{x_0})\|_E \\
&\qquad \le C\sup_{y\in E}\|\nabla^2 G_p(y)\|_{(2),0\to 0}\|\delta X_{ts}^{x_0}\|_E\|G_\ell(X_t^{x_0})\|_{\mathrm{dom}(A)} \\
&+ C\sup_{y\in E}\|\nabla G_p(y)\|\sup_{y\in E}\|\nabla G_\ell(y)\|_{E\to\mathrm{dom}(A)}\|\delta X_{ts}^{x_0}\|_E \\
&\qquad \le C\sup_{y\in E}\|\nabla^2 G_p(y)\|_{(2),0\to 0}\|\delta X_{ts}^{x_0}\|_E(1+\|X^{x_0}\|_{0,\mathrm{dom}(A)}) \\
&+ C\sup_{y\in E}\|\nabla G_p(y)\|\sup_{y\in E}\|\nabla G_\ell(y)\|_{E\to\mathrm{dom}(A)}\|\delta X_{ts}^{x_0}\|_E,
\end{aligned}
$$

$$
\begin{aligned}
\|I_{6,p,\ell}(t) - I_{6,p,\ell}(s)\|_E &\le \|\nabla^2 G_p(X_t^{x_0})(F(X_t^{x_0}), G_\ell(X_t^{x_0})) - \nabla^2 G_p(X_s^{x_0})(F(X_t^{x_0}), G_\ell(X_t^{x_0}))\|_E \\
&+ \|\nabla^2 G_p(X_s^{x_0})(F(X_t^{x_0}), G_\ell(X_t^{x_0})) - \nabla^2 G_p(X_s^{x_0})(F(X_s^{x_0}), G_\ell(X_s^{x_0}))\|_E \\
&\le C \sup_{y\in E} \|\nabla^3 G_p(y)\|_{(3),0\to 0} \|\delta X_{ts}^{x_0}\|_E (1 + \|X^{x_0}\|_{0,0})^2 \\
&+ C \sup_{y\in E} |\nabla^2 G_p(y)\|_{(2),0\to 0} \|\delta X_{ts}^{x_0}\|_E^2,
\end{aligned}
$$

$$
\begin{aligned}
\|I_{7,p,\ell}(t) - I_{7,p,\ell}(s)\|_E &\le \|\nabla G_p(X_t^{x_0}) \nabla F(X_t^{x_0}) G_\ell(X_t^{x_0}) - \nabla G_p(X_s^{x_0}) \nabla F(X_t^{x_0}) G_\ell(X_t^{x_0})\|_E \\
&+ \|\nabla G_p(X_s^{x_0}) \nabla F(X_t^{x_0}) G_\ell(X_t^{x_0}) - \nabla G_p(X_s^{x_0}) \nabla F(X_s^{x_0}) G_\ell(X_s^{x_0})\|_E \\
&\le C \sup_{y\in E} \|\nabla^2 G_p(y)\|_{(2),0\to 0} \|\delta X_{ts}^{x_0}\|_E \sup_{y\in E} \|\nabla F(y)\| (1 + \|X^{x_0}\|_{0,0}) \\
&+ C \sup_{y\in E} \|\nabla^2 F(y)\|_{(2),0\to 0} \|\delta X_{ts}^{x_0}\|_E (1 + \|X^{x_0}\|_{0,0}) \\
&+ C \sup_{y\in E} \|\nabla F(y)\| \|\delta X_{ts}^{x_0}\|_E.
\end{aligned}
$$

This shows that (3.15) holds true for vector fields of the $[G_0, G_p]; p = 1, 2, \dots$. A similar computation also shows (3.15) for vector fields of the form $[G_j, G_p]; j, p = 1, 2, \dots$. This shows that (3.12) holds for vectors fields $V \in \mathcal{V}_1$. By using (3.16) and (3.17) and iterating the argument, we recover (3.15) for vector fields $V \in \mathcal{V}_n; n \ge 0$ and hence we conclude the proof. ■

3.2 Doob-Meyer-type decomposition

Let us now turn our attention to a Doob-Meyer decomposition in the framework of integral equations involving a trace-class FBM. This will play a key step in the proof of the existence of density of Theorem 0.0.4. We recall the parameters $\tilde{\gamma}, \delta, \lambda$ are fixed according to (2.17). In a rather general situation, Friz and Schekar [16] have developed the concept of *true roughness* which plays a key role in determining the uniqueness of the Gubinelli's derivative in rough path theory. For sake of completeness, we recall the following concepts borrowed from [17] and adapted to our infinite-dimensional setting.

