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Newtonian Limit and General Radial Coordinate in the Brans-Dicke Theory of Gravity

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Abstract

We exhibit a general static spherically symmetric solution of the Brans-Dicke vacuum field equations, which is valid in the weak field approximation. From this solution, we investigated the Newtonian limit obtaining general expressions for gravitational force and the corresponding potential. In addition, we obtain the metric solution for a "Yukawa" radial coordinate.

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1 Introduction

The General Relativity is the standard theory of gravity, and if one considers the limit of weak field and low velocity the Newtonian gravitational force between two pointlike particles is obtained. Generally, in this case, the Schwarzschild radial coordinate is utilized in the spherical symmetry solution to the Einstein vacuum field equations. However, for convenience, different radial coordinates can be used [7]; the Fock radial coordinate is an example [6].

On the other hand, there are 4-dimensional alternative theories of gravity [10], such as the scalar-tensor theories, that are the simplest generalization of the General Relativity. In these theories, a scalar field ϕ joins to the metric of the space-time $g_{\mu\nu}$ to describe the gravitational effects of the matter. The scalar-tensor theories of the gravity admit a coupling parameter ω of the scalar field with the geometry, which is a function of the scalar field: $\omega = \omega(\phi)$. The Brans-Dicke theory corresponds to the case in that $\omega = \text{constant}$, being your value fixed from experimental observations [3]. More recently, scalar-tensor theories are investigated in several aspects, as for instance: they may be the limit of low-energy theories of unification, as string theory, since exhibit a dilaton-like gravitational scalar field [9]; also, they can be studied in order to quantization of gravity [11], as well as in the cosmological dark sector [1].

In this paper, we investigate the Newtonian limit in the context of the Brans-Dicke theory of gravity, obtaining expressions for gravitational force and the corresponding potential in terms of a general radial coordinate. In this way, the paper is organized as follows: in section 2, we exhibit a general spherical symmetry solution to the Brans-Dicke vacuum field equations considering a weak field regime; in the following section, we utilize low velocity approximation in order to obtain a formula for gravitational force. It is also shown that an appropriate selection of the radial coordinate leads to a "Yukawa" radial coordinate, because of the formal Yukawa-type term that appears in the resultant potential. Finally, we present our conclusion in section 4.

2 General Spherical Symmetry Solution to the Brans-Dicke Equations

The field equations of the Brans-Dicke theory are [3]:

$$G_{\mu\nu} = \frac{8\pi}{\phi c^4} T_{\mu\nu} + \frac{\omega}{\phi^2} \left(\phi_{,\mu} \phi_{,\nu} - \frac{1}{2} g_{\mu\nu} \phi_{,\alpha} \phi^{,\alpha} \right) + \frac{1}{\phi} (\phi_{;\mu;\nu} - g_{\mu\nu} \square \phi), \quad (1)$$

$$\square\phi = \frac{8\pi T}{(2\omega + 3)c^4}, \quad (2)$$

where c is the light velocity, $G_{\mu\nu}$ is the Einstein tensor and $\square\phi = \phi^{;\sigma}{}_{;\sigma} = g^{\sigma\gamma}\phi_{;\gamma;\sigma}$. The energy-momentum tensor associated with the material content is $T_{\mu\nu}$ and $T = T^\mu{}_\mu$.

In the weak field approximation of the Brans-Dicke theory, it is considered that the metric is given by

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad (3)$$

with $\eta_{\mu\nu}$ the metric of the plan and $h_{\mu\nu}$ is a small perturbation, such that only first-order terms in $h_{\mu\nu}$ are maintained. Also, in this approximation, the scalar field is

$$\phi = \phi_0 + \varepsilon = \phi_0 \left(1 + \frac{\varepsilon}{\phi_0}\right), \quad (4)$$

where ϕ_0 is a constant and ε a first-order term in the density of matter, so that $|\varepsilon/\phi_0| \ll 1$. Thus, we keep only the terms of first order in ε/ϕ_0 . Then, we rewrite Eq. (2) as

