

Modified Brans-Dicke theory of gravity from five-dimensional vacuum

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We investigate, in the context of five-dimensional (5D) Brans-Dicke theory of gravity, the idea that macroscopic matter configurations can be generated from pure vacuum in five dimensions, an approach first proposed in the framework of general relativity. We show that the 5D Brans-Dicke vacuum equations when reduced to four dimensions lead to a modified version of Brans-Dicke theory in four dimensions (4D). As an application of the formalism, we obtain two five-dimensional extensions of four-dimensional O'Hanlon and Tupper vacuum solution and show that they lead two different cosmological scenarios in 4D.

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I. INTRODUCTION

Recourse to extra dimensions, beyond space and time, has been considered during the last decades as probably the best if not the only way to achieve unification of gravity with the other interactions of physics. This idea, which goes back to the works of G. Nordström [1] and to the original form of Kaluza-Klein theory [2], has been the source of an enormous amount of research that have ultimately culminated in string theory and its recent generalization, the so-called brane theory [3]. Parallely to this development there has recently been a fair deal of work on a proposal by P. Wesson and collaborators, known as the induced-matter theory (IMT), a non-compactified approach to five-dimensional Kaluza-Klein gravity [4]. The basic principle of the IMT approach is that classical macroscopic properties of physical quantities, such as matter density and pressure, may be given a geometrical interpretation if one assumes the existence of a fundamental five-dimensional space M^5 , in which our usual four-dimensional (4D) spacetime is embedded. It is also assumed that the metric g_{AB} of M^5 is a solution of the vacuum Einstein field equations in five dimensions (5D) ¹

$$R_{AB} = 0.$$

The main idea of IMT theory is that the above equation may be re-expressed so that in 4D, the four-dimensional components of g_{AB} , which we denote by $g_{\alpha\beta}$, satisfy Einstein's equations

$$G_{\alpha\beta} = 8\pi G T_{\alpha\beta}$$

in which the "energy-momentum tensor" $T_{\alpha\beta}$ has a purely geometrical origin and this fact is interpreted as saying that four-dimensional matter is induced from geometry in five-dimensional space [4]. At the time the first papers on IMT were published there was no guarantee that any energy-momentum tensor could be induced in this way. Mathematically rephrased the question then was: Is it possible to isometrically embed any solution of Einstein's equations in a 5D Ricci-flat space? The answer to this question was provided by the Campbell-Magaard theorem, which asserts that any analytic n -dimensional Riemannian space can be locally embedded in a $(n+1)$ -dimensional Ricci-flat space [5, 6, 7]. The rediscovery of Campbell-Magaard theorem by physicists and its connection with modern spacetime embedding theories gave rise to a series of new results and extensions in the cases when the ambient space:

¹ Throughout capital Latin indices take value in the range (0,1,...4) while Greek indices run from (0,1,2,3). We shall also denote the fifth coordinate y^4 by l and the first four coordinates y^μ (the "spacetime" coordinates) by x^μ , that is, $y^A = (x^\mu, l)$, with $\mu = 0, 1, \dots, 3$.

a) is an Einstein space [8], b) has a nondegenerate Ricci-tensor [9], c) is sourced by scalar-fields [10]. The latter case includes embeddings in spaces that are solutions of the vacuum Brans-Dicke field equations.

The Brans-Dicke theory of gravity represents a natural extension of General Relativity where Newton's gravitational constant is replaced by a non-minimally coupled scalar field [11]. Brans-Dicke theory or, more generally, scalar-tensor theories in higher dimensions have also been studied in different contexts. For instance, cosmological questions related to quintessence, inflation, acceleration of the Universe, and the so-called braneworld scenario have been examined in the framework of higher-dimensional scalar-tensor theories [12]. Certainly, an attractive feature of the Brans-Dicke theory concerns the geometrical nature of the scalar field, as opposed to other models in which the scalar field has to be separately postulated and thereby have an *ad hoc* status.

In this paper we investigate a modified version of the induced-matter theory in which the role of General Relativity as the fundamental underlying theory is replaced by the Brans-Dicke theory of gravity. As we shall see, the five-dimensional Brans-Dicke vacuum equations when reduced to four dimensions lead to a modified 4D Brans-Dicke theory in which matter can still be viewed as generated by the fifth dimension.

The paper is organized as follows. In Section II we set up the Brans-Dicke theory in five dimensions and write the field equations in the absence of matter, a configuration to which we refer as the 5D vacuum state. We then proceed to Section III, where we show that the 5D vacuum field equations lead to a modified version of four-dimensional Brans-Dicke theory. In Sections IV and V we illustrate the formalism developed in the previous sections by extending the O'Hanlon and Tupper vacuum solution to five dimensions in two different ways and show that when viewed from a 4D perspective they lead to different cosmological scenarios. We conclude with some remarks in Section VI.