For a given $g \in \mathcal{W}^{\tilde{\gamma},\delta,\infty}_{\lambda,T}$, we write

$$\mathbf{G}_t = \sum_{j=1}^{\infty} \sqrt{\lambda_j} e_j g^j_t; 0 \leq t \leq T.$$

Of course, $\mathbf{G} \in \mathcal{C}^{\tilde{\gamma}}_1(U)$ for every $g \in \mathcal{W}^{\tilde{\gamma},\delta,\infty}_{\lambda,T}$.

Definition 3.2.1 *Given a path $\mathbf{G} \in \mathcal{C}^{\tilde{\gamma}}_1(U)$, we say that $Y \in \mathcal{C}^{\tilde{\gamma}}_1(\mathbb{R})$ is controlled by $\mathbf{G}$ if there exists $Y' \in \mathcal{C}^{\tilde{\gamma}}_1(U^*)$ so that the remainder term given implicitly through the relation*

$$\delta Y_{ts} = Y'_s \delta \mathbf{G}_{ts} + R^Y_{ts}, s < t$$

satisfies $\|R^Y\|_{2\tilde{\gamma}} < \infty$.

In our context, we restrict the analysis to the following class of derivatives. Let $C^{\beta,\infty}_1$ be the set of all sequences of real-valued functions on $[0,T]$, $(f_i)^{\infty}_{i=1}$ such that $\sup_{i\geq 1} \|\delta f_i\|_{\beta} < \infty$ for $0 < \beta \leq 1$. Let $Y' : [0,T] \to U^*$ be a U^*-valued path such that $(Y'^i)^{\infty}_{i=1} \in C^{\tilde{\gamma},\infty}_1$ where $Y'^i = Y'(e_i); i \geq 1$. We then observe if

$$\delta Y_{ts} = Y'_s \delta \mathbf{G}_{ts} + R^Y_{ts}, s < t \tag{3.18}$$

then, $\delta Y_{ts} = \int_s^t Y'_r d\mathbf{G}_r = \sum_{i=1}^{\infty} \sqrt{\lambda_i} \int_s^t Y'^i_r dg^i_r$ is a well defined Young integral, where the remainder is characterized by

$$R^Y_{ts} = \sum_{j=1}^{\infty} \sqrt{\lambda_j} \int_s^t \left(Y'^j_r - Y'^j_s\right) dg^j_r$$

and $\|R^Z\|_{2\tilde{\gamma}} < \infty$ due to Young-Loeve inequality. The class of all pairs (Y, Y') of the form (3.18) constitutes a subset of controlled paths which we denote it by $\mathscr{D}^{2\tilde{\gamma}}_{\mathbf{G}}([0,T]; U^*)$. Next, we recall the following concept of truly rough.

Definition 3.2.2 *For a fixed $s \in (0,T]$, we call a $\frac{1}{\tilde{\gamma}}$-rough path $\mathbf{G} : [0,T] \to U$, "**rough at time** s" if*

$$\forall v^* \in U^* \ \textit{non-null} \ : \ \limsup_{t \downarrow s} \frac{|\langle v^*, \delta \mathbf{G}_{ts} \rangle|}{|t-s|^{2\tilde{\gamma}}} = +\infty.$$

*If $\mathbf{G}$ is rough on some dense subset of $[0,T]$, then we call it **truly rough**.*

Lemma 3.2.3 *The U-valued trace-class FBM given by (1.9) is truly rough.*

Proof. The proof follows the same lines of Example 2 in [16] together with the law of iterated logarithm for Gaussian processes as described by Th 7.2.15 in [28]. We left the details to the reader. ■

The following result is given by Th. 6.5 in Friz and Hairer [15].