$$\square\varepsilon = \frac{8\pi T}{(2\omega + 3)c^4}. \quad (5)$$

In the weak field regime, the solutions of the Brans-Dicke equations are related to the solutions of the General Relativity equations with the same $T_{\mu\nu}$ [2]. To understand this result, let us make the transformation

$$g_{\mu\nu}^* = G_0\phi g_{\mu\nu}, \quad (6)$$

$$T_{\mu\nu}^* = G_0^{-1}\phi^{-1}T_{\mu\nu}, \quad (7)$$

where $G_0 = 1/\phi_0 = \left(\frac{2\omega+3}{2\omega+4}\right)G$ and G is the Newton's gravitational constant, so that the Brans-Dicke field equations can be written alternatively as [4]

$$G_{\mu\nu}^* = \frac{8\pi G_0}{c^4} \left[T_{\mu\nu}^* + \frac{(2\omega + 3)c^4}{16\pi G_0\phi^2} (\phi_{,\mu}\phi_{,\nu} - \frac{1}{2}g_{\mu\nu}^*g^{*\alpha\beta}\phi_{,\alpha}\phi_{,\beta}) \right], \quad (8)$$

$$\square^*(\ln G_0\phi) = \frac{8\pi G_0 T}{(2\omega + 3)c^4}, \quad (9)$$

being the quantities \square^* and $G_{\mu\nu}^*$ calculated with the metric $g_{\mu\nu}^*$. Now, using the weak field conditions (3) and (4), the expressions (6), (7) and (8) will be approximated by

$$g_{\mu\nu}^* = \eta_{\mu\nu} + h_{\mu\nu} + G_0\varepsilon\eta_{\mu\nu}, \quad (10)$$

$$T_{\mu\nu}^* = (1 - G_0\varepsilon)T_{\mu\nu} = T_{\mu\nu}, \quad (11)$$

$$G_{\mu\nu}^* = \frac{8\pi G_0}{c^4} T_{\mu\nu}. \quad (12)$$

As a consequence, it follows that the equations (12) are formally identical to the field equations of General Relativity, with G_0 replacing the Newton's gravitational constant G . Therefore, if the metric $g_{\mu\nu}^*(G, x)$ is a known solution of Einstein's equations in the weak field approximation for a given $T_{\mu\nu}$, then the Brans-Dicke solution in the weak field approximation, corresponding to the same $T_{\mu\nu}$, is given by

$$g_{\mu\nu}(x) = [1 - G_0\varepsilon(x)] g_{\mu\nu}^*(G_0, x), \quad (13)$$

in agreement with equations (4), (6) and (12).

Thus, according to (13), the general static spherically symmetric solution of the Brans-Dicke vacuum field equations must be

$$ds^2 = [1 - G_0\varepsilon(r)] [ds^*(G_0, x)]^2. \quad (14)$$

Here, due to the spherical symmetry, $\varepsilon(x) = \varepsilon(r)$. The factor $[ds^*(G_0, x)]^2$ represents the corresponding solution in the context of the General Relativity, but with replacement of G by G_0 . The exact solution obtained in the General Relativity case can be presented in the form [5]

$$\begin{aligned} [ds^*(G, x)]^2 = & \left(1 + \frac{\alpha_0 p(r)}{r}\right) c^2 dt^2 - \frac{P'^2}{1 + \frac{\alpha_0 p(r)}{r}} dr^2 \\ & - \left[\frac{r}{p(r)}\right]^2 (d\theta^2 + \sin^2\theta d\varphi^2), \end{aligned} \quad (15)$$

where $P = \frac{r}{p(r)}$ and α_0 is an arbitrary constant. The prime denotes the derivative with respect to r . In this solution, for $r \rightarrow \infty$, the following conditions are satisfied: $p(r) \rightarrow 1$, $p'(r) \rightarrow 0$, $P \rightarrow r$ and $P' \rightarrow 1$. Therefore, the metric (15) asymptotically becomes the metric of flat space-time. However, the function $p(r)$ is not fixed by the field equations, since that in the General Relativity the choice of the coordinates is arbitrary; as a consequence, four components of the metric $g_{\mu\nu}$ are arbitrarily fixed [8]. In this sense, if we choose for example $P(r) = 1$ and $\alpha_0 = -\frac{2GM}{c^2}$, where M is the mass of the central body, we obtain the metric in the Schwarzschild coordinates.