II. BRANS-DICKE FIELD EQUATIONS IN FIVE DIMENSIONS

In order to describe the Brans-Dicke (BD) theory of gravity in 5D in a general manner, we consider the 5D action

$$\mathcal{S}[g_{AB}, \varphi] = \frac{1}{16\pi} \int \sqrt{\left| \frac{{}^{(5)}g}{{}^{(5)}g_0} \right|} \left[\varphi^{(5)}\mathcal{R} + L_{\text{matt}} - \frac{\omega}{\varphi} g^{AB} \varphi_{,A} \varphi_{,B} \right] d^5x, \quad (1)$$

where φ is a scalar field that describes the gravitational coupling in 5D, ${}^{(5)}\mathcal{R}$ is the 5D Ricci scalar, ${}^{(5)}g$ is the determinant of the 5D metric here denoted by g_{AB} , ${}^{(5)}g_0$ is a dimensionalization constant, L_{matt} represents the matter lagrangian and ω is a constant parameter.

The field equations obtained from the action (1) are

$$G_{AB} = \frac{8\pi}{\varphi} T_{AB} + \frac{\omega^2}{\varphi^2} \left[\varphi_{,A} \varphi_{,B} - \frac{1}{2} g_{AB} \varphi^{;C} \varphi_{;C} \right] + \frac{1}{\varphi} [\varphi_{;A;B} - g_{AB} \varphi^{;C}{}_{;C}], \quad (2)$$

$$\frac{2\omega}{\varphi} \varphi^{;C}{}_{;C} - \frac{\omega}{\varphi^2} g^{AB} \varphi_{,A} \varphi_{,B} + {}^{(5)}\mathcal{R} = 0, \quad (3)$$

with $(;)$ denoting covariant derivative, T_{AB} being the energy-momentum tensor of matter and $G_{AB} = R_{AB} - (1/2) {}^{(5)}\mathcal{R} g_{AB}$ is the Einstein tensor in five dimensions. Taking the trace of the expression (2) gives

$${}^{(5)}\mathcal{R} = -\frac{16\pi}{3\varphi} T + \frac{\omega}{\varphi^2} \varphi^{;C} \varphi_{;C} + \frac{8}{3\varphi} \varphi^{;C}{}_{;C}, \quad (4)$$

where $T = T^A{}_A$. By substituting (4) in (3) one has

$${}^{(5)}\square\varphi = \frac{8\pi}{4 + 3\omega} T, \quad (5)$$

where ${}^{(5)}\square\varphi = \varphi^{;A}{}_{;A}$. Since (3) and (5) are not independent, then we may consider (2) and (5) as the 5D field equations.

Let us now consider the 5D vacuum case. We define the 5D vacuum state in BD theory as a configuration in which $T_{AB} = 0$, i.e. the 5D vacuum is defined as the absence of matter in 5D. This is equivalent to setting $L_{\text{matt}} = 0$ in the action (1). Thus, the 5D vacuum field equations will be given by

$$G_{AB} = \frac{\omega^2}{\varphi^2} \left[\varphi_{,A} \varphi_{,B} - \frac{1}{2} g_{AB} \varphi^{;C} \varphi_{;C} \right] + \frac{1}{\varphi} [\varphi_{;A;B} - g_{AB} \varphi^{;C}{}_{;C}], \quad (6)$$

$${}^{(5)}\square\varphi = 0. \quad (7)$$

while the equation (4) in vacuum becomes

$${}^{(5)}R = \frac{\omega}{\varphi^2} g^{AB} \varphi_{,A} \varphi_{,B}, \quad (8)$$

which implies that in vacuum the scalar curvature is generated only by the free scalar field φ .

III. THE EFFECTIVE FOUR-DIMENSIONAL MODIFIED BRANS-DICKE THEORY

Following the idea, as in the IMT approach, that matter can be generated from five-dimensional pure vacuum, we investigate the possibility of obtaining the Brans-Dicke four-dimensional field equations with matter as a subset of the field equations of a Brans-Dicke field theory in 5D vacuum. Our approach, in fact, will be more general in the sense that we shall deal with a somewhat modified version Brans-Dicke theory in 4D, that is, one in which a scalar potential $V(\varphi)$ is taken into account. In a cosmological setting it is well known that the scalar field in 4D Brans-Dicke theory can lead to cosmological acceleration only when the coupling parameter ω varies with time [13] or when there is a scalar potential which usually is added by means of an *ad hoc* assumption [14]. A scalar field obtained in this way has the desired property that it accelerates the universe, thereby leading to quintessence or K-essence. At this point we would like to call attention for the fact that in the formalism developed in the present paper a potential exhibiting the mentioned property can be geometrically induced due to the presence of the fifth coordinate, avoiding *ipso facto* its introduction as "by hand". In fact, the idea of inducing 4D scalar potentials $V(\varphi)$ geometrically from a 5D space-time has already been introduced in [15], in the context of inflationary models. Let us now develop a formalism which, in a way, generalises the induced-matter approach and which leads to a slightly modified Brans-Dicke theory in four dimensions, these modifications being caused essentially by the presence of the scalar potential.