Theorem 3.2.4 *Assume that* $\mathbf{G}$ *is a truly rough path. Let* (Y, Y') *and* $(\tilde{Y}, \tilde{Y}')$ *be controlled paths in* $\mathscr{D}_{\mathbf{G}}^{2\bar{\gamma}}([0,T];U^*)$ *and let* $N, \tilde{N}$ *be a pair of real-valued continuous paths. Assume that*

$$\int_0^{\cdot} Y d\mathbf{G} + \int_0^{\cdot} N dt = \int_0^{\cdot} \tilde{Y} d\mathbf{G} + \int_0^{\cdot} \tilde{N} dt$$

on $[0,T]$*. Then,* $\big(Y, Y'\big) = \big(\tilde{Y}, \tilde{Y}'\big)$ *and* $N = \tilde{N}$ *on* $[0,T]$*.*

3.3 Main Result: Proof of Theorem 0.0.4

We are now in position to proof the main result of this thesis.

Proof. Fix $x_0 \in \mathrm{dom}(A^\infty) \subset E$ and $t \in (0,T]$. By Lemma 3.1.6,

$$\gamma_t = \big(\mathcal{T} \circ \mathbf{J}_{0\to t}(x_0)\big)\ \mathcal{C}_t\ \big(\mathcal{T} \circ \mathbf{J}_{0\to t}(x_0)\big)^*$$

so that it is sufficient to prove that γ_t is positive definite a.s. For this purpose, we start by noticing that

$$\langle \gamma_t b, b\rangle_{\mathbb{R}^d} = \Big\langle \mathcal{C}_t\big(\mathcal{T} \circ \mathbf{J}_{0\to t}(x_0)\big)^* b, \big(\mathcal{T} \circ \mathbf{J}_{0\to t}(x_0)\big)^* b\Big\rangle_E;\ b \in \mathbb{R}^d.$$

We observe that $\big(\mathcal{T} \circ \mathbf{J}_{0\to t}(x_0)\big)^*$ is one-to-one. By assumption, $\mathrm{Ker}\mathcal{T}^* = \{\mathbf{0}\}$ and clearly $\mathrm{Ker}\mathbf{J}^*_{0\to t}(x_0) = \{\mathbf{0}\}$. Indeed, if $y \in \ker\mathbf{J}^*_{0\to t}(x_0)$, then for every $x \in E$

$$\begin{aligned}\langle y, S(t)x\rangle_E &= \langle y, \mathbf{J}_{0\to t}(x_0)\mathbf{J}^+_{0\to t}(x_0)S(t)x\rangle_E \\ &= \langle \mathbf{J}^{+,*}_{0\to t}(x_0)\mathbf{J}^*_{0\to t}(x_0)y, S(t)x\rangle_{S(t)E} = 0.\end{aligned}$$

This implies $y \in \big(S(t)E\big)^\perp = \{\mathbf{0}\}$ (the orthogonal complement in E). Therefore, it is sufficient to check

$$\mathcal{C}_t \text{ is positive definite a.s.} \tag{3.19}$$

This follows from the classical argument in the Brownian motion setting (see e.g Th 2.3.2 in [31] or Th 6.1 in [38]) combined with Theorem 3.2.4. For completeness, we provide the details. We argue by contradiction. Let us suppose there exists $\varphi_0 \neq \mathbf{0}$ such that

$$\mathbb{P}\big\{\langle \mathcal{C}_t\varphi_0, \varphi_0\rangle_E = 0\big\} > 0. \tag{3.20}$$

Take $\varphi \in E$. By (3.7), we have

$$\langle \mathcal{C}_t\varphi, \varphi\rangle_E = \alpha_H \sum_{\ell=1}^{\infty}\int_0^t\int_0^t \big\langle \mathbf{J}^{+}_{0\to u}(x_0)G_\ell(X^{x_0}_u), \varphi\big\rangle_E \big\langle \mathbf{J}^{+}_{0\to v}(x_0)G_\ell(X^{x_0}_v), \varphi\big\rangle_E$$
$$|u-v|^{2H-2}dudv. \tag{3.21}$$