Now, let us consider that $\alpha_0 = -\frac{2GM}{c^2}$, so that the weak field condition (3) for the metric (15) is satisfied assuming that $\frac{GM}{c^2 r} \ll 1$. Then, (14) can be written explicitly as

$$\begin{aligned}
 ds^2 = [1 - G_0\varepsilon(r)] & \left[\left(1 - \frac{2G_0M}{c^2r} p(r) \right) c^2 dt^2 - \frac{P'^2}{1 - \frac{2G_0M}{c^2r} p(r)} dr^2 \right. \\
 & \left. - \left[\frac{r}{p(r)} \right]^2 (d\theta^2 + \sin^2\theta d\varphi^2) \right]. \tag{16}
 \end{aligned}$$

Using (16), Eq. (5) becomes equal to

$$\frac{1}{P'^2} \varepsilon'' + \left(\frac{2}{PP'} - \frac{P''}{P^3} \right) \varepsilon' = 0, \tag{17}$$

since $T = 0$. The solution is

$$\varepsilon(r) = \frac{A}{P} + B = A \frac{p(r)}{r} + B, \tag{18}$$

where A and B are arbitrary constants. Making the choices $A = \frac{2M}{(2\omega + 3)c^2}$ and $B = 0$, we can retrieve from (16) the solution for a central body in the context of the Brans-Dicke theory, written in the Schwarzschild coordinates ($p(r) = 1$) [3]. Thus, with

$$\varepsilon(r) = \frac{2Mp(r)}{(2\omega + 3)c^2r}, \tag{19}$$

the Eq. (16) is the general spherical symmetry solution to the Brans-Dicke vacuum field equations.

3 The Newtonian Limit

Let us obtain the equation of motion for a particle of mass m in the field given by (16). If the velocity of the particle is small compared with the velocity of light, the geodesic equation is

$$\frac{d^2x^i}{dt^2} + \Gamma_{00}^i c^2 = 0. \tag{20}$$

For , $x^1 = r$ we obtain

$$\frac{d^2r}{dt^2} = \frac{c^2}{2} \eta^{11} h'_{00}. \tag{21}$$

From (16) with (19), we have $\eta^{11} = -\frac{1}{P'^2}$ and

$$h_{00} = -G_0\varepsilon(r) - \frac{2G_0M}{c^2r} p(r) = -\frac{2GM}{c^2r} p(r). \tag{22}$$

Therefore, the Eq. (21) takes the form

$$\frac{d^2r}{dt^2} = -\frac{GM}{r^2} \left(\frac{p(r)^4}{p(r) - rp'(r)} \right). \quad (23)$$

Hence, the gravitational force on the particle of mass m is

$$F_G = m \frac{d^2r}{dt^2} = -\frac{GMm}{r^2} \left(\frac{p(r)^4}{p(r) - rp'(r)} \right), \quad (24)$$

while the gravitational potential is given by

$$V_G \equiv - \int [-F^i dx_i] = - \int F_G P'^2 dr = -\frac{GMmp(r)}{r}. \quad (25)$$

It is interesting to note that the choice $p(r) = 1$ means that the Schwarzschild radial coordinate is used, and then the newtonian expressions of force and potential are recovered.

However, new choices for $p(r)$ imply different radial coordinates. If one considers, for example, that

$$p(r) = 1 + \alpha e^{-\frac{r}{\lambda}}, \quad (26)$$

where α and λ are constants, we get from (25) the potential with a formal Yukawa-type term

$$V_G = -\frac{GMm}{r} (1 + \alpha e^{-\frac{r}{\lambda}}). \quad (27)$$

Thus, the metric (16), with (19) and (26), represents a solution of the Brans-Dicke theory for a "Yukawa" radial coordinate.

4 Conclusion

We found, in the context of the Brans-Dicke theory, a general spherical symmetry solution to vacuum field equations considering a weak field regime. After, in additional low velocity approximation, we obtain expressions for gravitational force and potential depending on a general radial coordinate. As an application, we exhibit the metric solution written in terms of a "Yukawa" radial coordinate.

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