We start by considering the 5D line element written in local coordinates $y^A = (x^\mu, l)$

$$dS^2 = g_{AB} dx^A dx^B = g_{\mu\nu}(x^A) dx^\mu dx^\nu + g_{44}(x^A) (dl)^2. \quad (9)$$

Then we assume that the 5D space is foliated by a family of hypersurfaces Σ defined by $l = \text{const.}$ The metric induced on a generic hypersurface Σ_0 ($l = l_0$) has the form

$$ds^2 = g_{\mu\nu}(x^\alpha, l_0) dx^\mu dx^\nu. \quad (10)$$

On the other hand, the action (1) on Σ_0 takes the form

$$S_{\Sigma_0} = \frac{1}{16\pi} \int \sqrt{|{}^{(4)}g|} \left[\varphi {}^{(4)}R + L_{IM} - \frac{\omega}{\varphi} g^{\alpha\beta} \varphi_{,\alpha} \varphi_{,\beta} + V(\varphi) \right] d^4x, \quad (11)$$

where L_{IM} is the Lagrangian corresponding to the induced matter. Note that L_{IM} comes from the first term in action (1) when it is splitted in a 4D part plus a fifth coordinate contribution and finally evaluated on Σ_0 . Here we have made the formal identification

$$V(\varphi) = - \frac{\omega}{\varphi} g^{44} \varphi_{,4} \varphi_{,4} \Big|_{\Sigma_0}. \quad (12)$$

At a first view, $V(\varphi)$ looks like a kinetic term instead of a potential. However, as it has been shown in [15], depending on the metric background and considering separable scalar fields such an identification is valid.

By taking the space-time component ($A = \alpha, B = \beta$) of the 5D Brans-Dicke field equations (6) we obtain

$$\begin{aligned} \hat{G}_{\alpha\beta} = & \frac{\omega}{\varphi^2} \left[\varphi_{,\alpha} \varphi_{,\beta} - \frac{1}{2} g_{\alpha\beta} \varphi^{;\sigma} \varphi_{,\sigma} \right] + \frac{1}{\varphi} \left[\varphi_{;\alpha;\beta} - g_{\alpha\beta} ({}^{(4)}\square \varphi - \frac{1}{2} V(\varphi)) \right] - \\ & - \frac{g_{\alpha\beta}}{\varphi} \left[g^{44} \varphi^{**} + \frac{1}{2} \left(g^{\sigma\rho} g_{\rho\sigma}^{*44} + (g^{44})^2 g_{44}^{*} \right) \varphi^{*} + \frac{1}{2} g^{44} g_{44,\sigma} \varphi^{,\sigma} \right] \end{aligned} \quad (13)$$

where here (\wedge) denotes the 4D part of a 5D quantity, the semicolon indicates covariant derivative, the star stands for derivative with respect to the fifth coordinate l and ${}^{(4)}\square \varphi = \varphi^{;\sigma}{}_{;\sigma}$ denotes the four-dimensional D'Alembertian operator acting on the field φ . On the other hand, it can directly be shown that [16]

$$\hat{G}_{\alpha\beta} = G_{\alpha\beta} - \frac{8\pi}{\varphi} T_{\alpha\beta}^{(IMT)}, \quad (14)$$

where $T_{\alpha\beta}^{(IMT)}$ is given by

$$\frac{8\pi}{\varphi} T_{\alpha\beta}^{(IMT)} = g^{44}(g_{44,\alpha})_{;\beta} - \frac{1}{2}(g^{44})^2 \left[g^{44*} g_{44}^* g_{\alpha\beta} - g_{\alpha\beta}^{**} + g^{\lambda\mu} g_{\alpha\lambda}^* g_{\beta\mu}^* - \frac{1}{2} g^{\mu\nu} g_{\mu\nu}^* g_{\alpha\beta} + \frac{1}{4} g_{\alpha\beta} \left(g^{*\mu\nu} g_{\mu\nu}^* + (g^{\mu\nu} g_{\mu\nu}^*)^2 \right) \right]. \quad (15)$$