Let us define

$$K_s = \overline{\text{span}\{\mathbf{J}^{+}_{0\to r}(x_0)G_\ell(X^{x_0}_r); 0 \le r \le s, \ell \in \mathbb{N}\}}; 0 < s \le T,$$

and we set $K_{0+} = \cap_{s>0}K_s$. The Brownian filtration $\mathbb{F}$ allows us to make use of the Blumental zero-one law to infer that K_{0+} is deterministic[1] a.s. Let $N > 0$ be a natural number and let N_s be the (possibly infinite) dimension of the quotient space $\frac{K_s}{K_{0+}}$. Consider the non-decreasing adapted process $\big\{\min\{N, N_s\}, 0 < s \le T\big\}$ and the stopping time

$$S = \inf\big\{0 < s \le T; \min\{N, N_s\} > 0\big\}.$$

One should notice that $S > 0$ a.s. If $S = 0$ on a set A of positive probability, then for every $\epsilon > 0$ there exists $0 < s \le T$ such that

$$\epsilon > s > 0 \text{ and } \min\{N_s, N\} > 0$$

on A. This means that we should have $N_s > 0$ for every $s \in (0, T]$ on A. This implies that with a positive probability the dimension of $\frac{K_s}{K_{0+}}$ is strictly positive which is a contradiction.

We now claim that K_{0+} is a proper subset of E. Otherwise, $K_{0+} = E$ which implies $K_s = E$ for every $0 < s \le T$. In this case, if $\varphi \in E$ is such that $\langle \mathcal{C}_t\varphi, \varphi\rangle_E = 0$

[1] We say that a random subset $A \subset E_\kappa$ is deterministic a.s when all random elements $a \in A$ are constant a.s

with positive probability, then $\left\langle \mathbf{J}^{+}_{0\to r}(x_0)G_\ell(X^{x_0}_r), \varphi\right\rangle_E = 0$ for every $r \in [0,s]$ and $\ell \in \mathbb{N}$ with positive probability which in turn would imply that $\varphi \in K_s^{\perp} = E^{\perp}$ so that $\varphi = \mathbf{0}$. This contradicts (3.20) (see (3.21)). Now we are able to select a non-null $\varphi \in E^*$ such that $K_{0+} \subset \mathrm{Ker}\varphi$. At first, we observe $\varphi(K_s) = 0$ for every $0 \le s < S$ so that

$$\langle \mathbf{J}^{+}_{0\to s}G_\ell(X^{x_0}_s), \varphi\rangle_E = 0 \ \forall \ell \ge 1 \text{ and } 0 \le s < S. \tag{3.22}$$

We claim

$$\langle \mathbf{J}^{+}_{0\to s}(x_0)V(X^{x_0}_s), \varphi\rangle_E = 0 \text{ for every } 0 \le s < S, V \in \mathcal{V}_k, k \ge 0, \tag{3.23}$$

where we observe V in (3.23) takes values on $S(T)E$. We show (3.23) by induction. For $k = 0$, (3.22) implies (3.23). Let us assume (3.23) holds for $k-1$. Let $V \in \mathcal{V}_{k-1}$. By Proposition 3.1.8,

$$\begin{aligned} 0 &= \langle \mathbf{J}^{+}_{0\to s}(x_0)V(X^{x_0}_s), \varphi\rangle_E \\ &= \langle V(x_0), \varphi\rangle_E + \int_0^s \langle \mathbf{J}^{+}_{0\to r}(x_0)[G_0, V](X^{x_0}_r), \varphi\rangle_E dr \\ &\quad + \sum_{\ell=1}^{\infty} \sqrt{\lambda_\ell} \int_0^s \langle \mathbf{J}^{+}_{0\to r}(x_0)[G_\ell, V](X^{x_0}_r), \varphi\rangle_E d\beta^\ell_r \end{aligned}$$

where $\langle V(x_0), \varphi\rangle_E = 0$ by the induction hypothesis. By Theorem 3.2.4, we must have

$$\langle \mathbf{J}^{+}_{0\to r}(x_0)[G_0, V](X^{x_0}_r), \varphi\rangle_E = \langle \mathbf{J}^{+}_{0\to r}(x_0)[G_\ell, V](X^{x_0}_r), \varphi\rangle_E = 0$$

for every $0 \le r \le s$ and $0 \le s < S$ and $\ell \ge 1$. This proves (3.23). Clearly, (3.23) implies

$$\varphi(\mathcal{V}_k(x_0)) = 0 \text{ for every non-negative integer } k \tag{3.24}$$

and hence the Hörmander's bracket condition implies $\varphi = \mathbf{0}$. By Lemma 3.1.7, this concludes the proof. ■