Inserting (14) into (13) we obtain

$$G_{\alpha\beta} = \frac{8\pi}{\varphi} T_{\alpha\beta}^{(BD)} + \frac{\omega}{\varphi^2} \left[\varphi_{,\alpha} \varphi_{,\beta} - \frac{1}{2} g_{\alpha\beta} \varphi^{,\sigma} \varphi_{,\sigma} \right] + \frac{1}{\varphi} \left[\varphi_{;\alpha;\beta} - g_{\alpha\beta} ({}^{(4)}\square\varphi - \frac{1}{2} V(\varphi)) \right], \quad (16)$$

where $T_{\alpha\beta}^{(BD)}$ is the induced energy-momentum tensor on the effective 4D modified Brans-Dicke theory, which have the form

$$8\pi T_{\alpha\beta}^{(BD)} = 8\pi T_{\alpha\beta}^{(IMT)} - g_{\alpha\beta} \left[g^{44*} \dot{\varphi} + \left(\frac{1}{2} g^{\sigma\rho} g_{\rho\sigma}^* + \frac{1}{2} (g^{44})^2 g_{44}^* \right) \dot{\varphi} + \frac{1}{2} g^{44} g_{44,\sigma} \varphi^{,\sigma} \right]. \quad (17)$$

It is important to note that in comparison with the IMT in our case the induced energy-momentum tensor is composed of two parts. The first part is induced from the extra-dimensional part of the metric and it is the same as in the IMT approach. The second part adds a new contribution that depends on the scalar field φ and its derivatives with respect to the fifth coordinate.

Now setting $A = 4$ and $B = \mu$ in equation (6) yields

$$G_{4\mu} = \frac{\omega^2}{\varphi^2} \dot{\varphi} \varphi_{,\mu} + \frac{\dot{\varphi}_{,\mu}}{\varphi} - \frac{1}{2\varphi} \left[g^{\alpha\beta} g_{\beta\mu}^* \varphi_{,\alpha} - g^{44} g_{44,\mu} \dot{\varphi} \right]. \quad (18)$$

From the IMT we know that $G_{4\mu}$ can be expressed as $G_{4\mu} = \sqrt{g_{44}} P^{\beta}_{\mu;\beta}$, where $P_{\alpha\beta} = [1/(2\sqrt{g_{44}})](g_{\alpha\beta}^* - g_{\alpha\beta} g^{\mu\nu} g_{\mu\nu}^*)$ is a conserved quantity. Therefore, equation (18) becomes

$$\frac{\omega^2}{\varphi^2} \dot{\varphi} \varphi_{,\mu} + \frac{\dot{\varphi}_{,\mu}}{\varphi} - \frac{1}{2\varphi} \left[g^{\alpha\beta} g_{\beta\mu}^* \varphi_{,\alpha} - g^{44} g_{44,\mu} \dot{\varphi} \right] - \sqrt{g_{44}} P^{\beta}_{\mu;\beta} = 0. \quad (19)$$

The last component $A = 4$, $B = 4$ of (6) reads

$$G_{44} = \frac{\omega^2}{\varphi^2} \left[\dot{\varphi}^2 - \frac{1}{2} g_{44} \left(\varphi^{,\alpha} \varphi_{,\alpha} + g^{44} \dot{\varphi}^2 \right) \right] - \frac{1}{\varphi} \varphi^{;\alpha}_{;\alpha} - \frac{1}{2\varphi} g^{\sigma\rho} g_{\sigma\rho}^* \dot{\varphi}, \quad (20)$$

where $G_{44} = R_{44} - (1/2)g_{44} ({}^{(5)}R)$. On the other hand R_{44} is given by

$$R_{44} = -g_{44} g^{\mu\nu} (g_{44,\mu})_{;\nu} - \frac{1}{2} g^{*\lambda\beta} g_{\lambda\beta}^* - \frac{1}{2} g^{\lambda\beta} g_{\lambda\beta}^{**} + \frac{1}{2} g^{44*} g_{44} g^{\lambda\beta} g_{\lambda\beta}^* - \frac{1}{4} g^{\mu\beta} g^{\lambda\sigma} g_{\lambda\beta}^* g_{\mu\sigma}^*, \quad (21)$$

while the 5D scalar curvature is

$$({}^{(5)}R) = ({}^{(4)}R) - \frac{1}{4} (g^{44})^2 \left[g^{*\mu\nu} g_{\mu\nu}^* + (g^{\mu\nu} g_{\mu\nu}^*)^2 \right]. \quad (22)$$

Thus, the 5D component G_{44} becomes

$$G_{44} = -g_{44} g^{\mu\nu} (g_{44,\mu})_{;\nu} - \frac{1}{2} g^{*\lambda\beta} g_{\lambda\beta}^* - \frac{1}{2} g^{\lambda\beta} g_{\lambda\beta}^{**} + \frac{1}{2} g^{44*} g_{44} g^{\lambda\beta} g_{\lambda\beta}^* - \frac{1}{4} g^{\mu\beta} g^{\lambda\sigma} g_{\lambda\beta}^* g_{\mu\sigma}^* - \frac{1}{2} g_{44} ({}^{(4)}R) - \frac{1}{4} (g^{44})^2 \left[g^{*\mu\nu} g_{\mu\nu}^* + (g^{\mu\nu} g_{\mu\nu}^*)^2 \right] \quad (23)$$