Remark 3.3.1 *The assumption that $S(t)E$ is dense in E seems a bit restrictive but it covers a rather general class of examples. For instance, if $(A, dom\ (A))$ is a densely* *defined self-adjoint operator such that*

$$\sup_{x \in dom(A)/\{0\}} \frac{\langle x, Ax \rangle_E}{\|x\|_E} < \infty$$

then $(A, dom\ A)$ *is the generator of a self-adjoint analytic semigroup (see Th 7.3.4 and Example 7.4.5 in [5]). Since analytic semigroups are one-to-one,* $S^*(t)$ *is one-to-one for every* $t \geq 0$ *and hence,* $S(t)E$ *is dense in* E *for every* $t \geq 0$*. The heat semigroup on* L^2 *has dense range (see [13]). More generally, assume there exists a separable Hilbert space* W *densely and continuously embedded into* E *with compact imbedding. Assume that*

- $A : W \to W^*$ *is continuous and its restriction to* W*,* $A_E : dom(A_E) \to E$ *where* $dom(A_E) = \{u \in W; Au \in E\}$ *and* $A_E u = Au; u \in dom(A_E)$*, is a self-adjoint operator.*

- *There exists* $\lambda \in \mathbb{R}$ *and* $\eta > 0$ *such that*

  $$\big(Au, u\big)_{W,W^*} + \lambda \|u\|_E^2 \geq \eta \|u\|_W^2$$

  for each $u \in W$*.*

Then, $S(t)E$ *is dense in* E *for every* $t \in [0, T]$*. See e.g [3] for further details.*

References

[1] Baudoin, F. and Hairer, M. (2007). A version of Hörmander's theorem for the fractional Brownian motion. *Probab. Theory Related Fields*, **139**, 3-4, 373-395.

[2] Baudoin, F. and Teichmann, J. (2005). Hypoellipticity in infinite dimensions and an application in interest rate theory. *Ann. App. Probab*, **15**, 3, 1765-1777.

[3] Bejenaru, I., Ildefosno, J. and Vrabie, I. (2001). An abstract approximate controllability result and applications to elliptic and parabolic systems with dynamic boundary conditions. *Eletr. J. Diff. Eq*, **50**, 1-19.

[4] Bogachev, W. Gaussian measures. American Mathematical Society. Mathematical Surveys and Monographs. 1998.

[5] Buhler, T. and Salamon, D.A. *Functional Analysis.* Graduate Studies in Mathematics. American Mathematical Society, 2018.

[6] Cass, T., Litterer, C. and Lyons, T. (2013). Integrability and tail estimates for Gaussian rough differential equations. *Ann. Probab*, 41, no. 4, 3026-3050.

[7] Cass, T., Hairer, M., Litterer, C. and Tindel, S. (2015). Smoothness of the density for soluitions to Gaussian rough differential equations. *Ann. Probab*, **43**, 1, 188-239.

[8] Cass, T. and Friz, P. (2012). Densities for rough differential equations under Hörmander's condition. Ann. Math. 171, (2010), 2115-2141.

[9] Cass, T. and Friz, P. and Victoir, N. (20 Non-degeneracy of Wiener functionals arising from rough differential equations. Trans. Amer. Math. Soc. 361 (2009), no. 6, 3359-3371.

[10] Dieudonné, J. *Foundation of Modern Analysis*, Academic Press, 1969.

[11] Deya, A., Gubinelli, M. and Tindel, S. (2012). Non-linear rough heat equations. *Probab. Theory Related Fields*,**153**, 1-2, 97-147.

[12] Deya, A. and Tindel, S. (2013). Malliavin Calculus for Fractional Heat Equation. *Potential Analysis*, 361-384.

[13] Fabre, C., Puel, J.P. and Zuazua, E. (1994). On the density of the range of the semigroup for semilinear heat equations. In Control and optimal design of distributed parameter systems, Springer Verlag, New York, IMA Volume 70, 73-92.

[14] Filipovic, D. and Teichmann, J (2003). Existence of invariant Manifolds for Stochastic Equations in infinite dimension. *J. Funct. Anal.* **197**, 398-432.

[15] Friz, P and Hairer, M. A Course on Rough Paths. Springer-Verlag, 2014.

[16] Friz, P. and Shekhar, A. (2013). Doob-Meyer for rough paths. *Bulletin of the Institute of Mathematics Academia Sinica*, **8**, 1, 73-84.

[17] Gubinelli, M. (2004). Controlling rough paths. *J. Funct. Anal*, **216**, 86-140.

[18] Gubinelli, M and Tindel, S. (2010). Rough evolution equations. *Ann. Probab*, **38**, 1, 1-75.

[19] Gawarecki, L. and Mandreckar, V. Stochastic Differential Equations in Infinite Dimensions. Springer.

[20] Gerasimovics, and Hairer, M. Hörmander's theorem for semilinear SPDEs. arXiv: 1811-06339v1. 2018.

[21] Gross, Leonard (1967). Abstract Wiener spaces. Proc. Fifth Berkeley Sympos. Math. Statist. and Probability (Berkeley, Calif., 1965-66), Vol. II: Contributions to Probability Theory, Part 1. Berkeley, Calif. Univ. California Press, 31-42.

[22] Kusuoka, S. (1982). The non-linear transformation of Gaussian measure on Banach space and its absolute continuity (I), *J. Fac. Sci. Univ. Tokyo IA*,**29**, 567-597.

[23] Hairer, M. and Ohashi, A. (2007). Ergodic thepry for SDEs with extrinsic memory. *Ann. Probab*, **35**, 5, 1950-1977.

[24] Hu, Y. and D. Nualart, D. (2007). Differential equations driven by Hölder continuous functions of order greater than $\frac{1}{2}$. Abel Symp. 2, 349-413.

[25] Ito, K. and Kappel, F. Evolution equations and approximations. Series on Advances in Mathematics for Applied Sciences. Vol 61.

[26] Lunardi, A. Analytic Semigroups and Optimal Regularity in Parabolic Problems. Springer.

[27] Malliavin, P. Stochastic calculus of variations and hypoelliptic operators. *Proc. Intern. Symp. SDE*, 195-263.

[28] Marcus, Michael, B. and Rosen, J. *Markov Processes, Gaussian Processes, and Local Times*, Cambridge Studies in Advanced Mathematics, Cambridge University Press, 2010.

[29] Maslowski, B. and Nualart,D. (2003). Evolution equations driven by a fractional Brownian motion. *J. Funct. Anal.*, **202**, 1, 277-305.

[30] Mishura, Y. *Stochastic Calculus for Fractional Brownian Motion and Related Processes*. Lecture Notes in Maths. 1929.

[31] Nualart, D. The Malliavin calculus and related topics. Springer.

[32] Nualart, D. and Rascanu, A. (2002). Differential equations driven by fractional Brownian motion. *Collectanea Mathematica*, **53**, 1, 55-81.

[33] Nualart, D. and Saussereau, B. Malliavin calculus for stochastic differential equations driven by a fractional Brownian motion. *SPA*.

[34] Norris, J. (1986). Simplified Malliavin calculus. In Seminaire de Probabilités, XX, 1984/85, vol. 1204 of Lecture Notes in Math., 101-130. Springer, Berlin.

[35] Pazy, A. Semigroups of linear operators and applications to partial differential equations, Springer-Verlag, New York, 1983.

[36] Pronk, M., Veraar, M. (2014). Tools for Malliavin Calculus in UMD Banach spaces. *Potential Anal*, **40**, 4, 307-344.

[37] Samko, S.G., Kilbas, A.A., and Marichev, O.I. *Fractional Integrals and Derivatives. Theory and Applications*, Gordon and Breach, 1993.

[38] Shamarova, E. (2014). A version of the Hörmander-Malliavin theorem in 2-smooth Banach spaces. *Infin. Dimen. Dimens. Anal. Quantum Proba. Relat. Top*, **17**, 1, 1450004.

[39] Young, L.C. (1936). An inequality of Hölder type, connected with Stieltjes integration. *Acta Math*, **67**, 251-282.