From (23) and (20) we have

$$-g_{44} g^{\mu\nu} (g_{44,\mu})_{;\nu} - \frac{1}{2} g^{*\lambda\beta} g_{\lambda\beta}^* - \frac{1}{2} g^{\lambda\beta} g_{\lambda\beta}^{**} + \frac{1}{2} g^{44*} g_{44} g^{\lambda\beta} g_{\lambda\beta}^* - \frac{1}{4} g^{\mu\beta} g^{\lambda\sigma} g_{\lambda\beta}^* g_{\mu\sigma}^* - \frac{1}{2} g_{44} ({}^{(4)}R) + \frac{1}{8} g^{44} [g^{*\mu\nu} g_{\mu\nu}^* + (g^{\mu\nu} g_{\mu\nu}^*)^2] = \frac{\omega^2}{\varphi^2} \left[\dot{\varphi}^2 - \frac{1}{2} g_{44} (\varphi^{,\alpha} \varphi_{,\alpha} + g^{44} \dot{\varphi}^2) \right] - \frac{1}{\varphi} \varphi^{;\alpha}_{;\alpha} - \frac{1}{2\varphi} g^{\sigma\rho} g_{\sigma\rho}^* \dot{\varphi}. \quad (24)$$

We thus see that equation (6) is equivalent to the system of equations (16), (20) and (24). On the other hand, equation (7) evaluated on the hypersurface Σ_0 can be written as

$$\frac{2\omega}{\varphi} {}^{(4)}\square\varphi - \frac{\omega}{\varphi^2} g^{\alpha\beta} \varphi_{,\alpha} \varphi_{,\beta} + V'(\varphi) + {}^{(4)}R = 0 \quad (25)$$

where we have made the formal identification

$$V'(\varphi) \equiv \left[\frac{2\omega}{\varphi} \frac{1}{\sqrt{|{}^{(5)}g|}} \frac{\partial}{\partial l} \left(\sqrt{|{}^{(5)}g|} g^{44} \varphi_{,4} \right) - \frac{\omega}{\varphi^2} g^{44} \varphi_{,4} \varphi_{,4} \right] \Big|_{\Sigma_0} \quad (26)$$

It is clear that equations (16) and (25) are a subset of the field equations in 5D vacuum (6) and (3), while by taking the trace of (16) we have

$${}^{(4)}R = -\frac{8\pi}{\varphi} T^{(BD)} + \frac{\omega}{\varphi^2} \varphi^{\cdot\alpha} \varphi_{,\alpha} + \frac{1}{\varphi} \left[3 {}^{(4)}\square\varphi - 2V(\varphi) \right], \quad (27)$$

with $T^{(BD)} = g^{\alpha\beta} T_{\alpha\beta}^{(BD)}$. Now if we substitute (27) into (25) we get

$${}^{(4)}\square\varphi = \frac{8\pi}{3+2\omega} T^{(BD)} + \frac{2}{3+2\omega} V(\varphi) - \frac{\varphi}{3+2\omega} V'(\varphi), \quad (28)$$

It is now clear that (16) and (28) correspond to the field equations of a more general version of four-dimensional Brans-Dicke theory. The main difference with respect to the standard Brans-Dicke theory resides in the fact that here the potential is now geometrically induced by the fifth dimension in much the same way as the energy-momentum tensor comes from the 5D pure vacuum. In the next section we shall give an application of the above formalism in a cosmological context.

IV. THE FIVE-DIMENSIONAL ANALOGUE OF THE O'HANLON AND TUPPER SOLUTION

A most known vacuum solution in Brans-Dicke theory of gravity is the O'Hanlon and Tupper [17] solution. In this case $V(\phi) = 0$ and the range of values of parameter ω is restricted to $\omega > -3/2$, $\omega \neq 0$, $-4/3$ [18]. In this section we shall obtain the natural extension of this solution in five-dimensional Brans-Dicke theory. We start by considering the metric corresponding to a homogeneous and isotropic cosmological model in five-dimensional space

$$dS^2 = dt^2 - a^2(t)[dx^2 + dy^2 + dz^2 + dl^2], \quad (29)$$

where t is the cosmic time, (x, y, z) are Cartesian coordinates and l is the fifth coordinate. Given that we are assuming homogeneity and isotropy we should have $\varphi = \varphi(t)$, hence the field equations in vacuum (6) and (7) reduce to

$$6H^2 = \frac{\omega}{2} \frac{\dot{\varphi}^2}{\varphi^2} + \frac{\ddot{\varphi}}{\varphi}, \quad (30)$$

$$3\frac{\ddot{a}}{a} + 3H^2 = -\frac{\omega}{2} \frac{\dot{\varphi}^2}{\varphi^2} + H\frac{\dot{\varphi}}{\varphi}, \quad (31)$$

$$\ddot{\varphi} + 4H\dot{\varphi} = 0, \quad (32)$$

where $H(t) = \dot{a}(t)/a(t)$ is the Hubble parameter. From equation (8) and taking into account that ${}^{(5)}R = -8\ddot{a}/a - 12H^2$, we have

$$\dot{H} = -\frac{\omega}{8} \left(\frac{\dot{\varphi}}{\varphi} \right)^2 - \frac{10}{4} H^2. \quad (33)$$

From (30) and (32), the equation (33) becomes

$$\dot{H} = -\frac{\omega}{3} \left(\frac{\dot{\varphi}}{\varphi} \right)^2 + \frac{5}{3} H\frac{\dot{\varphi}}{\varphi}. \quad (34)$$

Let us try to obtain solutions with the following form

$$a(t) = a_0 \left(\frac{t}{t_0} \right)^{q\pm}, \quad \varphi(t) = \varphi_0 \left(\frac{t}{t_0} \right)^{s\pm}, \quad (35)$$

where q , s , a_0 and φ_0 are constants. Inserting (35) in (30), (32) and (34) gives

$$q_{\pm} = \frac{2 + 2\omega \pm \sqrt{4 + 3\omega}}{2(5 + 4\omega)} = \frac{1}{5 + 4\omega} \left[\omega + 1 \pm \sqrt{\frac{3\omega + 4}{4}} \right], \quad (36)$$

$$s_{\pm} = \frac{1 \mp 2\sqrt{4 + 3\omega}}{4\omega + 5}, \quad (37)$$

where s and q are algebraically related by the equation $s + 4q = 1$. On the hypersurfaces Σ_0 ($l = l_0$) the equations (30) and (31) read

$$3H^2 = \frac{8\pi}{\varphi} \rho^{(BD)} + \frac{\omega}{2} \left(\frac{\dot{\varphi}}{\varphi} \right)^2 - 3H \frac{\dot{\varphi}}{\varphi} \quad (38)$$

$$2\frac{\ddot{a}}{a} + H^2 = -\frac{8\pi}{\varphi} P^{(BD)} - \frac{\omega}{2} \left(\frac{\dot{\varphi}}{\varphi} \right)^2 - \frac{\ddot{\varphi}}{\varphi} - 2H \frac{\dot{\varphi}}{\varphi}, \quad (39)$$

where

$$\rho^{(BD)} \equiv T^{(BD) t}{}_{t} = T_{\varphi}^t{}_{t} + T^{(IMT) t}{}_{t} = -\frac{H}{8\pi} [\dot{\varphi} + 3H\varphi], \quad (40)$$

$$P^{(BD)} = -T^{(BD) i}{}_{i} = -\left(T_{\varphi}^i{}_{i} + T^{(IMT) i}{}_{i} \right) = \frac{1}{8\pi} \left[H\dot{\varphi} + \varphi(\dot{H} + 3H^2) \right]. \quad (41)$$

By employing (35) and the relation $s + 4q = 1$, the above equations reduce to

$$\rho^{(BD)} = \frac{1}{8\pi} \frac{q}{t^2} (q - 1)\varphi, \quad P^{(BD)} = -\frac{1}{8\pi} \frac{q^2}{t^2} \varphi. \quad (42)$$

Clearly, in order to have a physical induced energy density $\rho^{(BD)} > 0$ and a negative induce pressure $P^{(BD)} < 0$, the condition $q > 1$ is required. According to (36), this condition implies $-4/3 < \omega < -5/4$, which satisfies the weak energy condition $\omega \geq -4/3$ [19]. The 4D induced equation of state is then

$$P^{(BD)} = -\left(\frac{q}{q-1} \right) \rho^{(BD)}, \quad (43)$$

which for $4.33 < q < \infty$ gives $-1.3 < P^{(BD)}/\rho^{(BD)} < -1$. It turns out that this is in agreement with the values of the interval that comes from observational data: $-1.3 < P/\rho < -0.7$ whose values describe the present quintessential expansion of the universe [20].

V. EMBEDDING THE O'HANLON AND TUPPER SOLUTION IN FIVE-DIMENSIONAL VACUUM SPACE

Let us now drop the assumption of isotropy in all spatial dimensions and consider a five-dimensional space with the following metric

$$dS^2 = dt^2 - a^2(t)(dx^2 + dy^2 + dz^2) - dl^2, \quad (44)$$

where, as previously, t is the cosmic time, (x, y, z) are Cartesian coordinates, l denotes the fifth coordinate, and $a(t)$ is a cosmological scale factor. Assuming that $\varphi = \varphi(t, l)$ the field equations (6) and (7) now give

$$3H^2 = \frac{\omega}{2} \left(\frac{\dot{\varphi}}{\varphi} \right)^2 + \frac{\omega}{2} \left(\frac{\dot{\varphi}^*}{\varphi} \right)^2 + \frac{\ddot{\varphi}}{\varphi} \quad (45)$$

$$2\frac{\ddot{a}}{a} + H^2 = -\frac{\omega}{2} \left(\frac{\dot{\phi}}{\phi}\right)^2 + \frac{\omega}{2} \left(\frac{\dot{\phi}^*}{\phi}\right)^2 - H\frac{\dot{\phi}}{\phi} \quad (46)$$

$$\frac{\ddot{a}}{a} + 3H^2 = -\frac{\omega}{2} \left(\frac{\dot{\phi}}{\phi}\right)^2 - \frac{\omega}{2} \left(\frac{\dot{\phi}^*}{\phi}\right)^2 - \frac{\dot{\phi}^{**}}{\phi} \quad (47)$$

$$\ddot{\phi} + 3H\dot{\phi} - \dot{\phi}^{**} = 0. \quad (48)$$

Using (48), equation (45) can be rewritten as

$$H^2 = \frac{\omega}{6} \left(\frac{\dot{\phi}}{\phi}\right)^2 - H\frac{\dot{\phi}}{\phi} + \frac{\omega}{6} \left(\frac{\dot{\phi}^*}{\phi}\right)^2 + \frac{1}{3}\frac{\dot{\phi}^{**}}{\phi}. \quad (49)$$

The trace equation (8), with ${}^{(5)}R = -(6\dot{H} + 12H^2)$ yields

$$\dot{H} = -\frac{\omega}{6} \left(\frac{\dot{\phi}}{\phi}\right)^2 + \frac{\omega}{6} \left(\frac{\dot{\phi}^*}{\phi}\right)^2 - 2H^2. \quad (50)$$

Inserting (48) and (49) in (50), this expression becomes

$$\dot{H} = -\frac{\omega}{2} \left(\frac{\dot{\phi}}{\phi}\right)^2 + 2H\frac{\dot{\phi}}{\phi} - \frac{\omega}{6} \left(\frac{\dot{\phi}^*}{\phi}\right)^2 - \frac{2}{3}\frac{\dot{\phi}^{**}}{\phi}. \quad (51)$$

Now, the equations (48), (49) and (51) constitute an independent system of differential equations. In order to find solutions to this system we assume that the scalar field can be separated in the form $\varphi(t, l) = f(l)\phi(t)$. With this assumption the equations (48), (49) and (51) become

$$H^2 = \frac{\omega}{6} \left(\frac{\dot{\phi}}{\phi}\right)^2 - H\frac{\dot{\phi}}{\phi} + \frac{\omega}{6} \left(\frac{\dot{f}}{f}\right)^2 + \frac{1}{3}\frac{\dot{f}^{**}}{f}, \quad (52)$$

$$\dot{H} = -\frac{\omega}{2} \left(\frac{\dot{\phi}}{\phi}\right)^2 + 2H\frac{\dot{\phi}}{\phi} - \frac{\omega}{6} \left(\frac{\dot{f}}{f}\right)^2 - \frac{2}{3}\frac{\dot{f}^{**}}{f}, \quad (53)$$

$$\frac{\ddot{\phi}}{\phi} + 3H\frac{\dot{\phi}}{\phi} - \frac{\dot{f}^{**}}{f} = 0. \quad (54)$$

In the particular case when $f(l) = 1$ the system (52) to (54) reduces to

$$H^2 = \frac{\omega}{6} \left(\frac{\dot{\phi}}{\phi}\right)^2 - H\frac{\dot{\phi}}{\phi}, \quad (55)$$

$$\dot{H} = -\frac{\omega}{2} \left(\frac{\dot{\phi}}{\phi}\right)^2 + 2H\frac{\dot{\phi}}{\phi}, \quad (56)$$

$$\ddot{\phi} + 3H\dot{\phi} = 0. \quad (57)$$

Again, if we assume that the solutions have the form (35) we readily obtain

$$q_{\pm} = \frac{\omega}{3(\omega + 1) \mp \sqrt{3(2\omega + 3)}} = \frac{1}{3\omega + 4} \left[\omega + 1 \pm \sqrt{\frac{2\omega + 3}{3}} \right], \quad (58)$$

$$s_{\pm} = \frac{1 \mp \sqrt{3(2\omega + 3)}}{3\omega + 4}, \quad (59)$$

where now $3q + s = 1$. Note that this is the O'Hanlon and Tupper vacuum solution in four dimensions. Therefore we conclude that the each leaf Σ of the foliation defined by $l = \text{const}$ corresponds to nothing more than O'Hanlon and Tupper spacetime. Note, *en passant*, that even though they represent vacuum solutions, both the embedded and the ambient spaces are not Ricci-flat.

To obtain the induced 4D potential we proceed as follows. Carrying out a separation of variables in equation (54) yields

$$\ddot{\phi} + 3H(t)\dot{\phi} + |\alpha|\phi = 0, \quad (60)$$

$$\frac{d^2 f}{dl^2} + |\alpha|f(l) = 0, \quad (61)$$

where $\alpha < 0$ is a separation constant. Considering a constant Hubble parameter $H = H_0$, the general solutions of (60) and (61) are, respectively

$$\phi(t) = B_1 e^{-\frac{3}{2}H_0 t \pm \frac{1}{2}\sqrt{9H_0^2 - 4|\alpha|}t}, \quad (62)$$

$$f(l) = B_2 e^{\pm i\sqrt{|\alpha|}l}, \quad (63)$$

where B_1 and B_2 are integration constants. In the case of a power-law expanding universe where the scale factor takes the form given in (35) the solution of (61) remains unaltered, while the general solution of (60) now becomes

$$\phi(t) = t^{-\nu} \left[D_1 \mathcal{J}_\nu(\sqrt{|\alpha|}t) + D_2 \mathcal{Y}_\nu(\sqrt{|\alpha|}t) \right], \quad (64)$$

\mathcal{J}_ν and \mathcal{Y}_ν denoting the first and second kind Bessel functions, $\nu = (3q - 1)/2$, and D_1, D_2 integration constants. On the other hand, by using equation (12) for a separable Brans–Dicke scalar field we obtain

$$V[\phi] = \omega \left[\frac{f}{f} \right]_{l_0}^{\star 2} \phi. \quad (65)$$

Finally, inserting (63) into (65) lead to

$$V[\phi] = (2\omega + 3)|\alpha|\phi(t), \quad (66)$$

where we have chosen the integration constant $B_2 = -[(2\omega + 3)/\omega]e^{\mp i\sqrt{|\alpha|}l_0}$. This specification of B_2 allows us to write equation (60) as

$$\ddot{\phi} + 3H(t)\dot{\phi} - \frac{1}{2\omega + 3} [\phi V'(\phi) - 2V(\phi)] = 0, \quad (67)$$

with the prime (') denoting derivative with respect to the scalar field ϕ . On the hypersurfaces Σ the equations (45) and (46) are

$$H^2 = \frac{8\pi}{3\varphi}\rho^{(BD)} + \frac{\omega}{6} \left(\frac{\dot{\phi}}{\phi} \right)^2 - H \frac{\dot{\phi}}{\phi}, \quad (68)$$

$$2\frac{\ddot{a}}{a} + H^2 = -\frac{8\pi}{\varphi}P^{(BD)} - \frac{\omega}{2} \left(\frac{\dot{\phi}}{\phi} \right)^2 - H \frac{\dot{\phi}}{\phi} - \frac{1}{\varphi}[\ddot{\phi} + 3H\dot{\phi}], \quad (69)$$

where $P^{(BD)} = -\rho^{(BD)} = -\varphi^{\star\star}/(8\pi)$.

Therefore, we see that, in the present approach, it is possible to induce a linear potential and an equation of state for vacuum in four dimensions. This seems to be a remarkable result, although in a way it should be expected since the metric (44) does not induce matter on the hypersurfaces Σ . In other words, the induced energy-momentum tensor $T^{(IMT)}$ is null for this metric. However, in this case there is a contribution from the scalar field that makes $T^{(BD)}$ nonzero. On the other hand, it seems natural that we have obtained an equation of state that describes vacuum, since we do not have matter, just the scalar field varying with respect the fifth coordinate.

VI. FINAL REMARKS

In this paper we have developed a procedure in which we regard our spacetime as a hypersurface embedded in a five-dimensional space, solution of the Brans-Dicke vacuum field equations. The geometry as well as the energy-momentum tensor that acts as source of the curvature of the four-dimensional spacetime is determined by pure vacuum in five-dimensions, an idea which goes back to the old Kaluza-Klein theory [2] and that has been revived recently by the induced-matter theory or non-compactified Kaluza-Klein gravity [4]. Since the Brans-Dicke theory usually (but not always [21]) reduces to general relativity when $\omega \rightarrow \infty$, the formalism developed here is, in a certain sense, a generalization of the induced-matter approach. There are, however, from a geometrical point of view, significant differences. For instance, in the induced-matter theory *à la* Brans-Dicke the Campbell-Magaard theorem cannot any longer be invoked to guarantee the embedding of any four-dimensional spacetime in five-dimensional Brans-Dicke vacuum. This is because due to the presence of the Brans-Dicke scalar field the ambient space is not, in general, Ricci-flat. However, it can be shown that a new geometrical frame for five-dimensional Brans-Dicke theory exists and is supported by an extension of the Campbell-Magaard theorem [10]. On the other hand, as in the case of the induced-matter approach, the theory provides no way of obtaining a unique spacetime from a given five-dimensional metric, and that would require further mathematical conditions on the embedding, or a kind of new dynamical principle to select the possible choices of physically plausible spacetimes [22].

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